BY THE SAME AUTHOR

Three Dimensional Differential Geometry (2nd Edition)	*(1966)*
Mathematical Theory of Electromagnetism	*(1965)*
Elements of Tensor Theory	*(1967)*

THEORETICAL
HYDRODYNAMICS

A VECTORIAL TREATMENT

FOR POST-GRADUATE STUDENTS

BANSI LAL

Second Edition
Revised and Enlarged

1967

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To my

Bayji (MOTHER)

To whom I owe more than

I can possibly express

PREFACE TO THE SECOND EDITION

THEORETICAL HYDRODYNAMICS is the second edition of the work, originally published under the title 'ELEMENTARY HYDRODYNAMICS' in March 1961. In preparing the second edition of this book, I have been guided by suggestions kindly made to me by the users of the first edition of the book. These include a large number of teachers in the length and breadth of this country, as well as the students at the University of Delhi whom I taught out of this text for several years at the University Campus. There appeared to be no compelling reasons for making major changes in the structure of this work. The greatest change, however, consists in the reorganization of the material dealt with besides some minor changes made throughout the text, some of which should prove of great interest:

Chapter 0: "Some Subsidiary Results" has been inserted which provides the brief resume of results on vectors, notational devices, tensors, complex-variable, etc. This makes the book self-contained with regard to the use of subsidiary results from various mathematical disciplines. Chapter 1 collects all (?) the 'Hydrokinematics' and contains 'Reynolds transport theorem (p. 22) which I could use to develope Eulerian form of continuity-condition via Lagrangian system (p. 44). Hydrodynamical singularities (sources, vortices, etc.) are introduced much earlier (p. 38) than is usual and the continuity conditions are developed in the presence of all such singularities (p. 39, 45, 47, 48), the results obtained being more general than those ordinarilly available. A method of writing the continuity-conditions is also outlined (p. 53). Rotational and irrotational motions are regarded as two facets of the general motion as a whole and hence both these aspects are dealt with simultaneously. Chapter 2 introduces the 'hydrokinetics', and besides other vital concepts, exploits fully the "energy principle" (p. 102). The occurrence of vector potential in general fluid motion (p. 129) is also noted. Chapter 3, (pp. 124-234) is the largest in the text and is devoted to 'special methods for non-viscous liquids'. Here, an extensive use is made of complex-variable theory and conformal transformations, in dealing with rotational as well as irrotational fluid motions. Images in two dimensions are extensively dealt with, and 'Circle theorem' as well as 'Blasius theorem' is used with advantage. Stokes stream function (p. 221) is introduced and is subsequently used in some three dimensional images. Chapter 4 deals with motion of cylinders and spheres; study of circular cylinder being separated from that of elliptic cylinder only for the sake of simplicity. Motion of sphere, though a three-dimensional motion, is included here in this chapter, as Butlers' sphere theorem (p. 273) in three dimensions plays the same part as does the 'Milne-Thompson circle theorem in two-dimensions, thus making the treatments similar. Chepter 5 deals with gravity-controlled liquid waves, and is reprinted almost unaltered from the first edition of this book. Chapter 6 depicts further use of conformal transformations in fluid dynamics, with a particular reference to transformation of Joukowski, as well as those of Schwarz and Christoffel. The book ends with a short account of viscosity, where my main purpose is to obtain Navier-Stokes equations in various co-ordinate systems (e.g. curvilinear). Viscous flow between parallel planes and concentric pipes has also been studied. A brief account of 'Reynolds numbers' and boundary layer theory is also offered. In the treatment of viscosity, slight use of cartesion tensors is made of.

I have tried to refer to almost all the important results by proper names, e.g. Lagrange acceleration relation (p. 24), Cauchy-Stokes decomoposition theorem (p. 23), integrals of the Eulers' equations of motion under various conditions (p. 73) as Cauchy's pressure equation, Cauchy-Bernoulli equation, Cauchy-Euler equation, etc. I have combined the results of Cauchy-residue theorem and Blasius theorem and have called this result by the name of 'Cauchy-Blasius theorem (p. 158) thus offering shorter descriptions for calculations. Where proper names were not possible, the results are pin-pointed by suitable labelling, e.g. problem of complete stream lining (p. 240), motion without (or with) circulation (p. 258) etc.

Principal Symbols Used

Angular velocity	ω
Circulation	k, Γ
Coefficient of viscosity	μ
Complex potential	$w=\phi+i\psi$
Complex velocity	$-dw/dz=u-iv$
Current function (Lagrange)	ψ
Curvilinear coordinates	(α, β, γ)
Deformation tensor	e_{ij}
Density of fluid	ρ
Density of solid	σ
Differentiation following the fluid	$\frac{d}{dt}$
External force	$\mathbf{F}=(X, Y, Z)$
Force potential	χ
Force of viscosity	N
Impulsive pressure	ϖ
Intrinsic energy	E
Kinetic coefficient of viscosity	ν
Kinetic energy	T
Nebla (Del ; Atled)	∇
Potential energy	V
Pressure	p
Spherical polar coordinetes	(r, θ, φ)
Stoke's stream function	Ψ
Solid angle	ω
Strength of a doublet	μ
Strength of a source	m
Strength of a vortex filament	K, k
Stress Tensor	T_{ij}
Time	t
Unit normal	$\mathbf{n}$
Velocity of fluid	$\mathbf{q}=(u, v, w)$
Velocity of wave	c
Volume element	dv
Vorticity	$\omega=(\xi, \eta, \zeta)$
Vorticity tensor	ω_{ij}
Wave length	λ
Wave profile	η

Key to some University Abbreviations

Ag	Agra
Alig	Aligarh
Ald	Allahabad
Ban	Banaras
Bom	Bombay
Cal	Calcutta
Del	Delhi
Gti	Gauhati
Gor	Gorakhpur
Jab	Jabalpur
Jad	Jadavpur
Kr	Krachi
Kuru (Krk)	Kurukshetra
Lkn	Lucknow
Mad	Madras
Mar	Marathawada
Osm	Osmania
Pb	Panjab
Pbi	Panjabi
Pna	Poona
Raj	Rajasthan
Sag	Saugar
Ut	Utkal

I.A.S. : Indian Administrative Service.

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CONTENTS

0 : *Some Subsidiary Results*

1 : *Basic Concepts and Hydrokinematics*

6 : Conformal Representation. Viscous Liquids

PART A : APPLICATIONS OF CONFORMAL REPRESENTATION

0: *Some Subsidiary Results*

0.10. Brief résumé of vector analysis. Since the use of vectors not only simplifies and condenses the exposition of hydrodynamics but also makes mathematical and physical concepts more tangible and easy to grasp, it is proposed to give the vectorial treatment of what follows in these pages. Throughout this text book, bold face type is used to denote vector quantities.

If **a**, **b** and **c** are any vector functions (of position), then with the vector notation

$$\mathbf{a}=\mathbf{i}a_1+\mathbf{j}a_2+\mathbf{k}a_3\equiv(a_1, a_2, a_3) \; ; \; \mathbf{b}\equiv(b_1, b_2, b_3), \text{ etc.}$$

we have the following rules of vector algebra :

$$\mathbf{a}.\mathbf{b}=\mathbf{b}.\mathbf{a}=a_1b_1+a_2b_2+a_3b_3=ab\cos\theta$$
$$\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}=ab\sin\theta\,\mathbf{n}=\Sigma\,\mathbf{i}\,(a_2b_3-a_3b_2)$$
$$\mathbf{a}.(\mathbf{b}+\mathbf{c})=\mathbf{a}.\mathbf{b}+\mathbf{a}.\mathbf{c}\;;\;\;\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$$
$$\mathbf{a}.(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\times\mathbf{b}).\mathbf{c}=\mathbf{b}.(\mathbf{c}\times\mathbf{a})=\Sigma\,a_1(b_2c_3-b_3c_2)$$
$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{b}(\mathbf{a}.\mathbf{c})-\mathbf{c}(\mathbf{a}.\mathbf{b}).$$

The vector operator ∇ (called *del*) is defined by

$$\nabla\equiv\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\equiv\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

Then, if $\phi(x, y, z)$ and $\mathbf{a}(x, y, z)$ have continuous first partial derivatives in a *region*, we have the following definitions :

Gradient. The gradient of the scalar point function ϕ is defined by

$$\text{grad}\,\phi=\nabla\phi=\mathbf{i}\frac{\partial\phi}{\partial x}+\mathbf{j}\frac{\partial\phi}{\partial y}+\mathbf{k}\frac{\partial\phi}{\partial z}.$$

Divergence. The *divergence* of the vector point function **a** is defined by

$$\text{div}\,\mathbf{a}=\nabla.\mathbf{a}=\frac{\partial a_1}{\partial x}+\frac{\partial a_2}{\partial y}+\frac{\partial a_3}{\partial z}.$$

Curl. The curl of the vector point function **a** is defined by.

$$\text{Curl}\,\mathbf{a}=\nabla\times\mathbf{a}=\left(\frac{\partial a_3}{\partial y}-\frac{\partial a_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial a_1}{\partial z}-\frac{\partial a_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial a_2}{\partial x}-\frac{\partial a_1}{\partial y}\right)\mathbf{k}.$$

Laplacian. The divergence of the grad ϕ is called the *Laplacian* of ϕ; thus

$$\nabla.(\nabla\phi)\equiv\nabla^2\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}.$$

Then operator $\nabla^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ is called the *Laplacian operator.*

Formulae involving ∇ operator

$$\text{div}\,(\mathbf{a}\phi)=\phi\,\text{div}\,\mathbf{a}+\text{grad}\,\phi\,.\,\mathbf{a} \tag{1}$$

$$\text{div}\,(\mathbf{a}\times\mathbf{b})=\mathbf{b}\,.\,\text{curl}\,\mathbf{a}-\mathbf{a}\,.\,\text{curl}\,\mathbf{b} \tag{2}$$

$$\text{curl}\,(\mathbf{a}\phi)=\phi\,\text{curl}\,\mathbf{a}+\text{grad}\,\phi\times\mathbf{a} \tag{3}$$

$$\text{curl}\,(\mathbf{a}\times\mathbf{b})=\mathbf{a}\,\text{div}\,\mathbf{b}-\mathbf{b}\,\text{div}\,\mathbf{a}+(\mathbf{b}\,.\,\nabla)\,\mathbf{a}-(\mathbf{a}\,.\,\nabla)\,\mathbf{b} \tag{4}$$

$$\text{grad}\,(\mathbf{a}\,.\,\mathbf{b})=(\mathbf{a}\,.\,\nabla)\,\mathbf{b}+(\mathbf{b}\,.\,\nabla)\,\mathbf{a}+\mathbf{a}\times\text{curl}\,\mathbf{b}+\mathbf{b}\times\text{curl}\,\mathbf{a} \tag{5}$$

$$(\mathbf{a}\,.\,\nabla)\,\mathbf{a}=\nabla(a^2/2)-\mathbf{a}\times\text{curl}\,\mathbf{a} \tag{5'}$$

$$\text{curl grad}\,\phi\equiv 0, \qquad \text{div curl}\,\mathbf{a}\equiv 0 \tag{6}$$

$$\text{curl curl}\,\mathbf{a}\equiv\text{grad div}\,\mathbf{a}-\nabla^2\mathbf{a} \tag{7}$$

NOTE. The vector operator ∇^2 must be distinguished from the scalar ∇^2, meaning div grad.

In Cartesian coordinates *only*, $\nabla^2\mathbf{q}=(\nabla^2 q_1,\ \nabla^2 q_2,\ \nabla^2 q_3)$.

In *any other* coordinate system; $\nabla^2\mathbf{q}=\text{grad div}\,\mathbf{q}-\text{curl curl}\,\mathbf{q}$.

Vector integrals. Listed hereunder are the more common vector integral theorems ; $\mathbf{q}$ being any vector point function.

Gauss' divergence theorm : $$\int_S \mathbf{q}\,.\,d\mathbf{s}=\int_C \text{div}\,\mathbf{q}\,dv \tag{8}$$

$$\int_S \mathbf{n}\times\mathbf{q}\,ds=\int_V \text{curl}\,\mathbf{q}\,dv \tag{8'}$$

Green's theorem : $$\int_V \nabla\phi\,.\,\nabla\psi\,dv=\int_S \phi\nabla\psi\,.\,d\mathbf{s}-\int_V \phi\nabla^2\psi\,dv \tag{9}$$

$$=\int_S \psi\nabla\phi\,.\,d\mathbf{s}-\int_V \psi\nabla^2\phi\,dv \tag{9'}$$

and hence : $$\int_V (\phi\nabla^2\psi-\psi\nabla^2\phi)\,dv=\int_S\left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right) \tag{10}$$

Stokes' theorem : $$\int_C \mathbf{q}\,.\,d\mathbf{r}=\int_S \text{curl}\,\mathbf{q}\,.\,d\mathbf{s} \tag{11}$$

The proofs of the last two theorems are given in §0.40, p. 6 and §1.50, p. 30.

NOTES. (i) $\nabla\phi\,.\,d\mathbf{s}=(d\phi/\partial n)\,ds$

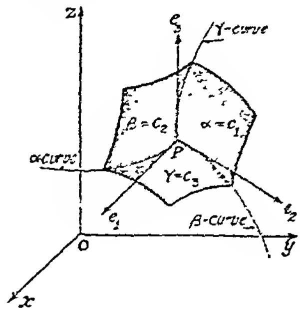
(ii) $d\mathbf{q}=\Sigma\frac{\partial\mathbf{q}}{\partial x}\,dx=\Sigma(\mathbf{i}\,dx)\,.\left(\mathbf{i}\,\frac{\partial}{\partial x}\right)\mathbf{q}=(d\mathbf{r}\,.\,\nabla)\,\mathbf{q}$

(iii) $d\phi=(d\mathbf{r}\,.\,\nabla)\,\phi=d\mathbf{r}\,.(\nabla\phi)=d\mathbf{r}\,.\,\nabla\phi$

(iv) $\frac{1}{\rho}\ \nabla p=\frac{\mathbf{n}}{\rho}\frac{\partial p}{\partial n}=\mathbf{n}\frac{\partial}{\partial n}\int\left(\frac{1}{\rho}\frac{\partial p}{\partial n}\right)dn=\mathbf{n}\frac{\partial}{\partial n}\int\frac{dp}{\rho}=\nabla\int\frac{dp}{\rho}.$

0.20 Brief rèsumè of general orthogonal curvilinear coordinates Consider three independent orthogonal families of surfaces

$$f_1(x, y, z)=\alpha,\ f_2(x, y, z)=\beta,$$
$$f_3(x, y, z)=\gamma \qquad (1)$$



where x, y, z, are the Cartesian coordinates. The surfaces

$$\alpha=\text{const.},\ \beta=\text{const.},\ \gamma=\text{const.},$$

form an orthogonal system and the values of α, β, γ may be used as the coordinates (*orthogonal curvilinear coordinates*) of a point in space. The relation between the two systems of coordinates x, y, z and α, β, γ may be expressed by equation (1) or

$$x=x(\alpha, \beta, \gamma),\ y=y(\alpha, \beta, \gamma),\ z=z(\alpha, \beta, \gamma).$$

The surfaces $\alpha=c_1$, $\beta=c_2$, $\gamma=c_3$ where c_1, c_2, c_3 are constants, are called *coordinate surfaces.*

Let $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, be the position vector of a point P. Then (1) can be written as $\mathbf{r}=\mathbf{r}\ (\alpha, \beta, \gamma)$. A tangent vector to the α-curve at P (for which β and γ are constants) is $\partial\mathbf{r}/\partial\alpha$; hence a unit tangent vector in this direction is

$$\mathbf{e}_1=\frac{\partial\mathbf{r}}{\partial\alpha}\bigg/\left|\frac{\partial\mathbf{r}}{\partial\alpha}\right| \Rightarrow \frac{\partial\mathbf{r}}{\partial\alpha}=h_1\mathbf{e}_1$$

where
$$h_1=\left|\frac{\partial\mathbf{r}}{\partial\alpha}\right|=\sqrt{\left[\left(\frac{\partial x}{\partial\alpha}\right)^2+\left(\frac{\partial y}{\partial\alpha}\right)^2+\left(\frac{\partial z}{\partial\alpha}\right)^2\right]}$$

Similarly, the unit vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ to the β-curve and γ-curve are given by $(\partial\mathbf{r}/\partial\beta)=h_2\mathbf{e}_2$ and $(\partial\mathbf{r}/\partial\gamma)=h_3\mathbf{e}_3$. The quantities h_1, h_2, h_3 are called *scale factors.*

From $\qquad \mathbf{r}=\mathbf{r}\ (\alpha, \beta, \gamma)$, we get

$$d\mathbf{r}=(\partial\mathbf{r}/\partial\alpha)\ d\alpha+(\partial\mathbf{r}/\partial\beta)d\beta+(\partial\mathbf{r}/\partial\gamma)\ d\gamma$$
$$=h_1d\alpha\ \mathbf{e}_1+h_2\ d\beta\ \mathbf{e}_2+h_3\ d\gamma\ \mathbf{e}_3.$$

Thus, $\qquad ds^2=d\mathbf{r}\cdot d\mathbf{r}=h_1^2\ d\alpha^2+h_2^2\ d\beta^2+h_3^2\ d\gamma^2.$

Also, along α-curve, β and γ are constants so that $d\mathbf{r}_1=h_1d\alpha\ \mathbf{e}_1$, whence $ds_1=h_1d\alpha$. Similarly, the differential arc lengths along β-curve and γ-curve at P are $ds_2=h_2d\beta$ and $ds_3=h_3d\gamma$. If then we draw the surfaces corresponding to α, $\alpha+d\alpha$; β, $\beta+d\beta$; γ, $\gamma+d\gamma$, we obtain an elementary curvilinear parallelopiped whose edges are $h_1d\alpha$, $h_2d\beta$, $h_3\ d\gamma$ (vide Fig. §1.73).

We shall now require the following vector definitions ; for any scalar ϕ and for any vector $\mathbf{q}=(q_1, q_2, q_3)$:

$$\text{grad}\ \phi=\left(\frac{1}{h_1}\frac{\partial\phi}{\partial\alpha},\ \frac{1}{h_2}\frac{\partial\phi}{\partial\beta},\ \frac{1}{h_3}\frac{\partial\phi}{\partial\gamma}\right) \qquad (1)$$

$$\operatorname{div} \mathbf{q}=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\alpha}(h_2h_3q_1)+\frac{\partial}{\partial\beta}(h_3h_1q_2)+\frac{\partial}{\partial\gamma}(h_1h_2q_3)\right] \qquad (2)$$

Denoting curl $\mathbf{q}$ by $\omega=(\omega_1, \omega_2, \omega_3)$;

$$\left.\begin{aligned}\omega_1&=\frac{1}{h_2h_3}\left[\frac{\partial}{\partial\beta}(h_3q_3)-\frac{\partial}{\partial\gamma}(h_2q_2)\right]\\ \omega_2&=\frac{1}{h_3h_1}\left[\frac{\partial}{\partial\gamma}(h_1q_1)-\frac{\partial}{\partial\alpha}(h_3q_3)\right]\\ \omega_3&=\frac{1}{h_1h_2}\left[\frac{\partial}{\partial\alpha}(h_2q_2)-\frac{\partial}{\partial\beta}(h_1q_1)\right]\end{aligned}\right\} \qquad (3)$$

Also required is the vector curl ω, whose components are of the form (3) with (q_1, q_2, q_3) replaced by $(\omega_1, \omega_2, \omega_3)$.

$$\nabla^2\phi=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\alpha}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{h_3h_1}{h_2}\frac{\partial\phi}{\partial\beta}\right)+\frac{\partial}{\partial\gamma}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial\gamma}\right)\right] \qquad (4)$$

Two special coordinate systems.

(A) Cylindrical polar coordinates (r, θ, z).

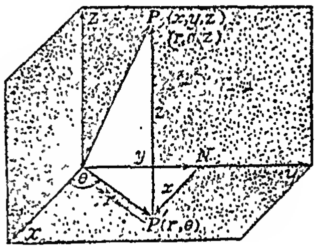
Here $(\alpha, \beta, \gamma)=(r, \theta, z)$; related to Cartesian coordinates by

$$x=r\cos\theta,\ y=r\sin\theta,\ z=z$$

$$(r\geqslant 0,\ 0\leqslant\theta\leqslant 2\pi,\ -\infty<z<\infty)$$

scale factors : $h_1=1,\ h_2=r,\ h_3=1.$

Volume element: $dx\,dy\,dz=r\,d\theta\,dr\,dz$



For the scalar ψ and vector $\mathbf{q}=(q_1, q_2, q_3)$ in the directions (r, θ, z):

$$\operatorname{grad}\psi=\left(\frac{\partial\psi}{\partial r},\ \frac{1}{r}\frac{\partial\psi}{\partial\theta},\ \frac{\partial\psi}{\partial z}\right) \qquad (A_1)$$

$$\operatorname{div}\mathbf{q}=\frac{1}{r}\left[\frac{\partial}{\partial r}(rq_1)+\frac{\partial q_2}{\partial\theta}+\frac{\partial q_3}{\partial z}\right] \qquad (A_2)$$

$$\operatorname{curl}\mathbf{q}=\omega=\left(\frac{1}{r}\frac{\partial q_3}{\partial\theta}-\frac{\partial q_2}{\partial z},\ \frac{\partial q_1}{\partial z}-\frac{\partial q_3}{\partial r},\ \frac{1}{r}\frac{\partial}{\partial r}(rq_2)-\frac{1}{r}\frac{\partial q_1}{\partial\theta}\right) \qquad (A_3)$$

$$\nabla^2\psi=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{\partial^2\psi}{\partial z^2} \qquad (A_4)$$

(B) Spherical polar coordinates (r, θ, φ)

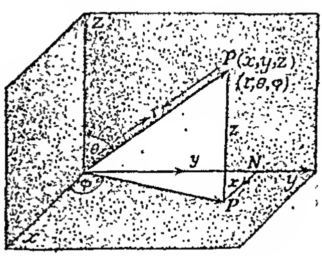
Here $(\alpha, \beta, \gamma)\equiv(r, \theta, \varphi)$; relatad to Cartesion coordinates by

$$x=r\sin\theta\cos\varphi,\ y=r\sin\theta\sin\varphi,$$
$$z=r\cos\theta$$

$$(r\geqslant 0,\ 0\leqslant\theta\leqslant\pi,\ 0\leqslant\varphi\leqslant 2\pi)$$

Scale factors : $h_1=1,\ h_2=r,\ h_3=r\sin\theta$

Volume element : $dx\,dy\,dz=r^2\sin\theta\,dr\,d\theta\,d\varphi$



For the scalar ψ and vector $\mathbf{q}=(q_1, q_2, q_3)$ in the directions (r, θ, φ):

$$\text{grad}\,\psi=\left(\frac{\partial\psi}{\partial r},\ \frac{1}{r}\frac{\partial\psi}{\partial\theta},\ \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\right) \tag{B$_1$}$$

$$\text{div}\,\mathbf{q}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2q_1)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ q_2)+\frac{1}{r\sin\theta}\frac{\partial q_3}{\partial\phi} \tag{B$_2$}$$

$$\text{curl}\,\mathbf{q}=\omega=(\omega_1,\omega_2,\omega_3),$$

where

$$\omega_1=\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(q_3\sin\theta)-\frac{\partial q_2}{\partial\phi}\right],\quad \omega_2=\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial q_1}{\partial\phi}-\frac{\partial}{\partial r}(rq_3)\right]$$

$$\omega_3=\frac{1}{r}\left[\frac{\partial}{\partial r}(rq_2)-\frac{\partial q_1}{\partial\theta}\right] \tag{B$_3$}$$

$$\nabla^2\psi=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \tag{B$_4$}$$

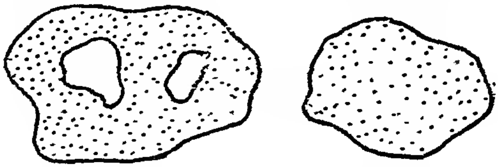
0.30. A note on connectivity. A region of space is said to be connected if a path joining any two points of the same lies *entirely* in the given region.

When the two paths taken together form a reducible circuit, they are termed *reconcilable.* And when one circuit can be continuously varied so as to coincide with another circuit without leaving the region, the two circuits are called *reconcilable.*

A simply connected (*acyclic*) region is one in which all paths connecting any two points within the region can be deformed into one another without passing outside the region. Obviously, in simply connected region, every circuit is *reducible,* i.e., it can be contracted to a point of the region without ever passing out of it.

Examples of simply connected region :

(i) The region between two concentric spheres.
(ii) Un-bounded space.
(iii) Region interior to sphere and region exterior to a sphere, etc.



Triply Connected *Simply Connected*

A region is said to be doubly connected if it can be made simply connected by the insertion of *one* barrier.

Examples of Doubly Connected region :

(i) Region between two co-axial infinitely long cylinders.
(ii) Region exterior to an infinitely long cylinder.

(iii) Region interior to an anchor ring; region exterior to an anchor ring, etc.

In general, a region is said to be **r-ply connected** if it can be made simply connected by the insertion of $(r-1)$ barriers.

The above definitions can also be expressed as under:

A domain is called **simply connected**, if the frontier thereof consists of a single continuum. Generally, a domain is called **r-ply connected** if the frontier of the same consists of r distinct continuua.

NOTE: Fortunately, the multiply-connected regions which occur in most hydrodynamical problems are of an extremely simple kind, and that it is not necessary to develop a formal topological theory (i.e. *the study of figures which survive twisting and stretching*: rubber sheet geometry).

0.40. Green's theorem. *If ϕ_1, ϕ_2 are two continuously differentiable scalar point functions such that $\nabla\phi_1$ and $\nabla\phi_2$ are also continuously differentiable and S denotes a closed surface bounding any singly-connected region of space, then*

$$\int_V (\nabla\phi_1 . \nabla\phi_2)\, dv = -\int_V \phi_1 \nabla^2\phi_2 dv - \int_S \phi_1 \frac{\partial\phi_2}{\partial n}\, ds$$

where V is the region enclosed by S and δn *an element of the normal at any point on the boundary drawn* into *the region considered.*

Proof. From vector calculus

$$\nabla . (\phi \mathbf{F}) = \phi(\nabla . \mathbf{F}) + \mathbf{F}.(\nabla\phi) \tag{1}$$

where ϕ is a scalar and **F** a vector point function.

In (1), we put $\mathbf{F} = \nabla\phi_2$, $\phi = \phi_1$; and integrate it over V; we get

$$\int_V \nabla . (\phi_1 \nabla\phi_2)\, dv = \int_V \phi_1 (\nabla . \nabla\phi_2) dv + \int_V (\nabla\phi_1) . (\nabla\phi_2) dv \tag{2}$$

By *Gauss's divergence theorem*, (2) reduces to

$$-\int_S \mathbf{n} . (\phi_1 \nabla\phi_2) ds = \int_V \phi_1 \nabla^2\phi_2 dv + \int_V (\nabla\phi_1) . (\nabla\phi_2) dv \tag{3}$$

Since $\mathbf{n} . \nabla\phi_2 = \partial\phi_2/\partial n$; therefore, (3) reduces to

$$\int_V (\nabla\phi_1) . (\nabla\phi_2) dv = -\int_V \phi_1 \nabla^2\phi_2 dv - \int_S \phi_1 \frac{\partial\phi_2}{\partial n}\, ds \tag{4}$$

The form embodied in (4) is known as Green's theorem in *non-symmetric form.*

Cor. Since interchanging ϕ_1 and ϕ_2 does not alter the left hand side of (4), another expression for the right hand side of (4) follows in which ϕ_1 and ϕ_2 are interchanged. Further, equating the right sides of (4) thus obtained gives

$$\int_V (\phi_1 \nabla^2\phi_2 - \phi_2 \nabla^2\phi_1) dv = \int_S \left(\phi_1 \frac{\partial\phi_2}{\partial n} - \phi_2 \frac{\partial\phi_1}{\partial n}\right) ds \quad \textit{(Symmetric form)}$$

If however, $\nabla^2\phi_1=0$, $\nabla^2\phi_2=0$, we get

$$\int_S \phi_1 \frac{\partial \phi_2}{\partial n}\, ds = \int_S \phi_2 \frac{\partial \phi_1}{\partial n}\, ds \qquad \text{(Reciprocal theorem)}$$

0.50. Material or total derivative of a functional determinant (i.e. Jacobian). Let

$$J=\frac{\partial(x, y, z)}{\partial(a, b, c)}=\begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial x}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial x}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{vmatrix}=\nabla x \times \nabla y \,.\, \nabla z$$

where the operator ∇ stands for $(\partial/\partial a, \partial/\partial b, \partial/\partial c)$.

We shall assume the validity of the operator

$$\frac{d}{dt}\left(\frac{\partial x}{\partial a}\right)=\frac{\partial}{\partial a}\left(\frac{dx}{dt}\right)=\frac{\partial u}{\partial a}, \text{ etc.}$$

Then the rule of differentiating products provides

$$\dot{J}=\nabla \dot{x} \times \nabla y \,.\, \nabla z+\nabla x \times \nabla \dot{y} \,.\, \nabla z+\nabla x \times \nabla y \,.\, \nabla \dot{z} \qquad (\dot{x}=dx/dt)$$

or
$$\frac{dJ}{dt}=\frac{\partial(u, y, z)}{\partial(a, b, c)}+\frac{\partial(x, v, z)}{\partial(a, b, c)}+\frac{\partial(x, y, w)}{\partial(a, b, c)} \tag{1}$$

Now, $\dfrac{\partial u}{\partial a}=\dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial a}+\dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial a}+\dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial a}$; with two more like equations for $\partial u/\partial b$ and $\partial u/\partial c$. Transposing $\partial u/\partial a$, $\partial u/\partial b$, and $\partial u/\partial c$ on the right sides of the equations and then eliminating $\partial u/\partial v$ and $\partial u/\partial z$ from these three equations provide

$$\begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial a}-\frac{\partial u}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial u}{\partial x}\frac{\partial x}{\partial b}-\frac{\partial u}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial u}{\partial x}\frac{\partial x}{\partial c}-\frac{\partial u}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{vmatrix}=0.$$

Splitting this determinant into two we get

$$\frac{\partial u}{\partial x}\,\frac{\partial(x, y, z)}{\partial(a, b, c)}=\frac{\partial(u, y, z)}{\partial(a, b, c)} \text{ or } \frac{\partial u}{\partial x}J=\frac{\partial(u, y, z)}{\partial(a, b, c)}$$

Similarly : $\dfrac{\partial v}{\partial y}J=\dfrac{\partial(x, v, z)}{\partial(a, b, c)}$ and $\dfrac{\partial w}{\partial z}J=\dfrac{\partial(x, y, w)}{\partial(a, b, c)}$.

Adding these three equations and using (1) we obtain

$$\frac{dJ}{dt}=J\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) \Rightarrow \frac{dJ}{dt}=J \operatorname{div} \mathbf{q}.$$

0.60. A brief note on dyadic products. The *indefinite* or *dyadic* product of two vectors **a** and **b** is given by **a b** or **a ; b** and has no geometrical interpretation in contradiction with geometrically significant scalar and vector products **a.b** and **a**$\times$**b**. However, this type of product is very useful in vector transformations.

A sum of dyadic products is called a *dyad* or *tensor* of rank two, e.g.

$$(\mathbf{a} ; \mathbf{b})+(\mathbf{c} ; \mathbf{d}) \text{ or } \mathbf{a} ; \mathbf{b}+\mathbf{c} ; \mathbf{d}$$

is a dyad.

The *scalar* (or *direct*) product of vector **c** with the dyad **a ; b** is the vector defined by

$$\mathbf{c}(\mathbf{a} ; \mathbf{b})=(\mathbf{c}.\ \mathbf{a})\mathbf{b} \text{ and } (\mathbf{a} ; \mathbf{b})\mathbf{c}=\mathbf{a}(\mathbf{b}.\ \mathbf{c})$$

according as the vector **c** is a prefactor or post factor. The *idem factor* or *unit dyad* I is a dyad such that, for any vector **a**

$$\mathrm{I}\mathbf{a}=\mathbf{a}\mathrm{I}=\mathbf{a} \tag{1}$$

and may be expressed in terms of usual unit vector **i, j, k** as

$$\mathrm{I}=(\mathbf{i} ; \mathbf{i})+(\mathbf{j} ; \mathbf{j})+(\mathbf{k} ; \mathbf{k}) \text{ or simply } \mathbf{ii}+\mathbf{jj}+\mathbf{kk}.$$

The equation (1) may be easily verified by taking $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$.

We may define the differerential operator

$$\nabla=\frac{\partial}{\partial \mathbf{r}}=\mathbf{i}\,\frac{\partial}{\partial x}+\mathbf{j}\,\frac{\partial}{\partial y}+\mathbf{k}\,\frac{\partial}{\partial z}\ ; \qquad \mathbf{r}=\mathbf{i}x+\mathbf{j}y+\mathbf{k}z.$$

The dyadic product is

$$\nabla ; \mathbf{r}=\frac{\partial}{\partial \mathbf{r}} ; \mathbf{r}=\left(\mathbf{i}\,\frac{\partial}{\partial x}+\ldots\right) ; (\mathbf{i}x+\ldots)=\mathbf{i} ; \mathbf{i}+\mathbf{j} ; \mathbf{j}+\mathbf{k} ; \mathbf{k}=\mathrm{I}$$

For a constant vector **a**, the scalar product is

$$\nabla(\mathbf{r}.\ \mathbf{a})=(\nabla ; \mathbf{r})\mathbf{a}=\mathrm{I}\mathbf{a}=\mathbf{a}.$$

Further, since $\dfrac{\partial\phi}{\partial x}=\dfrac{\partial\phi}{\partial \mathbf{r}}.\ \mathbf{i}=\dfrac{\partial\phi}{\partial \mathbf{r}}.\ \dfrac{\partial \mathbf{r}}{\partial x}$, we may conclude

$$\frac{\partial}{\partial \mathbf{N}}=\mathbf{i}\,\frac{\partial}{\partial l}+\mathbf{j}\,\frac{\partial}{\partial m}+\mathbf{k}\,\frac{\partial}{\partial n} \text{ where } \mathbf{N}=\mathbf{i}l+\mathbf{j}m+\mathbf{k}n\,;$$

and hence if **a** is a constant vector, then

$$\frac{\partial}{\partial \mathbf{N}}\,(\mathbf{N}.\ \mathbf{a})=\left(\frac{\partial}{\partial \mathbf{N}} ; \mathbf{N}\right)\mathbf{a}=\mathrm{I}\mathbf{a}=\mathbf{a}.$$

We may also verify, $\dfrac{\partial ; \mathbf{r}}{\partial \mathbf{r}_0}\ \dfrac{\partial ; \mathbf{r}_0}{\partial \mathbf{r}}=\mathrm{I}, \ \Rightarrow\ \dfrac{\partial ; \mathbf{r}}{\partial \mathbf{r}_0}\ \dfrac{\partial}{\partial \mathbf{r}}=\dfrac{\partial}{\partial \mathbf{r}_0}$ and

$$\nabla_0 f=(\nabla_0 ; \mathbf{r}).\nabla f.$$

We may further note that if $\partial x/\partial a=x_a$, $\nabla_0=\partial/\partial\mathbf{r}_0$, $\mathbf{i} ; \mathbf{i}=\mathbf{ii}$, etc. then

$$\nabla_0 ; \mathbf{r}=x_a\mathbf{ii}+y_a\mathbf{ij}+z_a\mathbf{ik}+x_b\mathbf{ji}+y_b\mathbf{jj}+z_b\mathbf{jk}+x_c\mathbf{ki}+y_c\mathbf{kj}+z_c\mathbf{kk}.$$

0.70. Brief résumé of complex function theory. A very powerful technique for dealing with two-dimensional problems in theoretical hydrodynamics is furnished by the properties of *analytic functions* (i.e. functions possessing derivatives for all values of $z=x+iy$ in a region) of a complex variable z. Thus, if $f(z)$ is *regular* (analytic) in a domain D of the complex z-plane, and if we write

$$W=f(z)=\phi(x, y)+i\psi(x, y)$$

then it is shown in all texts on 'complex variable' that if $f(z)$ is to possess a unique derivative, then it is necessary as well as sufficient that

$$\partial\phi/\partial x=\partial\psi/\partial y\ ;\ \partial\phi/\partial y=-\partial\psi/\partial x \tag{1}$$

where it is supposed that these partial derivatives are continuous. These are called *Cauchy-Riemann partial differential equations.* The vector equivalent to (1) is

$$\text{grad }\phi=(\text{grad }\psi)\times\mathbf{k}\ ;\ \ \mathbf{k}=(0, 0, 1) \tag{2}$$

An alternative single equivalent expression to (1) is $\partial\phi/\partial n=\partial\psi/\partial s$ where n and s are perpendicular directions related to each other in the anti-clockwise sense.

If we eliminate ψ and ϕ in succession between equations (1), we get

$$\frac{\partial^2\phi}{\partial x^2}=\frac{\partial^2\psi}{\partial x\partial y}=-\frac{\partial^2\phi}{\partial y^2},\ \text{i.e. } \nabla^2\phi=0\ ;\ \text{likewise } \nabla^2\psi=0,$$

where $\nabla^2=(\partial^2/\partial x^2)+(\partial^2/\partial y^2)$. These two conjugate functions, ϕ and ψ, are called the *velocity potential* and *stream function* (or *current function*), though we could, of course, interchange ϕ and ψ and as such write $f(z)=\psi+i\phi$.

Since ϕ and ψ are *harmonic functions* (i.e. functions which satisfy the Laplace's equations $\nabla^2\phi=0$, $\nabla^2\psi=0$) these will be the *possible* velocity potential and stream function, and provided the necessary boundary conditions for a problem are satisfied, these will yield a unique solution to the problem.

If $W=f(z)$ provides the solution to a hydrodynamical problem, it is called the *complex potenial* characterizing the given fluid flow.

It may be observed that equations (1) imply that the family of curves, $\phi(x, y)=\text{const.}$ (*a set of equipotentials*) and $\psi(x, y)=\text{const.}$ (*a set of stream lines*), are orthogonal families, since

$$(\partial\phi/\partial x)(\partial\psi/\partial x)+(\partial\phi/\partial y)(\partial\psi/\partial y)=0.$$

We now include clear *statements* of pertinent definitions, principles, and theorems which are relevant to the study of Hydrodynamics.

Cauchy's theorem. If $f(z)$ is analytic *within* the region bounded by C (a simple *closed* curve) as well as *on* C, then

$$\int_C f(z)\,dz=\oint_C f(z)dz=0.$$

A simple consequence of this theorem is that $\int_{z_1}^{z_2} f(z)dz$ has a value *independent of path* joining z_1 and z_2.

Cauchy's integral formulae. If $f(z)$ is analytic within and on a simple *closed* positively oriented curve C, and z_0 is any point interior to C, then

$$f(z_0)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)}\,dz\ ;\ f^{(n)}(z_0)=\frac{n\,!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{(n+1)}}\,dz$$

where $f^{(n)}(z_0)$ is the nth derivative of $f(z)$ at $z=z_0$.

Taylor's series. Let $f(z)$ be analytic inside and on a circle having its centre at $z=a$. Then for all points z in the circle

$$f(z)=f(a)+(z-a)f'(a)+[(z-a)^2/2\,!]f''(a)+[(z-a)^3/3\,!]\,f'''(a)+\ldots\ldots$$

If $a=0$, there results *Maclaurin series.*

Singular points. A singular point of a function $f(z)$ is a value of z at which $f(z)$ ceases to be analytic. If $f(z)$ is analytic everywhere in some domain except at an interior point $z=a$, then $z=a$ is called an *isolated singularity* of $f(z)$: [obviously $z=1$ is an isolated singularity of $f(z)=1/(z-1)^2$].

Poles. If $f(z)=F(z)/(z-a)^n$; $F(a)\neq 0$, where $F(z)$ is analytic everywhere in a region *including* $z=a$, and if n is a positive integer, then $f(z)$ has an isolated singularity at $z=a$. This isolated singularity is called a *pole of order n.* If $n=1$, the pole is called a simple pole ; if $n=2$, it is called a *double* pole, and so on.

Laurent's series. If $f(z)$ is analytic inside and on the boundary of the ring-shaped region R bounded by two concentric circles (positively oriented) C_1, C_2 with centre at a and respective radii R_1 and R_2 $(R_1>R_2)$, then for all z in R

$$f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n+\sum_{n=1}^{\infty} a_{-n}(z-a)^{-n}$$

where
$$a_n=\frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{(z-a)^{n+1}}\,dz\ ;\quad n=0,\,1,\,2,\ldots$$

$$a_{-n}=\frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{(z-a)^{-n+1}}\,dz\ ;\quad n=1,\,2,\ldots$$

With an appropriate change of notation and replacing C_1, C_2 by some concentric circle C between C_1 and C_2, we can write the above as

$$f(z)=\sum_{n=-\infty}^{\infty} a_n\,(z-a)^n$$

where
$$a_n=\frac{1}{2\pi i}\int_C\frac{f(z)}{(z-a)^n}\,dz \qquad n=0,\,\pm 1,\,\pm 2,\ldots$$

The part $\sum_{n=0}^{\infty} a_n\,(z-a)^n$ is called the *analytic part* and the remainder

$\sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}$ is called the *principal part*. If the principal part is zero, the Laurent series reduces to a Taylor series.

Residues. If $f(z)$ be single-valued and analytic inside and on a circle C except at the point $z=a$, chosen as the centre of C, then Laurent series is given by

$$f(z)=a_0+a_1(z-a)+a_2(z-a)^2+\dots+a_{-1}(z-a)^{-1}+a_{-2}(z-a)^{-2}+\dots$$

where $$a_n=\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz \qquad n=0, \pm 1, \pm 2, \dots$$

Clearly, $$a_{-1}=\frac{1}{2\pi i}\int_C f(z)\,dz=\lim_{z\to a}\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\{(z-a)^n f(z)\}$$
where n is the order of the pole.

The coefficient a_{-1} is called the *residue* of $f(z)$ at the pole $z=a$. For simple poles $(n=1)$, $a_{-1}=\text{Lim}\,(z-a)f(z)$ as $z\to a$.

Cauchy's residue theorem. If $f(z)$ is analytic on the boundary C of a region R except at a finite number of poles *within* R, then

$$\int_C f(z)\,dz=2\pi i[\text{sum of the residues of } f(z) \text{ at its poles}].$$

Cauchy's theorem and Cauchy's integral formulae are special cases of this theorem.

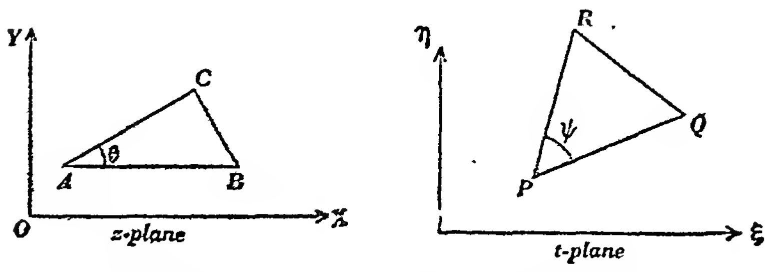
0.71. Conformal Representation. Suppose a bi-uniform mapping of a region of the z-plane on a region of the t-plane is connected by the relation

$$t=f(z) \tag{1}$$

Let z, z_1 and z_2 be represented by the points A, B, and C respectively in the z-plane and let the corresponding values t, t_1 and t_2 be represented by points P, Q and R in t-plane.

Then

$$\frac{t_1-t}{z_1-z}=\frac{f(z_1)-f(z)}{z_1-z}\,;\quad \frac{t_2-t}{z_2-z}=\frac{f(z_2)-f(z)}{z_2-z}.$$



Provided that AB and AC are *small enough*, we have

$$[(t_1-t)/(z_1-z)]=f'(z)\,;[(t_2-t)/(z_2-z]=f'(z)$$

and consequently $$\frac{t_1-t}{z_1-z}=\frac{t_2-t}{z_2-z}=f'(z)=\frac{dt}{dz}.$$

It follows, by taking arguments and modulus that

$$\frac{AC}{AB}=\frac{PR}{PQ} \text{ and } 0=\psi.$$

Therefore, the triangles ABC and PQR are *directly similar.*

Also $\left|\frac{t_1-t}{z_1-z}\right|=\left|\frac{f'(z)}{1}\right|=\left|\frac{dt/dz}{1}\right|$, it follows that the linear dimensions are in the ratio of $1 : |dt/dz|$, and the ratio of the corresponding small areas, i.e.

$$\frac{\triangle PQR}{\triangle ABC}=|f'(z)|^2=f'(z).\bar{f}'(\bar{z})=\frac{dt}{dz}\cdot\frac{d\bar{t}}{d\bar{z}}$$

are in the ratio $1 : dt/dz$.

Thus the mapping given by (1) is such that *an infinitesimal triangle in one plane maps into a directly similar infinitesimal triangle in the other plane, preserving the angles and the similarity of the corresponding infinitesimal triangles.*

Since small elements of area are unaltered in shape, the transformation is said to be **Conformal.** The factor $|dt/dz|$ is often referred to as the **linear magnification.**

By a proper choice of formulae of transformation, motion with a complicated boundary can be deduced from that with a simpler boundary. An extensive use is made of several sets of transformations and applied successfully to potential flow in two dimensions. Thus a problem which stands unsolved in one physical configuration (say z-plane) may be solved into another configuration (say t-plane) by some suitable transformation. *The problem may thus be regarded not as that of finding a direct solution, but of finding a proper transformation into a configuration which admits of an immediate solution.* Though not always applicable, it is the most reliable method to derive exact solutions.

NOTE. Some of the important conformal transformations which map the given regions into the *upper half of* ζ-plane are as under :

Semi-infinite Region	*Transformation*	*Infinite Region*	*Transformation*
$x=0, y=0, y=a$	$\zeta=\cosh(\pi z/a)$	$y=0, y=a$	$\zeta=e^{\pi z/a}$
$x=0, y=\pm a/2$	$\zeta=i\sinh(\pi z/a)$	$y=\pm a/2$	$\zeta=ie^{\pi z/a}$
$x=0, y=0, x=a$	$\zeta=-\cos(\pi z/a)$	$x=0, x=a$	$\zeta=e^{i\pi z/a}$
$y=0, x=\pm a/2$	$\zeta=\sin(\pi z/a)$	$x=\pm a/2$	$\zeta=ie^{\pi z/a}$

These results are simple consequences of Schwarz-Christoffel transformation, vide Chapter VI.

Similarly ; $\zeta=\frac{a}{2}\left(z+\frac{1}{z}\right)$ maps, *half-plane with semicircle removed* on the upper ζ-plane ; and $\zeta=[(1+z^m)/(1-z^m)]^2$, $m\geq 1/2$ maps the sector of a circle of unit radius bounded by $0\leq\theta\leq\pi/m$.

0.80. Some notational devices

Component notation. Sometimes, it will be convenient to adopt the following notations :

$$(x, y, z)=(x_1, x_2, x_3) \; ; \; (u, v, w)=(v_1, v_2, v_3)$$

so that $$\text{div } \mathbf{q}=\Sigma(\partial v_i/\partial x_i), \quad (\mathbf{q} \cdot \nabla)=\Sigma v_i\,(\partial/\partial x_i).$$

Summation convention. *Whenever in a term, a repeated suffix occurs, then a summation over all possible values of that suffix is implied.* Thus

$$a_1b_1+a_2b_2+\ldots+a_Nb_N=a_ib_i \; ;$$
$$v_1\,(\partial/\partial x_1)+v_2\,(\partial/\partial x_2)+v_3\,(\partial/\partial x_3)=v_i\,(\partial/\partial x_i), \text{ etc.}$$

Consequently, the summation sign Σ is dispensed with.

Kronecker delta. The two-suffix symbal δ_{ij}, called Kronecker delta is defined by

$$\delta_{ij}=0 \text{ (if } i\neq j) \; ; \; \delta_{ij}=1 \text{ (if } i=j \text{ and no summation over } i).$$

Thus, $\delta_{11}=1$, $\delta_{22}=1$, $\delta_{33}=1$, $\delta_{12}=0=\delta_{31}=\delta_{23}$, etc.

And $\delta_{ii}=\delta_{11}+\delta_{22}+\delta_{33}=3$. Obviously. $\delta_{ij}\,a_{jk}=a_{ik}$, etc.

Permutation symbol. *This is defined by*

$$\epsilon_{ijk}=1, -1, 0$$

according as i, j, k is an even, or odd or non-distinct permutation of 1, 2, 3. Thus,

$$\epsilon_{123}=\epsilon_{231}=\epsilon_{312}=1 \; ; \; \epsilon_{213}=\epsilon_{321}=\epsilon_{132}=-1 \; ;$$
$$\epsilon_{111}=\epsilon_{122}=\epsilon_{323}=\ldots=\epsilon_{333}=0$$

If $|\,a_{ij}\,|=a$ is a third order determinant, then it may be shown that

$$\epsilon_{ijk}\,a=\epsilon_{lmn}\,a_{il}\,a_{jm}\,a_{kn}.$$

By a direct evalution, it may be further shown that

$$\epsilon_{ijk}\,\epsilon_{ilm}=\delta_{jl}\,\delta_{km}-\delta_{jm}\,\delta_{kl} \; ; \; \epsilon_{ijl}\,\epsilon_{ijm}=2\,\delta_{lm}.$$

These concepts may be fruitfully utilized in coordinate transformations. Thus, if $\mathbf{R}$ and $\mathbf{r}$ are the position vectors of a point referred to a moving and a fixed orthonormal triad, then

$$\mathbf{R}=\mathbf{r}-\mathbf{b} \text{ yielding } y_i\mathbf{e}_i=(x_j-b_j)\,\mathbf{i}_j \; ; \quad \text{(Vide §0.81, p. 14)}$$

Thus, $$y_i=\mathbf{e}_i.\,\mathbf{i}_j\,(x_j-b_j)=l_{ij}\,(x_j-b_j)$$

where l_{ij} is cosine of the angle between the ith rotating axis and the jth fixed axis. The large number of relations which l_{ij} satisfy are embodied into a single expression

$$l_{ij}\,l_{ik}=\delta_{jk}.$$

Definition of a tensor. A tensor is a quantity T_{ijk}...with n suffixes, which obeys the transformation law

$$T'_{ijk\cdots}=(l_{ip}\, l_{jq}\, l_{kr}\cdots\cdots)\, T_{pqr\cdots}$$

and respresents a *physical invariant.*

Stress tensor. It is a nine-component physical system T_{ij}, independent of any coordinate system, and so designed that the force on any small element of area ds at a field point is

$$\delta\mathbf{F}=T_{ij}\,.\,d\mathbf{s}=T_{ij}\,.\,\mathbf{n}ds \qquad \text{or} \qquad \delta F_j=T_{ij}\, n_i ds.$$

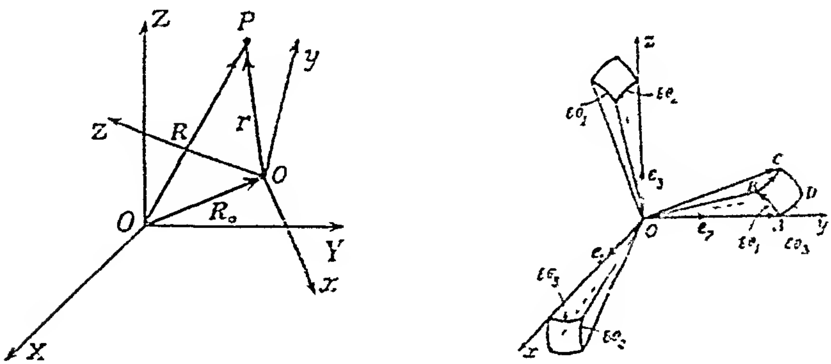
This definition implies that the *total force over a closed surface* is

$$\mathbf{F}=\int_S T_{ij}\,.\,d\mathbf{s}=\int_V (\nabla\,.\,T_{ij})dV \quad \text{(by Gauss's theorem).}$$

NOTE: In T_{ij}, the first suffix refers to the direction of the normal to the plane surface; the second suffix refers to the direction of the stress-component. Matrix representation of T_{ij} is extremely useful.

0.81*. Velocity and acceleration in Eulerian coordinates with regard to rotating frame of reference. Let the position vector of a fluid particle at P with regard to axes moving with constant angular velocity $\omega=(\omega_1, \omega_2, \omega_3)$ be $\mathbf{r}=x\mathbf{e}_1+y\mathbf{e}_2+z\mathbf{e}_3$ where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are unit *orthonormal vectors along the moving axes. Let $\mathbf{R}$ be the position* vector of P with regard to axes fixed in space; and let the origin O of the moving system have the position vector $\mathbf{R}_0$ and translate with velocity $\mathbf{q}_0$. Then with obvious notation (see Fig.)

$$\mathbf{R}=\mathbf{R}_0+\mathbf{r} \tag{1}$$



(i) (ii)

If we use dots for total differentiation with regard to time t, we immediately get the absolute velocity of the particle

$$\dot{\mathbf{R}}=\dot{\mathbf{R}}_0+\dot{\mathbf{r}} \tag{2}$$

Now $\qquad \mathbf{r}=x\mathbf{e}_1+y\mathbf{e}_2+z\mathbf{e}_3,$

$$\therefore \quad \dot{\mathbf{r}}=(\dot{x}\mathbf{e}_1+\dot{y}\mathbf{e}_2+\dot{z}\mathbf{e}_3)+(x\dot{\mathbf{e}}_1+y\dot{\mathbf{e}}_2+z\dot{\mathbf{e}}_3) \tag{3}$$

* This article is placed here for convenience, and *must be read only after* §1.31, p. 23.

To find $\dot{\mathbf{e}}_1$, $\dot{\mathbf{e}}_2$, $\dot{\mathbf{e}}_3$ we observe that if the xyz system turns through an angle $\delta\theta_1$ about the x-axis, $\delta\theta_2$ about the y-axis and $\delta\theta_3$ about the z-axis, [vide Fig. (ii)]

$$OC=\overline{OA}+AB+\overline{BC}$$

i.e. $$\mathbf{e}_2+\delta\mathbf{e}_2=\mathbf{e}_2+\delta\theta_1\mathbf{e}_3-\delta\theta_3\mathbf{e}_1.$$

Now cancelling $\mathbf{e}_2$, dividing by δt and proceeding to the limit we get

$$\dot{\mathbf{e}}_2=\mathbf{e}_3\dot{\theta}_1-\mathbf{e}_1\dot{\theta}_3=\mathbf{e}_3\omega_1-\mathbf{e}_1\omega_3$$

Similarly, $$\dot{\mathbf{e}}_3=\mathbf{e}_1\omega_2-\mathbf{e}_2\omega_1\ ;\ \dot{\mathbf{e}}_1=\mathbf{e}_2\omega_3-\mathbf{e}_3\omega_2.$$

Thus, $$x\dot{\mathbf{e}}_1+y\dot{\mathbf{e}}_2+z\dot{\mathbf{e}}_3=(\omega_2 z-\omega_3 y)\mathbf{e}_1+(\omega_3 x-\omega_1 z)\mathbf{e}_2+(\omega_1 y-\omega_2 x)\mathbf{e}_3$$
$$=\boldsymbol{\omega}\times\mathbf{r}$$

Thus (3) yields, $\mathbf{r}=q'+\boldsymbol{\omega}\times\mathbf{r}$ and thereby (2) gives

$$\mathbf{q}=\mathbf{q}_0+\mathbf{q}'+\boldsymbol{\omega}\times\mathbf{r} \tag{4}$$

i.e. *absolute velocity=velocity of the moving system+relative velocity due to motion within the system+velocity due to turning within the system.*

To find the absolute acceleration $\mathbf{a}$, we need differentiate (4) with regard to time t ; since $\mathbf{q}_0$ and $\boldsymbol{\omega}$ are constants, $\mathbf{q}_0=0$, $\dot{\boldsymbol{\omega}}=0$. Hence we get

$$\mathbf{a}=\dot{\mathbf{q}}'+\boldsymbol{\omega}\times\mathbf{r} \tag{5}$$

Now $\mathbf{r}=\mathbf{q}'+\boldsymbol{\omega}\times\mathbf{r}$ and $\dot{\mathbf{q}}'=(\dot{x}\mathbf{e}_1+\dot{y}\mathbf{e}_2+\dot{z}\mathbf{e}_3)^{\cdot}=\mathbf{a}'+\boldsymbol{\omega}\times\mathbf{q}'$,

$$\therefore\quad \mathbf{a}=\mathbf{a}'+\boldsymbol{\omega}\times\mathbf{q}'+\boldsymbol{\omega}\times(\mathbf{q}'+\boldsymbol{\omega}\times\mathbf{r})\quad\text{by (5)}$$
$$=\frac{\partial\mathbf{q}'}{\partial t}+(\mathbf{q}'.\nabla)\mathbf{q}'+2\boldsymbol{\omega}\times\mathbf{q}'+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \tag{6}$$

In Eulerian coordinates, we have the following identities :

(i) $\dfrac{\partial}{\partial t}(\boldsymbol{\omega}\times\mathbf{r})=0$, (ii) $\dfrac{\partial\mathbf{q}}{\partial t}=\dfrac{\partial}{\partial t}(\mathbf{q}_0+\mathbf{q}'+\boldsymbol{\omega}\times\mathbf{r})=\dfrac{\partial\mathbf{q}'}{\partial t}$

(ii) *$(\mathbf{q}'.\nabla)\mathbf{q}=(\mathbf{q}'.\nabla)\mathbf{q}'+\boldsymbol{\omega}\times\mathbf{q}'$

Using these indentities in (6) we get

$$\mathbf{a}=\frac{\partial\mathbf{q}}{\partial t}+(\mathbf{q}'.\nabla)\mathbf{q}+\boldsymbol{\omega}\times(\mathbf{q}'+\boldsymbol{\omega}\times\mathbf{r})$$
$$=\frac{\partial\mathbf{q}}{\partial t}+(\mathbf{q}'.\nabla)\mathbf{q}+\boldsymbol{\omega}\times\mathbf{q},\ \text{(neglecting } \mathbf{q}_0).$$

0.90. Boundary value problems. Scientific problems are often formulated mathematically which lead to *partial differential equations* and associated conditions called *boundary conditions*. Consequently, the *existance* and *uniqueness* of the problem is of fundamental impo tance, from a mathematical as well as physical point of view.

*$(\mathbf{q}.\nabla)\mathbf{q}=(\mathbf{q}'.\nabla)(\mathbf{q}_0+\mathbf{q}'+\boldsymbol{\omega}\times\mathbf{r})=(\mathbf{q}'.\nabla)\mathbf{q}'+\boldsymbol{\omega}\times(\mathbf{q}'.\nabla)\mathbf{r}$

Since $\boldsymbol{\omega}\times(\mathbf{q}'.\nabla)\mathbf{r}=\boldsymbol{\omega}\times\left(u'\dfrac{\partial\mathbf{r}}{\partial x}+v'\dfrac{\partial\mathbf{r}}{\partial y}+w'\dfrac{\partial\mathbf{r}}{\partial z}\right)=\boldsymbol{\omega}\times\mathbf{q}'$ [$\therefore$

Thus $(\mathbf{q}'.\nabla)\mathbf{q}=(\mathbf{q}'.\nabla)\mathbf{q}'+\boldsymbol{\omega}\times\mathbf{q}'$

Two types of boundary, (i) *the open boundary* (whre the region of interest extends indefinitely in one or more directions, without any specification of the solution in these directions), and (ii) *the closed boundary* (where the region of interest is completely surrounded, with boundary conditions specified in all directions) are usually considered along with three types of boundary conditions.

(1) *Dirichlet's conditions* : require the determination of a function ϕ, satisfying Laplace's equation in R and taking prescribed values on the boundary C.

(2) *Neumann's conditions* : require the determination of a function ϕ, satisfying Laplace's equation in R and taking prescribed values of normal derivative $(\partial\phi/\partial n)$ on the boundary C.

(3) *Cauchy's conditions* : require the determination of a function ϕ satisfying Laplace's equation in R and taking prescribed values of ϕ as well as $(\partial\phi/\partial n)$ on the boundary C.

Here R may be a simply-connected region bounded by a simple closed curve C, or R may be unbounded region $(y \geqslant 0)$.

The general partial differential equation of the second order, viz.

$$Rr+Ss+Tt+f(x, y, z, p, q)=0$$

is classed under three heads :

Hyperbolic if $S^2-4RT>0$, [e.g. $(\partial^2z/\partial x^2)=(\partial^2z/\partial y^2)$: Wave Eqn].

Parabolic if $S^2-4RT=0$, [e.g. $(\partial^2z/\partial x^2)=(\partial z/\partial y)$: Diffusion Eqn].

Elliptic if $S^2-4RT<0$, [e.g. $(\partial^2z/\partial x^2)+(\partial^2z/\partial y^2)=0$, Harmonic Eqn.]

The field of boundary-value problems is extensive and in the present text, only a few hydrodynamical problems based on the elliptic equation are dealt with. These problems originate from the fact that ψ=Const., is a rigid boundary and physical problem has other specified conditions, e.g. $(\partial\phi/\partial n)=0$, etc. For an extensive mathematical formulation of physical problems as differential equations with initial and boundary conditions, with clear explanations is given in "*A collection of problems on Mathematical Physics*" by Budak, Samarskii and Tikhonov ; Pergamon Press London.

NOTE : *Elliptic equations* are characterized by the fact that the differential operators are homogeneous in the second partial derivatives : they all occur with the same sign, e.g. Laplace's and Poisson's equations and Helmholtz's equation.

Hyperbolic equations are those where one of the second-order derivatives is opposite in sign to the others, e.g. the wave equation.

Parabolic equations are those where the derivative with respect to one of the variables occurs only in the first order, e.g. the diffusion equation.

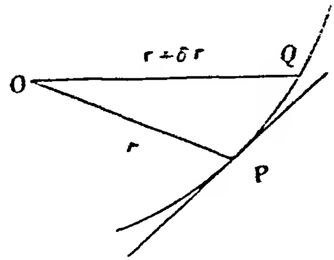
A fluid particle is a volume whose linear dimensions are negligible and it may be thought of as a geometrical point for the purpose of investigating its velocity and acceleration.

1.11. Velocity of a fluid particle. Let the particle be at P at any time 't' where

$$\overline{OP}=\mathbf{r}$$

and at time $(t+\delta t)$, it be at Q, where

$$\overline{OQ}=\mathbf{r}+\delta\mathbf{r}.$$



The velocity of the particle at P is then defined by the vector

$$\mathbf{q}=\underset{\delta t\to 0}{\mathrm{Lt}}\frac{\mathbf{r}+\delta\mathbf{r}-\mathbf{r}}{t+\delta t-t}$$

$$=\underset{\delta t\to 0}{\mathrm{Lt}}\frac{\delta\mathbf{r}}{\delta t}=\frac{d\mathbf{r}}{dt}.$$

Thus the velocity $\mathbf{q}$ is a function of $\mathbf{r}$ and t and can be expressed mathematically as $\mathbf{q}=f(\mathbf{r}, t)$. The form of the function f gives the motion of the fluid.

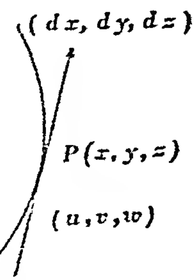
To detect the motion of an individual particle, we illuminate it by some convenient device so as to make it a luminous particle. If the tracks of the luminous particle form the parts of the regular system of curves. the motion is termed as *stream line motion*; and if the tracks are wildly irregular, the motion is called *turbulent.*

NOTES : (1) Flow across a surface. The instantaneous mass rate of flow across any surface S is defined by

$$\int_S \rho\,(\mathbf{q}\,.\,\mathbf{n})\,dS.$$

(2) Velocity at a point by the Flux Method is measured thus : Hold a small plane surface at the point perpendicular to the direction of flow : then the velocity at the point is measured by the time rate of flow of volume of fluid per unit area across the surface.

1.12. Stream lines : *A stream line is a curve drawn in the fluid such that, at any time, the direction of the tangent at any point of the curve coincides with the direction of the velocity of the fluid particle at that point.* Thus if u, v, w be the components of velocity of the fluid particle at $P(x, y, z)$. the direction ratios of the tangent being $d\mathbf{r}=(dx, dy, dz)$ at that point, the differential equations of 'stream lines' are



$$\mathbf{q}\times d\mathbf{r}=0$$

or
$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w} \tag{1}$$

where
$$\mathbf{q}=u\mathbf{i}+v\mathbf{j}+w\mathbf{k}.$$

Stream lines form a doubly infinite set at any time t. They are generally not material curves : a stream line through

$\mathbf{r}_0$ at time t_0 does not, in general, consist of the same particles as the stream line which goes through $\mathbf{r}_0$ at any other time t. The aggregate of all stream lines is called the *stream-pattern.*

The appearance and form of the stream-pattern is altered completely if a uniform velocity is superimposed on the fluid as a whole, e.g. the solutions of

$$\frac{dx}{u-u_0}=\frac{dy}{v}=\frac{dz}{w} \tag{2}$$

differ markedly from those of (1). Thus, the stream lines due to a *fixed* sphere in an infinite uniform stream are very different from those occasioned by the motion of a sphere in a *still* stream; although the two systems are dynamically equivalent.

The point where $\mathbf{q}=0$ (the *stagnation point*) is such that the stream lines are not well-defined there at; a stream line may divide into two branches at such a point.

Path lines: *A path line is a curve which a particular fluid particle describes during its motion.* The differential equations of the path lines are

$$\dot{x}=u\,; \qquad \dot{y}=v\,; \qquad \dot{z}=w \qquad (\dot{x}=dx/dt)$$

where the dot denotes differentiation with respect to time.

Path lines form a triply infinite set.

Difference between the stream lines and path lines: Consider a particular stream line and take any three consecutive points A, B and C on it. Since the velocity $\mathbf{q}$ is a function of $\mathbf{r}$ and t, any particle through A at time t will move along AB, but when it reaches B in time δt, BC shall no longer be the direction of velocity at B. Consequently the particle will not move in the direction of the new velocity at B. However, in the case of steady motion, the stream lines remain unchanged as the time passes, and so these are the same as the actual paths of the fluid particle. In passing we may note that stream lines reveal *how each fluid particle is moving at a given instant*, whereas the path lines show *how a given particle is moving at each instant.*

Stream tube: The stream lines drawn through each point of a closed curve enclose a tubular surface in the fluid, called a stream tube or tube of flow. A stream tube of infinitesimal cross-section is called a stream filament.

Ex. *What are stream lines? Are stream lines and the paths of particles of a fluid always the same? Give reasons.* (Ag. 1956)

Exp. *Find the stream lines and the paths of the particles for the two-dimensional velocity field*

$$u=x/(1+t),\; v=y,\; w=0.$$

Sol. The stream lines at time t are the solutions of

$$\frac{dx}{ds}=\frac{x}{1+t},\quad \frac{dy}{ds}=y,\quad \frac{dz}{ds}=0.$$

Thus, keeping t constant, (i.e. at a particular instant), the stream line through $\mathbf{r}_0$ (a, b, c) is

$$x=ae^{s/(1+t)},\ y=be^{s},\ z=c.$$

This is a curve (in the plane $z=c$) ; $y/b=(x/a)^{(1+t)}$.

The particle paths are solutions of

$$\frac{dx}{dt}=\frac{x}{1+t},\ \frac{dy}{dt}=y,\ \frac{dz}{dt}=0$$

These are $x=a(1+t)$, $y=be^{t}$, $z=c$: or the curves in the plane $z=c$ given by $y=be^{(x-a)/a}$.

Ex. 1. Show that if $\mathbf{q}/|\mathbf{q}|$ is independent of t, then stream lines coincide with path lines.

Ex. 2. Find the stream lines when

(i) $u=ax,\ v=-ay,\ w=c$ [Intersections of $xy=\lambda,\ x=\mu e^{az/c}$]

(ii) $u=ax,\ v=ay,\ w=-2az$ [Intersection of $y^2z=k_1,\ x=k_2y$]

(iii) $u=-ay/(x^2+y^2),\ v=ax/(x^2+y^2),\ w=c$ [Helices on the cylinders $x^2+y^2=b^2$]

(iv) $u=k(y^2+z^2-2x^2)/r^5,\ v=-3kxy/r^5,\ w=-3kxz/r^5$ [$(x^2+y^2)^3=cy^4$]

Ex. 3 Show that the stream lines and the particle paths coincide for the velocity field $[x/(1+t),\ y/(1+t),\ z/(1+t)]$.

Ex. 4. Find the stream lines and particle paths for velocity field

$$[x/(1+at),\ y/(1+bt),\ z/(1+ct)].$$

1.20. The Eulerian and Lagrangian methods. Now we describe two methods by which the general problem of Hydro-dynamics can be dealt with. These are Eulerian (Flux) and Lagrangian Methods and refer to '*Local time-rate*' of change and '*individual time-rate*' of change.

(1) Euler's method : In this method we select any point fixed in space occupied by the fluid and observe the changes which take place in velocity, density and pressure as the fluid passes through this point. Obviously, the point being fixed x, y, z and t are independent variables and so $\dot{x}$, x, etc. are meaningless in this method.

Let us consider any scalar point function

$$\phi\,(x, y, z, t)=\phi\,(\mathbf{r}, t)$$

associated with a fluid in motion. Then keeping the point $P(x, y, z)$ as fixed, the change is

$$\phi\,(\mathbf{r}, t+\delta t)-\phi\,(\mathbf{r}, t)$$

whence the local time-rate of change, $(\partial\phi/\partial t)$ is

$$\lim_{\delta t\to 0}\frac{\phi(\mathbf{r}, t+\delta t)-\phi(\mathbf{r}, t)}{\delta t}.$$

Thus
$$\frac{\partial\phi}{\partial t}=\lim_{\delta t\to 0}\frac{\phi(\mathbf{r}, t+\delta t)-\phi(\mathbf{r}, t)}{\delta t}.$$

A similar expression can be established for a vector point function, i.e.

$$\frac{\partial\mathbf{f}}{\partial t}=\lim_{\delta t\to 0}\frac{\mathbf{f}(\mathbf{r}, t+\delta t)-\mathbf{f}(\mathbf{r}, t)}{\delta t}.$$

(2) Lagrangian method : In this method we seek to determine the the history of every fluid particle, i e. we select any particle of the fluid and pursue it on its onward course making observations of changes in velocity, density and pressure at each instant and at each point.

Thus the expressions x, $\dot{x}$, etc. have definite significance, and to specify a particular fluid-particle we need its initial position co-ordinates, say (a, b, c) or $(\mathbf{r}_0)$ so that there are altogether four independent variables (a, b, c, t) in Cartesian treatment and $(\mathbf{r}_0, t)$ in vector treatment.

Let us now consider any scalar point function $\phi(x, y, z, t)$, i.e. $\phi(\mathbf{r}, t)$ associated with a fluid in motion. Then keeping the particle fixed, the change is

$$\phi(\mathbf{r}+\delta\mathbf{r}, t+\delta t)-\phi(\mathbf{r}, t).$$

The change $\delta\mathbf{r}$ in the position of the particle during the time δt depends upon $\mathbf{q}$, the velocity of the particle at time t. Thus

$$\delta\mathbf{r}=\mathbf{q}\delta t.$$

Then $$\lim_{\delta t\to 0}\frac{\phi(\mathbf{r}+\mathbf{q}\delta t, t+\delta t)-\phi(\mathbf{r}, t)}{\delta t}=\frac{d\phi}{dt}$$

is the individual time-rate of change.

1.201. Relation between the local and individual time-rates : Let (u, v, w) be the components of velocity $\mathbf{q}$ along the co-ordinate axes, so that

$$\mathbf{q}=u\mathbf{i}+v\mathbf{j}+w\mathbf{k}\ ;\ \text{where } dx/dt=u, \text{ etc.}$$

Now $$\phi=\phi(x, y, z, t)$$

$$\begin{aligned}\therefore\quad \frac{d\phi}{dt}&=\frac{\partial\phi}{\partial x}\frac{dx}{dt}+\frac{\partial\phi}{\partial y}\frac{dy}{dt}+\frac{\partial\phi}{\partial z}\frac{dz}{dt}+\frac{\partial\phi}{\partial t}\\ &=u\frac{\partial\phi}{\partial x}+v\frac{\partial\phi}{\partial y}+w\frac{\partial\phi}{\partial t}+\frac{\partial\phi}{\partial t}\\ &=(u\mathbf{i}+v\mathbf{j}+w\mathbf{k})\cdot\left(\mathbf{i}\frac{\partial\phi}{\partial x}+\mathbf{j}\frac{\partial\phi}{\partial y}+\mathbf{k}\frac{\partial\phi}{\partial z}\right)+\frac{\partial\phi}{\partial t}\\ &=\mathbf{q}\cdot(\nabla\phi)+\frac{\partial\phi}{\partial t}.\end{aligned}$$

Thus, $$\frac{d\phi}{dt}=\mathbf{q}\cdot(\nabla\phi)+\frac{\partial\phi}{\partial t}=(\mathbf{q}\cdot\nabla)\phi+\frac{\partial\phi}{\partial t}.$$

A similar expression for a vector point function $\mathbf{f}$ can be established in the form

$$\frac{d\mathbf{f}}{dt}=\frac{\partial\mathbf{f}}{\partial t}+(\mathbf{q}\cdot\nabla)\mathbf{f}$$

1.30. Differentiation following the fluid motion. Consider some property of the fluid (e.g. density, fluid boundary, fluid velocity) typified

by some function $G(x, y, z, t)$; a scalar (or vector) point function. Then

$$G=G(x, y, z, t)=G(\mathbf{r}, t).$$

The position vector $\mathbf{r}$ may depend upon time t and hence we may calculate dG/dt.

Now $\quad G+\delta G=G(\mathbf{r}+\delta\mathbf{r}, t+\delta t)$

$$\therefore \quad \delta G=G(\mathbf{r}+\delta\mathbf{r}, t+\delta t)-G(\mathbf{r}, t)$$

$$=[G(\mathbf{r}+\delta\mathbf{r}, t+\delta t)-G(\mathbf{r}, t+\delta t)]+[G(\mathbf{r}, t+\delta t)-G(\mathbf{r}, t)]$$

i.e. $\quad \delta G \doteq \delta\mathbf{r}\,.\,\nabla G(\mathbf{r}, t+\delta t)+\delta t\ \partial G(\mathbf{r}, t)/\partial t \qquad$ [to first order]

Dividing both sides by δt and proceeding to limits, we get

$$dG/dt=\mathbf{q}.\nabla G+\partial G/\partial t \qquad (\because d\mathbf{r}/dt=\mathbf{q}) \qquad (1)$$

This equation indicates the time rate of change of the quantity G as a fluid particle moves about, but is written in terms of quantity observed at a point.

The operator $d/dt \equiv (\mathbf{q}.\nabla)+\partial/\partial t$ is known as *differentiation following the motion of the fluid* or the material derivative. Often d/dt is denoted by D/Dt.

NOTES: (1) The term $(\mathbf{q}.\nabla)G$ represents the rate of change of G at a *fixed time* t due to the change of position from one point to the other ; and the term $\partial G/\partial t$ gives the rate of change of G at a *fixed point*.

(2) $d\rho/dt=0$ implies incompressible fluid but not steady flow, but $\partial\rho/\partial t=0$ implies ρ is independent of t at a *fixed point*. Similarly, the fluid boundary $f(\mathbf{r}, t)=0$, [boundary particles being distinguished from all others, since they lie on $f=0$] always consists of the same fluid particles, we must have $df/dt=0$ (vide §1.80).

(3) If G is replaced by the velocity vector $\mathbf{q}$, we obtain particle acceleration, viz.

$$\mathbf{a}=d\mathbf{q}/dt=(\mathbf{q}.\nabla)\ \mathbf{q}+(\partial\mathbf{q}/\partial t).$$

Since $\mathbf{q}.\nabla=u\,\partial/\partial x+v\,\partial/\partial y+w\,\partial/\partial z$; [$\because$ $\mathbf{q}=(u, v, w)$] the acceleration components (a_x, a_y, a_z) are given by

$$a_x=\frac{du}{dt}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}$$

with two more expressions for dv/dt, dw/dt, etc.

For details, refer to §1.31 p. 23.

1·301. Reynold's transport theorem. Let $F(\mathbf{r}, t)$ be any function (scalar or tensor component) and $V(t)$ be a material* volume (not necessarily infinitesimal) moving with the fluid. Then

$$f(t)=\int_{V(t)} F(\mathbf{r}, t)dv \qquad (1)$$

* i.e. consisting of the same fluid particles.

is a function of t that can be calculated. Because the integral is over the varying volume $V(t)$, we cannot differentiate under the integral sign. However, if the integration was with regard to a volume in $\mathbf{r}_0$-space (Lagrangian or initial coordinates), it would be possible to interchange differentiation and integration for the simple reason that d/dt is material derivative keeping the initial vector $\mathbf{r}_0$ constant.

If V_0 is at $t=0$, what $V(t)$ is at time t, then $\mathbf{r}=\mathbf{r}(\mathbf{r}_0, t)$ and $dv=J dv_0$ provide

$$\begin{aligned}
\frac{d}{dt}\int_{V(t)} F(\mathbf{r}, t)\, dv &= \frac{d}{dt}\int_{V_0} F[\mathbf{r}(\mathbf{r}_0, t), t]\, J dv_0 \\
&= \int_{V_0}\left(\frac{dF}{dt} J + F\frac{dJ}{dt}\right) dv_0 \\
&= \int_{V_0}\left(\frac{dF}{dt} + F \operatorname{div} \mathbf{q}\right) J dv_0 \quad \left[\because \frac{dJ}{dt} = J \operatorname{div}\mathbf{q}, §0.50\right] \\
&= \int_{V(t)}\left(\frac{dF}{dt} + F \operatorname{div} \mathbf{q}\right) dv \\
&= \int_{V(t)}\left[\frac{\partial F}{\partial t} + \operatorname{div}(F\mathbf{q})\right] dv \quad \left\{\because \frac{dF}{dt} = \frac{\partial F}{\partial t} + (\mathbf{q}\cdot\nabla)F\right\} \qquad (2)
\end{aligned}$$

The second term on the right can be transformed to a surface integral over $S(t)$: the surface of $V(t)$.

1.31. Particle acceleration. Since the velocity field vector $\mathbf{q}$ is a function of both position and time, (i.e. of four independent variables), we may write it as, say

$$\mathbf{q}=\mathbf{q}(\mathbf{r}, t) \qquad (1)$$

Suppose that the value of the velocity at time $t+\delta t$ when the particle has moved to a neighbouring position is $\mathbf{q}+\delta\mathbf{q}$. Then

$$\begin{aligned}
\delta\mathbf{q} &= \mathbf{q}(\mathbf{r}+\delta\mathbf{r}, t+\delta t)-\mathbf{q}(\mathbf{r}, t) \\
&= [\mathbf{q}(\mathbf{r}+\delta\mathbf{r}, t+\delta t)-\mathbf{q}(\mathbf{r}, t+\delta t)]+[\mathbf{q}(\mathbf{r}, t+\delta t)-\mathbf{q}(\mathbf{r}, t)] \qquad (2)
\end{aligned}$$

Now, to the first order of approximations

$$\mathbf{q}(\mathbf{r}+\delta\mathbf{r}, t+\delta t)-\mathbf{q}(\mathbf{r}, t+\delta t) \doteq (\delta\mathbf{r}\cdot\nabla)\, \mathbf{q}(\mathbf{r}, t+\delta t) \qquad (3)$$

$$\mathbf{q}(\mathbf{r}, t+\delta t)-\mathbf{q}(\mathbf{r}, t) \doteq \delta t\, \partial\mathbf{q}(\mathbf{r}, t)/\partial t \qquad (4)$$

The acceleration vector $\mathbf{a}$ of the fluid particle at a point being $\operatorname{Lim}(\delta\mathbf{q}/\delta t)$, as $\delta t\to 0$; we divide (2) by δt, use (3) and (4) and proceed to the limits. This yields

$$\mathbf{a}=\frac{d\mathbf{q}}{dt}=\frac{\partial\mathbf{q}}{\partial t}+(\mathbf{q}\cdot\nabla)\mathbf{q} \qquad (5)$$

NOTES: (1) The expression (5) is in reality the Lagrangian acceleration. In the Eulerian concepts, it is composed of two factors : one a *temporal* acceleration ($\partial\mathbf{q}/\partial t$) at the point, and the other a *convective* acceleration, $(\mathbf{q}\cdot\nabla)\mathbf{q}$, resulting from flow entering the fluid element from regions having different velocities.

(2) **Lagrange's acceleration relation.** Since

$$(\mathbf{q}.\nabla)\mathbf{q}=\nabla(\tfrac{1}{2}\mathbf{q}^2)+\boldsymbol{\omega}\times\mathbf{q}, \qquad (\boldsymbol{\omega}=\text{curl } \mathbf{q})$$

$$\therefore \qquad \mathbf{a}=d\mathbf{q}/dt=(\partial\mathbf{q}/\partial t)+\nabla(\tfrac{1}{2}q^2)+\boldsymbol{\omega}\times\mathbf{q} \text{ [vide §0.10(5')]} \qquad (6)$$

The acceleration vector given by (6) is *Lagrange's acceleration relation* and its chief merit is that whereas the form (5) is not invariant under a change of coordinate system, the form (6) is invariant under change of coordinate system.

The vector $\mathbf{q}\times\boldsymbol{\omega}$ is called *Lamb vector*.

(3) **Particle acceleration in curvilinear coordinates.** With velocity components (q_1, q_2, q_3) in the (α, β, γ)-directions and using the vector definitions

$$\nabla=\left(\frac{1}{h_1}\frac{\partial}{\partial\alpha}, \frac{1}{h_2}\frac{\partial}{\partial\beta}, \frac{1}{h_3}\frac{\partial}{\partial\gamma}\right), \quad \text{curl } \mathbf{q}=\boldsymbol{\omega}=(\omega_1, \omega_2, \omega_3)$$

where
$$\omega_1=\frac{1}{h_1h_2}\left[\frac{\partial}{\partial\beta}(h_3q_3)-\frac{\partial}{\partial\gamma}(h_2q_2)\right], \text{ etc.}$$

we get the acceleration components (a_1, a_2, a_3) from Lagrange's acceleration relation

$$\mathbf{a}=(d\mathbf{q}/dt)=(\partial\mathbf{q}/\partial t)+\nabla(\tfrac{1}{2}q^2)+\boldsymbol{\omega}\times\mathbf{q}$$

$$a_1=(\partial q_1/\partial t)+\frac{1}{h_1}\frac{\partial}{\partial\alpha}(q_1^2+q_2^2+q_3^3)+(\omega_2q_3-\omega_3q_2) \qquad (7)$$

with similar expressions for a_2 and a_3.

(4) **Particle acceleration in cylindrical coordinates.** With velocity components (u, v, w) in the (r, θ, z)-directions and using the vector definitions

$$\mathbf{q}=(u, v, w)\,;\ \mathbf{q}^2=u^2+v^2+w^2\,;\ \nabla=\left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{\partial}{\partial z}\right)$$

$$\text{curl } \mathbf{q}=\left[\frac{1}{r}\frac{\partial w}{\partial\theta}-\frac{\partial v}{\partial z}, \frac{\partial u}{\partial z}-\frac{\partial w}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(rv)-\frac{1}{r}\frac{\partial u}{\partial\theta}\right]$$

in the Lagrange acceleration relation, we get

$$\mathbf{a}=\frac{\partial\mathbf{q}}{\partial t}+\tfrac{1}{2}\left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{\partial}{\partial z}\right)(u^2+v^2+w^2)+(\omega_1, \omega_2, \omega_3)\times(u, v, w) \qquad \text{(i)}$$

Putting
$$\frac{\partial}{\partial t}+u\frac{\partial}{\partial r}+\frac{v}{r}\frac{\partial}{\partial\theta}+w\frac{\partial}{\partial z}=\frac{d}{dt}$$

and splitting the three components in (i), we get

$$\mathbf{a}=\left(\frac{du}{dt}-\frac{v^2}{r}, \frac{dv}{dt}+\frac{uv}{r}, \frac{dw}{dt}\right). \qquad (8)$$

(5) **Particle acceleration in space polar coordinates.** With velocity components (u, v, w) in the (r, θ, φ)-directions and using the vector definitions

$$\mathbf{q}=(u, v, w),\ q^2=u^2+v^2+w^2\ ;\ \nabla=\left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

curl $\mathbf{q}=\boldsymbol{\omega}=(\omega_1, \omega_2, \omega_3)$ where

$$\omega_1=\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(w\sin\theta)-\frac{\partial v}{\partial\phi}\right],\ \omega_2=\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial u}{\partial\phi}-\frac{\partial}{\partial r}(rw)\right]$$

$$\omega_3=\frac{1}{r}\left[\frac{\partial}{\partial r}(rv)-\frac{\partial u}{\partial\theta}\right];$$

in the Lagrange acceleration relation, we get

$$\mathbf{a}=\frac{\partial\mathbf{q}}{\partial t}+\frac{1}{2}\left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)(u^2+v^2+w^2)+ (\omega_1, \omega_2, \omega_3)\times(u, v, w) \qquad \text{(i)}$$

Putting $$\frac{\partial}{\partial t}+u\frac{\partial}{\partial r}+\frac{v}{r}\frac{\partial}{\partial\theta}+\frac{w}{r\sin\theta}\frac{\partial}{\partial\phi}=\frac{d}{dt}$$

and splitting the three components in (i), we get

$$\mathbf{a}=\left(\frac{du}{dt}-\frac{v^2+w^2}{r}, \frac{dv}{dt}-\frac{w^2\cot\theta}{r}+\frac{uv}{r}, \frac{dw}{dt}+\frac{vw\cot\theta}{r}\right) \qquad (9)$$

Ex. 1. Obtain the components of acceleration in the cylindrical and space polar coordinates by evaluating $\partial\mathbf{q}/\partial t+(\mathbf{q}\cdot\nabla)\mathbf{q}$ by direct differentiation.

[HINT: Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be the unit vectors in the (r, θ, z) directions; then with obvious notations:

$$(\partial/\partial r, \partial/\partial\theta, \partial/\partial z)(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)=[(0, 0, 0)\ ;\ (\mathbf{e}_2, -\mathbf{e}_1, 0)\ ;\ (0, 0, 0)],$$

with similar meanings for space polar coordinates:

$$(\partial/\partial r, \partial/\partial\theta, \partial/\partial\phi)(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)=[(0, 0, 0)\ ;\ (\mathbf{e}_2, -\mathbf{e}_1, 0)\ ;\ (\mathbf{e}_3\sin\theta, \mathbf{e}_3\cos\theta, -\mathbf{e}_1\sin\theta-\mathbf{e}_2\cos\theta)]$$

Substitutions into expansion of $(\mathbf{q}\cdot\nabla)\mathbf{q}$ and simplyfying we get the above results (8) and (9)].

Ex. 2. Show that the rate of change of quantity λ appertaining to a particle of fluid, in plane polar coordinates is

$$d\lambda/dt=u_r(\partial\lambda/\partial r)+(u_t/r)(\partial\lambda/\partial\theta)+(\partial\lambda/\partial t)$$

where u_r, u_t are the radial and transverse (or circumferential) components of velocity, respectively.

Hence find the radial and transverse components of the acceleration of a fluid particle in 2-dimensional motion.

1.40. Rotational and irrotational motion

Vorticity. If $\mathbf{q}$ be the velocity vector of a fluid particle, then the vector quantity.*

$$\boldsymbol{\omega}=\nabla\times\mathbf{q}=\text{curl } \mathbf{q}, \text{ or, rot } \mathbf{q}$$

*Among the writers using the definition ω=curl q are Lamb, Milne-thompson, Dryden. Rutherford, Howarth, Goldstein. Aris. Temple. Birkhoff, Wilson, Robertson and Hunt. Authors using $\omega=\frac{1}{2}$ curl q include Prandtle, Pai, Sneddon, and Sommerfeld.

is called the *vorticity vector* or simply the *vorticity* and is a measure* of the angular velocity of an infinitesimal element. The components of spin are given by (ξ, η, ζ) where

$$\xi=\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right);\ \eta=\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right);\ \zeta=\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

A Vortex line *is a curve drawn in the fluid such that the tangent to it at each point is in the direction of the vorticity vector* ω *at that point. The vortex line is often abbreviated into* ω*-line.*

The definition of the vortex line implies that its analytical expression is given by $d\mathbf{r}\times\omega=0$, or its equivalent in Cartesian form by the differential equations

$$\frac{dx}{\xi}=\frac{dy}{\eta}=\frac{dz}{\zeta} \qquad [\omega=(\xi, \eta, \zeta)].$$

Vortex tube. *The vortex lines drawn through each point of a closed curve enclose a tubular space in the fluid called a vortex tube.* Avotex tube of infinitesimal cross-section is called a vortex filament or simply a vortex.

The motion of a fluid is said to be irrotational when the vorticity ω of every fluid particle is zero so that $\xi=0$, $\eta=0$, $\zeta=0$. When the vorticity is different from zero, the motion is said to be rotational.

Rotational motion is also called vortex motion. The definition implies that in an irrotational motion of the fluid, there are no vertex lines.

Velocity potential. When the motion is irrotation, i.e. when $\omega=$ curl $\mathbf{q}=0$, then since curl grad$\equiv 0$, it follows that the velocity vector $\mathbf{q}$ must be grad of some scalar point function ϕ (say); i.e.

$$\text{curl } \mathbf{q}=0 \Rightarrow \mathbf{q}=-\nabla\phi.\dagger$$

Usually, ϕ is called the velocity potential or velocity function.

We may observe that whenever velocity potential exists, the system of surfaces given by the differential equation

$$\mathbf{q}.\,d\mathbf{r}=0 \quad \text{or} \quad u\,dx+v\,dy+w\,dz=0 \tag{1}$$

possess solution $\phi(\mathbf{r})=$const., for

$$0=\mathbf{q}.\,d\mathbf{r}=-\nabla\phi.d\mathbf{r}=-d\phi \Rightarrow \phi(\mathbf{r})=\text{const.}$$

Further, these surfaces cut the stream lines $\mathbf{q}\times d\mathbf{r}=0$ orthogonally, since the velocity vector which isparallel to $d\mathbf{r}$ for the stream lines, is perpendicular to $d\mathbf{r}$ in (1).

NOTE. Vortex is flow in circles about a central point. It is termed *free* when motion is such that the tangential velocity q is inversely proportional to the radius, i.e $q \propto 1/r \Rightarrow qr=$constant. Motion is irrotational and vorticity is zero; stream lines are circles and circulation is constant.

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* See §1.41 p. 27 for the physical interpretation of vorticity vector ω

† The negative sign is due to convention; many authors define $\mathbf{q}=\nabla\phi$.

A vortex is termed *forced* when the motion is the result of some external force and the motion is such that $q \propto r$. Here vorticity is constant.

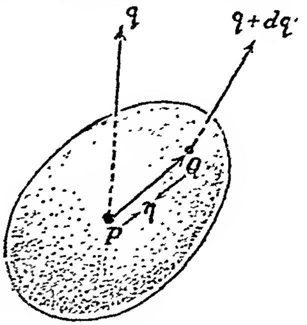
Ex. 1. The velocity potential of a two-dimensional motion is $\phi = cxy$. Find the stream lines. [*Ans:* $x^2-y^2=a^2$]

Ex. 2. A body of liquid is revolving about a vertical axis with angular speed which is a function of the perpendicular distance r from the axis. Show that, if the motion is *irrotational*,

$$\mathbf{q}=(\mu/r)\mathbf{e}_2,\ \omega=(\mu/r^2)\mathbf{e}_3$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the unit vectors in the directions of (r, θ, z) with cylindrical coordinates; ω denotes angular velocity.

1.41. Physical interpretation of the vorticity vector ω. Consider an element of fluid of mass m, whose centre of gravity is at P, with velocity $\mathbf{q}$ and whose *principal moment of interia at P are equal.* This ensures that the moments of inertia about all axes through P are equal. Let us denote their common value by I. If Q be any neighbouring point within the element, with position vector η relative to P, its velocity will be $\mathbf{q}+d\mathbf{q}$. Since



$$d\mathbf{q}=\Sigma(\partial\mathbf{q}/\partial x)dx=[(\Sigma\mathbf{i}\,dx).\,(\Sigma\mathbf{i}\,\partial/\partial x)]\mathbf{q}$$
$$=(d\mathbf{r}.\,\nabla)\mathbf{q}=(\eta.\,\nabla)\mathbf{q}$$

the velocity of Q shall be $\mathbf{q}+(\eta\,.\,\nabla)\mathbf{q}$. Now, the angular momentum $\mathbf{h}$ of the element about P is

$$\mathbf{h}=\Sigma m\eta\times[\mathbf{q}+(\eta.\,\nabla)\mathbf{q}]=\Sigma m\eta\times\mathbf{q}+\Sigma m\eta\times(\eta\,.\,\nabla)\mathbf{q} \qquad (1)$$

where summations extend to all the particles of the fluid within the element. Also, the centroid is at P, which when taken as the origin provides $\Sigma m\eta\times\mathbf{q}=(\Sigma m\eta)\times\mathbf{q}=0$; hence (1) reduces to

$$\mathbf{h}=\Sigma m\eta\times(\eta.\,\nabla)\mathbf{q} \qquad (2)$$

Let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be the unit vectors parallel to the principal axes of the element and $\eta=\mathbf{i}x'+\mathbf{j}y'+\mathbf{k}z'$. We may note that the partial derivatives have their values at P, the centroid of the element, and so may be regarded as constant in this discussion. We now have

$$\mathbf{h}=\Sigma m(\mathbf{i}x'+\mathbf{j}y'+\mathbf{k}z')\times\left(x'\frac{\partial}{\partial x}+y'\frac{\partial}{\partial y}+z'\frac{\partial}{\partial z}\right)\mathbf{q}$$
$$=\Sigma m\left(x'^2\,\mathbf{i}\frac{\partial}{\partial x}+y'^2\,\mathbf{j}\frac{\partial}{\partial y}+z'^2\,\mathbf{k}\frac{\partial}{\partial z}\right)\times\mathbf{q} \qquad (3)$$

because, the products of inertia are all zero by hypothesis.

Further, $\Sigma mx'^2=\Sigma my'^2=\Sigma mz'^2=\frac{1}{2}I$, hence (3) may be rewritten as

$$\mathbf{h}=\tfrac{1}{2}I\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times\mathbf{q}=\tfrac{1}{2}I\,\nabla\times\mathbf{q}=\tfrac{1}{2}I\omega.$$

If now the element is suddenly solidified and detached from the rest of the fluid, the angular momentum about P would remain

unaltered, *for* the impulses for his soldification are absolutely internal. Thus the motion of the element relative to P after solidification is an angular velocity ω, such that

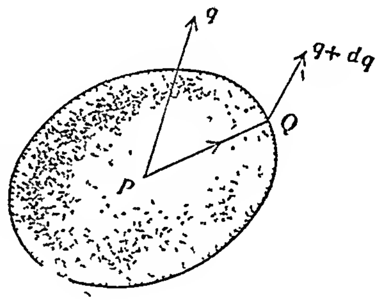
$$I\,\omega=\tfrac{1}{2}\,I\,\text{curl}\,\mathbf{q},$$

$$\therefore \qquad \omega=\tfrac{1}{2}\,\text{curl}\,\mathbf{q}.$$

Thus the *curl of the velocity of any particle of a rigid body equals twice the angular velocity.*

Ex. Prove that through any point P of a fluid in motion there is in general at any instant one set of three straight lines at right angles to each other such that, if the lines move with the fluid, then after a short time δt, the angle between them remain right angles to the first order in δt, and that the angular velocity of this triad of lines, as it moves with the fluid is $\frac{1}{2}$ curl $\mathbf{q}$, where $\mathbf{q}$ is the fluid velocity at P. Prove also that, if a small portion of the fluid with its mass centre at P be instantaneously solidified without change of angular momentum, then its angular velocity immediately after slidification is $\frac{1}{2}$ curl $\mathbf{q}$, if and only if, the principal axes of inertia for the resulting solid lie along the above triad of line. *(London)*

1.42. Cauchy-Stokes' decomposition theorem : analysis of the most general displacement of a fluid element. Let P be the centroid of an infinitesimal fluid element and Q be any point of the element with position vector η relative to P. Let $\mathbf{q}$ be the fluid velocity at P and $\mathbf{q}+d\mathbf{q}$ that at Q. Then



$$\mathbf{q}+d\mathbf{q}=\mathbf{q}+(PQ\,.\,\nabla)\mathbf{q}=\mathbf{q}+(\eta\,.\,\nabla)\mathbf{q} \qquad (1)$$

To find $(\eta\,.\,\nabla)\mathbf{q}$ we proceed as under :

Consider the equation*

$$[(\eta\,.\,\nabla)\mathbf{q}]\,.\,\eta=C \text{ (constant)}. \qquad (2)$$

The left side of this equation is homogeneous and quadratic in the components of η and so represents a quadratic surface. It is a central quadric, for replacing η by $-\eta$ does not alter the equation. If $R(\eta+d\eta)$ is a neighbouring point to Q *on* the surface, then the equation of the surface will be satisfied by $\eta+d\eta$. Hence replacing η by $\eta+d\eta$ in (2) we get

$$\{[(\eta+d\eta)\,.\,\nabla]\mathbf{q}\}\,.\,(\eta+d\eta)=C$$

or $$[(\eta\,.\,\nabla)\mathbf{q}+(d\eta\,.\,\nabla)\mathbf{q}]\,.\,(\eta+d\eta)=C$$

or $$[(\eta\,.\,\nabla)\mathbf{q}]\,.\,\eta+[(d\eta\,.\,\nabla)\mathbf{q}]\,.\,\eta +[(\eta\,.\,\nabla)\mathbf{q}]\,.\,d\eta+[(d\eta\,.\,\nabla)\mathbf{q}]\,.\,d\eta=C \qquad (3)$$

Subtracting (2) from (3) and neglecting the term containing $d\eta\,.\,d\eta$, we get

$$[(d\eta\,.\,\nabla)\mathbf{q}]\,.\,\eta+[(\eta\,.\,\nabla)\mathbf{q}]\,.\,d\eta=0 \qquad (4)$$

* Thus if $P=(x, y, z)$, $Q=(x+X, y+Y, z+Z)$, $\eta=(X, Y, Z)$, then

$$[(\eta\,.\,\nabla)\mathbf{q}]\,.\,\eta=(\partial u/\partial x)X^2+(\partial v/\partial y)Y^2+(\partial w/\partial z)Z^2+(\partial v/\partial z+\partial w/\partial y)YZ$$
$$+(\partial u/\partial z+\partial w/\partial x)ZX+(\partial u/\partial y+\partial v/\partial x)XY$$
$$=aX^2+bY^2+cZ^2+2fYZ+2gZX+2hXY \text{ (say).}$$

It may be noted that ∇ operates *only* on $\mathbf{q}$ and *not* on η.

Since ∇ operates on q but not on η (i.e. η is treated as constant),

$\therefore$ $$[(d\eta\,.\,\nabla)\mathbf{q}]\,.\,\eta=(d\eta\,.\,\nabla)(\mathbf{q}\,.\,\eta)=d\eta\,.\,\nabla(\mathbf{q}\,.\,\eta) \tag{5}$$

Also, $$\nabla(\mathbf{q}\,.\,\eta)=(\mathbf{q}\,.\,\nabla)\,\eta+(\eta\,.\,\nabla)\,\mathbf{q}+\mathbf{q}\times\text{curl}\,\mathbf{q}+\eta\times\text{curl}\,\mathbf{q}$$
$$=(\eta\,.\,\nabla)\,\mathbf{q}+\eta\times\text{curl}\,\mathbf{q} \tag{6}$$

as ∇ operates on q but not on η. Hence from (4), (5) and (6), we have

$$[(\eta\,.\,\nabla)\mathbf{q}]\,.\,d\eta+[(\eta\,.\,\nabla)\mathbf{q}-(\nabla\times\mathbf{q})\times\eta]\,.\,d\eta=0$$

or $$[2(\eta\,.\,\nabla)\mathbf{q}-(\nabla\times\mathbf{q})\times\eta]\,.\,d\eta=0.$$

But the normal at $Q(\eta)$ is perpendicular to the tangent vector $d\eta$, therefore, it must be in the direction of the vector

$$2(\eta\,.\,\nabla)\mathbf{q}-(\nabla\times\mathbf{q})\times\eta=2f(\eta)=2\eta f(\mathbf{n}) \qquad \text{(say)}$$

hence $$(\eta\,.\,\nabla)\mathbf{q}=f(\eta)+\tfrac{1}{2}(\nabla\times\mathbf{q})\times\eta. \tag{7}$$

The unit vector n is in the direction of η; i.e. is in the direction of the normal at Q.

From (1) and (7), it follows that the velocity at Q is

$$\mathbf{q}+d\mathbf{q}=\mathbf{q}+\tfrac{1}{2}(\nabla\times\mathbf{q})\times\eta+f(\eta) \tag{8}$$

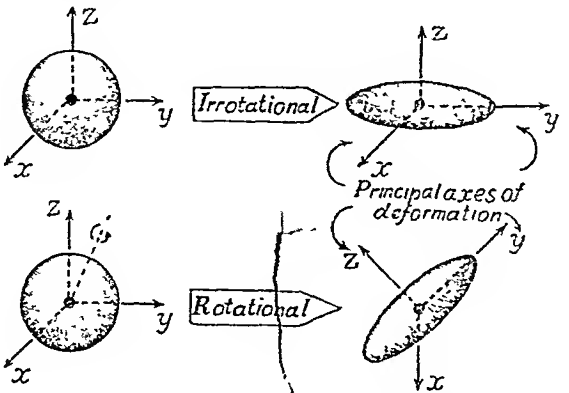
Thus the velocity at Q consists of the following three parts :

(1) The velocity q at P, the velocity of translation which carries the element as a whole.

(2) The velocity $\frac{1}{2}(\nabla\times\mathbf{q})\times\eta$ due to rotation, carries the element as a whole with angular velocity $\frac{1}{2}\nabla\times\mathbf{q}$.

(3) A velocity $f(\eta)$ relative to P, which is in the direction of the normal $f(\eta)$ to the quadric of the system of central quadrics $\{(\eta.\nabla)\mathbf{q}\}.\eta$ =const., on which Q lies. This part of the motion is called a *pure strain* and is characteristic of any deformable substance.

NOTE. The pure strain motion is such that lines drawn parallel to the three perpendicular axes of symmetry of a central quadric undergo elongation at a uniform rate. Figure illustrates the deformation of a sphere to an ellipsoid.



When the motion is irrotational, the relative displacement of the fluid element consists of a *pure strain* and that too on account of translation only.

Ex. (i) *Show that the most general displacement of a fluid element consists of a pure strain compounded with a rotation.*

(ii) *Give a physical interpretation when the motion of the fluid is irrotational.*

(*Del 1953*)

[(ii) When the motion is irrotational, curl $\mathbf{q}=\omega=0$. Physically interpreted, it means that the relative displacement of a fluid element consists of a pure strain together with motion of translation only.]

1.50. Flow. *If A, P be any two points in a fluid, then* $\int_A^P \mathbf{q}\cdot d\mathbf{r}$ *is called the flow along the path from A to P.* It is evident that when velocity potential exists, i.e. $\mathbf{q}=-\nabla\phi$, then

$$\text{Flow}=-\int_A^P \nabla\phi\cdot d\mathbf{r}=\phi_A-\phi_P.$$

Circulation. *The flow round a closed curve C is known as circulation; and is usually denoted by Γ. Thus*

$$\Gamma=\int_C \mathbf{q}\cdot d\mathbf{r}.$$

Obviously, when a single-valued velocity potential ϕ exists, circulation round C is zero ; it being equal to $\phi_A-\phi_A$.

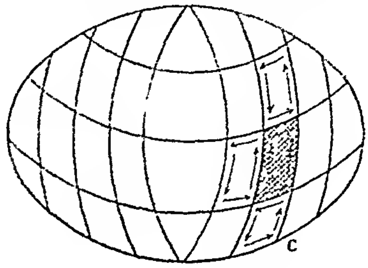
Stokes' theorem. This theorem deals with the concept of *rotation in terms of circulation* and states as under :

If $\mathbf{q}$ is the velocity vector point function and S is a surface bounded by a curve C, then

$$\int_C \mathbf{q}\cdot d\mathbf{r}=\int_S \text{curl } \mathbf{q}\cdot d\mathbf{S}\ ;\ \text{i.e. } \Gamma=\int_S \omega\cdot\mathbf{n}\,dS$$

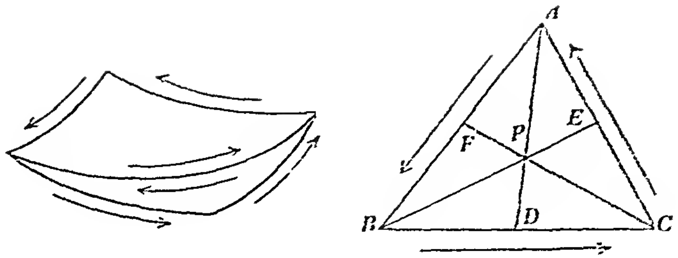
where the unit normal vector $\mathbf{n}$ at any point of S is drawn in the sense in which a right-handed screw would move when rotated in the sense of description of C.

Proof. Firstly, we observe that any surface can be divided up into a network of small meshes or curvilinear parallelograms and integrating round all these parallelograms, all the integrals will cancel, because each integration-path is described in two opposite directions except those round the bounding curve C. Thus,



circulation round C=sum of the circulations in all the meshes.

Therefore, it is sufficient to consider the theorem for a single mesh, or since each mesh can be divided into two curvilinear triangles, it suffices to consider the theorem for a single triangular mesh ABC (say) whose sides are of infinitesimal length. Let D, E, F be the mid-points of the sides and P the centroid of this triangle. [Fig. p. 31]. Let us set



$$\overline{AB}=\mathbf{x}\,;\ \overline{BC}=\mathbf{y}\,;\ CA=-(\mathbf{x}+\mathbf{y}).$$

Let $\mathbf{q}_M$ stand for the value of $\mathbf{q}$ at M. Then setting contour $ABC=l$, we get

$$\int_l \mathbf{q}.d\mathbf{r}=\overline{AB}.\mathbf{q}_F+\overline{BC}.\mathbf{q}_D+\overline{CA}.\mathbf{q}_E$$
$$=\mathbf{x}.\mathbf{q}_F+\mathbf{y}.\mathbf{q}_D-(\mathbf{x}+\mathbf{y})\,\mathbf{q}_E$$
$$=\mathbf{x}.(\mathbf{q}_F-\mathbf{q}_E)+\mathbf{y}.\,(\mathbf{q}_D-\mathbf{q}_E).$$

Since $\qquad \mathbf{q}_F-\mathbf{q}_P\doteq(\overline{\mathbf{PF}}.\nabla)\mathbf{q}_P:\ \mathbf{q}_E-\mathbf{q}P\doteq(\overline{\mathbf{PE}}.\nabla)\,\mathbf{q}_P$

therefore, by subtraction, $\mathbf{q}_F-\mathbf{q}_E=(\mathbf{EF}.\nabla)\mathbf{q}_P=-\frac{1}{2}(\mathbf{y}.\nabla)\mathbf{q}_P$.

Similarly $\mathbf{q}_D-\mathbf{q}_E=\frac{1}{2}(\mathbf{x}.\nabla)\mathbf{q}_P$; replacing F by D.

With these values, the last integral can be written as

$$\int_l \mathbf{q}.d\mathbf{r}=\tfrac{1}{2}[\mathbf{y}(\mathbf{x}.\nabla)-\mathbf{x}(\mathbf{y}.\nabla)].\,\mathbf{q}_P$$
$$=\tfrac{1}{2}\,[(\mathbf{x}\times\mathbf{y})\times\nabla]\,.\mathbf{q}_P$$
$$=\tfrac{1}{2}(\mathbf{x}\times\mathbf{y}).(\nabla\times\mathbf{q}_P) \tag{1}$$

Now $\qquad \mathbf{n}dS=\frac{1}{2}(\mathbf{x}\times\mathbf{y})$, where dS is the area of $\triangle\ ABC$

$$\therefore \qquad \int_l \mathbf{n}.(\nabla\times\mathbf{q})\;dS=\tfrac{1}{2}(\mathbf{x}\times\mathbf{y}).(\nabla\times\mathbf{q}) \tag{2}$$

From (1) and (2)

$$\int_l \mathbf{q}\,.\,d\mathbf{r}=\int_l \mathbf{n}.(\nabla\times\mathbf{q})dS$$

Hence by summation, we get the circulation round any closed curve C as

$$\int_C \mathbf{q}\,.\,d\mathbf{r}=\int_S \mathbf{n}.(\nabla\times\mathbf{q})dS.$$

NOTE. Stokes' theorem is true for *any* vector F ; the use of velocity vector $\mathbf{q}$ in place of F simply reflects its hydrodynamical interpretation.

Ex. The circle $x^2+y^2-2ax=0$ is situated in a two-dimensional shear flow with $u=0,\ v=cy$. Find the circulation in the circle (i) by direct calculation and (ii) by integration of the vorticity.

1.51. Kelvin's circulation theorem. *The circulation Γ around any material closed curve moving with the inviscid fluid is constant for all times,* **provided** *the external forces are conservative and derivable from a single valued potential function χ and the density is a function of the pressure only.*

For a closed material curve C, the circulation Γ is defined by $\Gamma = \int_C \mathbf{q} \cdot d\mathbf{r}$.

$$\text{Hence} \qquad \frac{d\Gamma}{dt} = \frac{d}{dt} \oint_C \mathbf{q} \cdot d\mathbf{r} = \oint_C \left[\frac{d\mathbf{q}}{dt} \cdot d\mathbf{r} + \mathbf{q} \cdot d\left(\frac{d\mathbf{r}}{dt}\right) \right]. \tag{1}$$

For barotropic flow of a perfect fluid under conservative system of forces, the acceleration vector is irrotational* so that $d\mathbf{q}/dt = -\nabla H$, hence (1) reduces to

$$\frac{d\Gamma}{dt} = -\oint_C [\nabla H \cdot d\mathbf{r} - d(\tfrac{1}{2} q^2)] = -\oint_C d(H - \tfrac{1}{2} q^2) = [\tfrac{1}{2} q^2 - H]_C \tag{2}$$

Since all the quantities involved in the integral (2) are single valued, the change expressed in $[H - \tfrac{1}{2} q^2]_C$ on passing once round the circuit is zero. Hence

$$d\Gamma/dt = 0 \Rightarrow \Gamma = \text{const} \; ; \textit{ independent of } t. \tag{3}$$

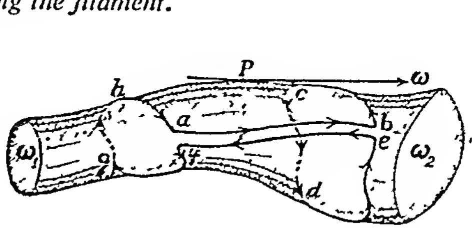
As the substantial derivative d/dt is involved, this means that the constancy of circulation is a *fluid-bound property*, i.e. one that is carried about with the fluid particles. The fluid contour C involved in the Γ-definition moves with the fluid particles.

NOTE : This theorem may be generalized, vide Chapter VI.

1.52. Helmholtz's vorticity theorems

(1) *The product of the cross-section and vorticity at any point on a vortex filament is constant along the filament.*

Let C be a reducible circuit $a b c d e f g h a$ drawn on the surface S of a vortex tube. Then



$$\Gamma = \int_C \mathbf{q} \cdot d\mathbf{r} = \int_S \operatorname{curl} \mathbf{q} \cdot \mathbf{n} \, dS$$

Since $\boldsymbol{\omega} \cdot \mathbf{n} = 0$, we have $\Gamma = 0$; i.e. the circulation in any reducible circuit drawn *on* the surface of a vortex tube is zero. Such a circuit $f g h a b c d e f$ can be taken to consist of two cross-sections of a vortex tube joined by close parallel lines (See Fig.). Thus,

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* This statement needs a proof (must be inserted here) which follows readily from Euler's equation of motion : $d\mathbf{q}/dt = \mathbf{F} - (1/\rho)\nabla p = -\nabla \chi - \nabla P = -\nabla(\chi + P)$ so that $d\mathbf{q}/dt = -\nabla H$ Although Euler's equation of motion is dealt with later on, (vide §2.10), the above theorem is placed here for convenience ; for its frequent use is made of in some sections which follow.

$$0=\int q.d\mathbf{r}=\int_{fgha} q.d\mathbf{r}+\int_{ab} q.d\mathbf{r}+\int_{bcde} q.d\mathbf{r}+\int_{ef} q.d\mathbf{r}.$$

In the limit, the flows along *ab* and *ef* cancel out ; the two cross-sections reduce to two oppositely oriented circuits C_1 and C_2 enclosing the vortex tube. Hence

$$0=\int_{fghaf} q.d\mathbf{r}-\int_{edcbe} q.d\mathbf{r} \Rightarrow \Gamma_{C_1}=\Gamma_{C_2} \tag{1}$$

Thus, the circulation along any circuit enclosing the vortex tube is the same. *The strength of a vortex tube is defined as the circulation around any circuit enclosing the tube.* If ω is the vorticity and A the normal cross-section of the vortex tube, supposed small, then

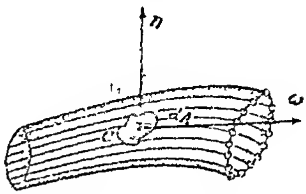
$$\Gamma=\int_C q.d\mathbf{r}=\int_S \omega.d\mathbf{s}=|\omega|A=\omega A=\text{constant, by (1).}$$

Hence, in particular, $\omega_1 A_1=\omega_2 A_2$

NOTE. We have proved that *at the instant considered*, circulation in every simple circuit embracing the tube and described on its surface is constant: and *not* that circulation is time-invariant. However, time-independence of circulation is guaranteed only for barotropic inviscid fluids under conservative body forces : a consequence of Kelvin's circulation theorem.

(2) *When pressure is a function of density and inviscid fluid moves under conservative forces, the vortex lines move with the fluid.*

Consider a vortex tube and take any small simple circuit C_1 on its wall [of surface area $S_1=dA$], not encircling it. Then

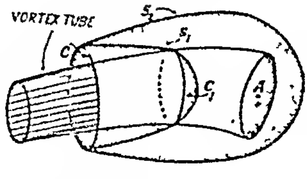


$$\Gamma=\int_{C_1} q.d\mathbf{r}=\int_{S_1} \text{curl } q.\mathbf{n}dS=0$$

$$(\because\ \omega.\mathbf{n}=0).$$

Thus, the flux of rotation vector across this small element dA, i.e. the circulation along its contour has the constant value zero. But according to Kelvin's circulation theorem, constancy of circulation is a fluid-bound property ; therefore circulation and hence the flow across this surface element dA, considered as a fluid surface, must always be zero for all times. By piecing together all these areas dA and so obtaining the whole vortex tube, we see that the total flow across its wall, considered as a fluid surface, must always remain zero. In other words, those fluid elements which at one time form a vortex tube form it for all times, i.e. the fluid particles within the vortex tube always remain there. Thus a vortex tube (and hence a vortex filament) is always composed of the same fluid elements and so moves with the fluid as if vorticity in it is frozen.

(3) *A vortex tube cannot have an end within the fluid.* Suppose the vortex tube terminates abruptly at A. Let C be a circuit embracing the vortex tube and lying wholly in the fluid. With C as an edge describe surfaces S_1 and S_2 lying wholly within the fluid ; intersecting the vortex tube in the circuit C_1 and not intersecting it at all. Now



$$\Gamma_C=\Gamma_{C_1}=\Gamma \text{ : the strength of the vortex tube.}$$

But $\Gamma_C=0$, from the consideration of S_2 as there is no vortex tube. We have therefore arrived at a contradiction. Hence the vortex tube cannot end in the fluid : it must either be a closed loop (e.g. torus) lying wholly in the fluid or end on the boundary of the fluid.

Ex. (*a*) Define (i) Vorticity, (ii) Vortex line, (iii) Vortex filament, and prove the following:

(1) The product of the cross-section and angular velocity at any point on a vortex filament is constant all along the vortex filament and for all times.

(2) The vortex lines move with the fluid. [*Alig 1963, 56* ; *Mad 59* ; *Osm 59*]

(*b*) If $u=\dfrac{ax-by}{x^2+y^2}$, $v=\dfrac{ay+bx}{x^2+y^2}$, $w=0$; investigate the nature of the motion of the liquid. [*Alig 1963* ; *Bom 63*]

Exp. 1. (i) *Prove that the necessary and sufficient condition that the vortex lines may be at right angles to the stream lines is*

$$u, v, w=\mu\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

where μ and ϕ are functions of x, y, z and t.

(ii) *Find the necessary and sufficient condition that votex lines may be at right angles to the stream lines.*

[*Bom 1963* ; *Mad 59* ; *Osm 63*]

Sol. The vortex lines and the stream lines are the field lines of the vectors $\mathbf{q}$ and curl $\mathbf{q}$ $(=\boldsymbol{\omega})$ To establish the necessity, consider the differential equation $\mathbf{q}\cdot d\mathbf{r}=0$. The condition that this differential equation is integrable is known to be

$$\mathbf{q}\cdot\text{curl }\mathbf{q}=0 \;\Rightarrow\; \mathbf{q}\cdot\boldsymbol{\omega}=0.$$

Thus, under this condition, which implies that vortex lines are at right angles to the stream lines, the equation $\mathbf{q}\cdot d\mathbf{r}=0$ is integrable. However, it is certainly not an exact equation ; because its exactness implies $\mathbf{q}=\text{grad }\phi$ and hence, curl $\mathbf{q}=0$; i.e. vortex lines cease to exist. Hence, there exists an integrating factor μ^{-1} (say) such that $\mu^{-1}\mathbf{q}\cdot d\mathbf{r}=0$ is an exact differential equation. Thus

$$\mu^{-1}\mathbf{q}\cdot d\mathbf{r}=0=d\phi \;\Rightarrow\; \mathbf{q}=\mu\nabla\phi. \tag{1}$$

The three components of (1) are, what is the result required.

To establish sufficiency*, we let

$$\mathbf{q}=\mu\nabla\phi, \text{ so that } \nabla\times\mathbf{q}=\nabla\times(\mu\nabla\phi)$$

or

$$\boldsymbol{\omega}=\mu\text{ curl grad }\phi+\text{grad }\mu\times\nabla\phi=\text{grad }\mu\times\nabla\phi$$

Thus, $\quad \mathbf{q}\cdot\boldsymbol{\omega}=\text{grad }\mu\times\nabla\phi\cdot\mathbf{q}=\text{grad }\mu\cdot\mathbf{q}\times\mathbf{q}\mu^{-1}=0 \quad$ [by (1)]

Hence vortex lines cut stream lines at right angles.

---

* To prove that $\mathbf{q}\cdot\text{curl }\mathbf{q}=0$ when $\mathbf{q}=\mu\nabla\phi$, we prove that $\mathbf{q}\cdot d\mathbf{r}=0$ is integrable.

Now $\quad \mathbf{q}\cdot d\mathbf{r}=0 \;\Rightarrow\; \mu\nabla\phi\cdot d\mathbf{r}=0$, i.e. $d\phi=0$ $(\mu\neq 0)$.

Hence $\phi=$const. is the solution.

Exp. 2. *In an incompressible fluid the vorticity at every point is constant in magnitude and direction. Prove that the components of velocity (u, v, w) are solutions of Laplace's equation.* [Del 1955 ; Kuru 64 ; Mad 59, 53]

Sol. We are given that $\omega = \text{curl}\,\mathbf{q}$ is constant, say $\mathbf{c}$; so that

$$\text{curl curl}\,\mathbf{q} = \text{curl}\,\mathbf{c} = 0$$

or
$$\text{grad div}\,\mathbf{q} - \nabla^2 \mathbf{q} = 0$$

Since div $\mathbf{q}=0$, for incompressible fluids, by virtue of continuity equation (p. 40, the preceding equation reduces to

$$\nabla^2 \mathbf{q} = 0 \;\Rightarrow\; \nabla^2 u = 0,\ \nabla v^2 = 0,\ \nabla^2 w = 0.$$

Ex. If $\mathbf{q}$ and $\mathbf{q}'$ are the velocities of some fluid particle relative to two frames of reference S, S' moving in any manner, the angular velocity of S' relative to S being ω, show that if the vorticities at P relative to S and S' are ζ and ζ', then

$$\zeta = \zeta' + \omega.$$

Show also that if Γ, Γ' be the circulations in the same closed contour C, relative to S, S', then

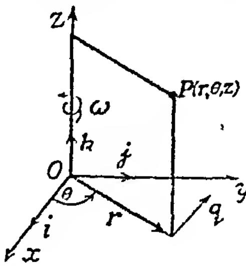
$$\Gamma = \Gamma' + 2 \mid \omega \mid A,$$

where A is the area enclosed by the projection of C on plane perpendicular to ω.

Exp. 3. *Every particle of a mass of liquid is revolving uniformly about a fixed axis with the angular velocity varying as the nth power of the distance from the axis. Show that the motion is irrotational only if $n+2=0$.*

If a very small spherical portion of the liquid be suddenly solidified, prove that it will begin to rotate about a diameter with an angular velocity $(n+2)/2$ of that with which it was revolving about the fixed axis. [Alig 1962 ; Bom 52 ; Del 63]

Sol. Since motion is symmetrical about z-axis, the problem may be considered as 2-dimensional. If ω is the angular velocity, we have $\omega = Ar^n \mathbf{k}$, whence from ordinary kinematics of Dynamics, we have



$$\mathbf{q} = \omega \times \mathbf{r} = Ar^n \mathbf{k} \times (x\mathbf{i} + y\mathbf{j})$$

Thus $\quad u = -Ar^n y,\ v = Ar^n x,\ r^2 = x^2 + y^2.$

Now motion is irrotational if curl $\mathbf{q}=0$,

i.e.
$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0. \qquad (1)$$

Since
$$\frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},$$

$$\frac{\partial v}{\partial x} = A(r^n + nx^2 r^{n-2}),\ \frac{\partial u}{\partial y} = -A(r^n + ny^2 r^{n-2})$$

substitutions in (1) provide

$$2Ar^n + nAr^{n-2}(x^2+y^2) = 0 \text{ or } A(n+2)r^n = 0.$$

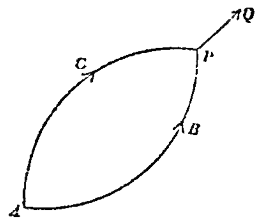
Since $r \neq 0$, we must have $n+2=0$.

The required angular velocity $\omega_1 = \frac{1}{2}$ curl $\mathbf{q} = \frac{1}{2}(n+2)(Ar^n \mathbf{k})$.

Thus,
$$\omega_1 = \tfrac{1}{2}(n+2)\omega.$$

1.60. Irrotational motion and velocity potential. *If the motion is irrotational, the velocity potential necessarily exists, and conversely, if the velocity potential exists, the motion is necessarily irrotational.*

Let the motion of a fluid in a simply connected region be irrotational. Let A be any *fixed* point and P any arbitrary point in the region under consideration. We join A and P by two different routes ABP and ACP both lying in the region in question.



By *Stokes' Theorem*, applied to the closed curve $ABPCA$

$$\int_{ABP} \mathbf{q}\cdot d\mathbf{r}+\int_{PCA} \mathbf{q}\cdot d\mathbf{r}=\int_{S} \mathbf{n}\cdot(\nabla\times\mathbf{q})dS \tag{1}$$

where S is the surface lying entirely in the fluid having $ABPCA$ for its rim. Since the motion is irrotational, $\nabla\times\mathbf{q}=0$ so that from (1), we get

$$\int_{ABP} \mathbf{q}\cdot d\mathbf{r}=\int_{ACP} \mathbf{q}\cdot d\mathbf{r}=-\phi_P \text{ (say)}. \tag{2}$$

Clearly ϕ_P is a *scalar point function depending on the position of P alone* (for A is fixed) *and not on the various routes from A to P.*

Consider now a point Q very close to P such that the velocity vector $\mathbf{q}$ may be *constant* along PQ.

Let η be the position vector of Q with respect to P so that $\overline{PQ}=\eta$. Then by (2),

$$\int_{ABPQ} \mathbf{q}\cdot d\mathbf{r}=-\phi_Q. \text{ Thus, } \quad -\phi_Q-(-\phi_P)=\int_{ABPQ} \mathbf{q}\cdot d\mathbf{r}-\int_{ABP} \mathbf{q}\cdot d\mathbf{r}$$

Hence,
$$-(\phi_Q-\phi_P)=\int_{PQ} \mathbf{q}\cdot d\mathbf{r}=\eta\cdot\mathbf{q} \tag{3}$$

for $\mathbf{q}$ is constant along PQ. Since $\phi_Q-\phi_P \doteq \eta\cdot\nabla\phi_P$ (by Taylor's series), we get from (3)

$$\eta\cdot\mathbf{q}=-\eta\cdot\nabla\phi_P \Rightarrow \mathbf{q}=-\nabla\phi_P=-\nabla\phi \text{ (say)} \tag{4}$$

because η is an arbitrary vector as Q is an arbitrary point.

Thus when the motion is irrotational, there does exist a velocity potential connected with velocity vector by the relation (2).

Conversely, let the velocity potential exist, so that $\mathbf{q}=-\nabla\phi$.

But
$$\nabla\times\mathbf{q}=[-\nabla\times(\nabla\phi)]=0$$

because the **curl** of the **gradient** of a scalar point function is zero.

Therefore, the condition curl $\mathbf{q}=0$, *ensures that the motion is irrotational,*

Cor. *When the external forces are conservative and are derived from a single-valued potential and pressure is a function of density only, then if once the motion of a non-viscous fluid is irrotational, it remains irrotational ever afterwards.*

If the motion is initially irrotational, the circulation is zero for every closed circuit. But the circulation in any such circuit is constant for all times and therefore remains zero for all times. Hence, at any subsequent time, by *Stokes' Theorem,*

$$\int_S \mathbf{n} \cdot (\nabla \times \mathbf{q})\, dS = 0, \text{ or, curl } \mathbf{q} = 0\,; \text{ i.e. } \omega = 0$$

Hence the irrotational motion is permanent.

Ex. 1. Point out the important physical distinction that exists in the character of rotational and irrotational motion of a perfect fluid, and prove the theorem that if in a perfect fluid under the action of a conservative system of forces a velocity potential ever exists there will always exist a velocity potential.

[*Ag 1952 ; Jad 59*]

Ex. 2. State and prove Stokes' theorem and use it to obtain the theorem of the permanence of irrotational motion. [*Pb 1954*]

Exp. 1. *Distinguish between* (i) *stream lines,* (ii) *paths of the particles in the case of motion of a perfect fluid. Obtain the equations of stream lines in* (a) *three-dimensional polar co-ordinates,* (b) *cylindrical co-ordinates when the motion is irrotational.* [*Del 1960, 56*]

Sol. Stream lines are the field lines of the velocity vector $\mathbf{q}$ and are determined by the equation

$$\mathbf{q} \times d\mathbf{r} = 0. \tag{1}$$

If the motion is irrotational, $\mathbf{q} = -\text{grad } \phi$. For the two systems we get

space polar co-ordinates (r, θ, φ) : $d\mathbf{r} = (dr, r\,d\theta, r \sin\theta\, d\varphi)$,

$$\text{grad } \phi = \left(\frac{\partial\phi}{\partial r}, \ \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \ \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\right)$$

cylindrical co-ordinates (r, θ, z) : $d\mathbf{r} = (dr, r\,d\theta, dz)$,

$$\text{grad } \phi = \left(\frac{\partial\phi}{\partial r}, \ \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \ \frac{\partial\phi}{\partial z}\right)$$

Substitutions in (1) yield the results

$$\frac{dr}{-(\partial\phi/\partial r)} = \frac{r\,d\theta}{-(1/r)(\partial\phi/\partial\theta)} = \frac{r\sin\theta\, d\varphi}{-(1/r\sin\theta)(\partial\phi/\partial\varphi)}$$

$$\frac{dr}{-(\partial\phi/\partial r)} = \frac{r\,d\theta}{-(1/r)(\partial\phi/\partial\theta)} = \frac{dz}{-(\partial\phi/\partial z)}.$$

Exp. 2. *Define circulation and prove that the circulation in any closed path moving with the fluid is independent of time.*

Show that if the velocity potential of an irrotational fluid motion is $(A/r^2)\, z \cos\theta$, *where* (r, θ, z) *are the polar co-ordinates of any point, the lines of flow lie on the series of surfaces*

$$r = k \sin^2\theta.$$ [*Pb 1948*]

Sol. The differential equations for the stream lines in polar spherical co-ordinates are

$$\frac{dr}{q_r}=\frac{rd\theta}{q_\theta}=\frac{r\sin\theta d\varphi}{q_\varphi} \tag{1}$$

Here $V=\frac{A}{r^2}\varphi\cos\theta.$ $\therefore\ q_r=-\frac{\partial V}{\partial r}=\frac{2A}{r^3}\varphi\cos\theta$

$$q_\theta=-\frac{1}{r}\frac{\partial V}{\partial\theta}=\frac{A}{r^3}\varphi\sin\theta,\qquad q_\varphi=-\frac{1}{r\sin\theta}\frac{\partial V}{\partial\varphi}=\frac{A}{r^3}\frac{\cos\theta}{\sin\theta}$$

Thus (1) is equivalent to

$$\frac{dr}{2\varphi\cos\theta}=\frac{rd\theta}{\varphi\sin\theta}=\frac{r\sin\theta\,d\varphi}{-\cot\theta}.$$

The first two members give

$$\frac{dr}{r}=\frac{2\cos\theta}{\sin\theta}d\theta$$

Integration gives : $\log r=\log k\sin^2\theta$, or $r=k\sin^2\theta$.

Ex. 1. Show that if the velocity potential of an irrotational fluid motion is equal to $A(x^2+y^2+z^2)^{-\frac{3}{2}}z\tan^{-1}(y/x)$; the lines of flow will lie on the series of surfaces

$$(x^2+y^2+z^2)=k^{\frac{2}{3}}(x^2+y^2)^{\frac{2}{3}}$$ [*Alig 1957 ; Jab 62*]

Ex. 2. The vorticity vector of every fluid particle *inside* an infinitely long circular cylinder of fluid parallel to Oz, is $\omega\mathbf{k}$, ω constant, and zero *outside* the cylinder. Calculate the fluid velocity inside and outside this cylindrical vortex.

1.61. Hydrodynamical singularities : three-dimensional sources and dipoles.

If the motion of a liquid consists of an outward symmetrical radial flow in all directions, issuing from a point, the point is called a simple **source.**

If the total flow across a small surface surrounding the point source is $4\pi m$, then m is called the strength of the source.

A sink is a negative source and hence m is negative. *The ultimate combination of a source and a sink, placed indefinitely close to each other, is called the* **doublet.**

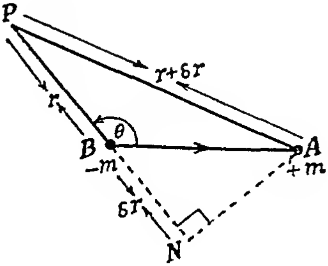
For a sink of strength $-m$ at B and source of strength m at A, where $BA=\delta s$, when $\delta s\to 0$, $m\to\infty$, but $m\delta s\to$ a finite limit μ, then μ is termed the strength of the doublet. The direction BA gives the axis of the doublet.

Velocity potential of a source. The velocity q_r distant r from the source situated at O, will be purely radial and hence a function of r only. With centre o and radius r, draw a sphere and consider a small area dS which subtends a small solid angle ϖ at O. Then the flow through dS gives $\varpi\, m=\varpi r^2 q_r$, hence

$$q_r=\frac{m}{r^2},\ \text{i.e.}\ \mathbf{q}=\frac{m}{r^2}\mathbf{n}=-\frac{\partial}{\partial r}\left(\frac{m}{r}\right)\mathbf{n} \tag{1}$$

where $\mathbf{n}$ is a unit radial vector. We infer from (1) that $\mathbf{q}=-\nabla(m/r)$ and hence motion is irrotational. Thus, ϕ exists and is given by

$$\phi=m/r. \tag{2}$$



Velocity potential of a point doublet. For a sink $-m$ at B and source m at A, by superimposing their fields, we get

$$\phi=\frac{m}{r+\delta r}-\frac{m}{r}=-\frac{m\delta r}{r(r+\delta r)}.$$

Let AN be perpendicular to PB produced, then

$$\delta r\doteq BN=-AB\cos\theta.$$

Thus, $$\phi=m\,AB.\cos\theta/r(r+\delta r). \tag{3}$$

Now, as $A\to B$ along BA, $m\to\infty$, $AB\to 0$, but $m.AB\to\mu$, then the velocity potential of a doublet is

$$\phi=\mu\cos\theta/r^2. \tag{4}$$

Cor. Since $$\phi=\frac{\mu\cos\theta}{r^2}=\mu.\ \frac{\partial}{\partial s}\left(\frac{1}{r}\right), \tag{5}$$

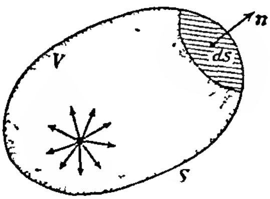
it follows that velocity potential due to a doublet may be obtained from that of a source by differentiation in the direction of the axis of the doublet.

NOTE. *Radial Velocity* $=-\dfrac{\partial\phi}{\partial r}=\dfrac{2\mu\cos\theta}{r^3}$.

Cross-radial Velocity $=-\dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta}=\dfrac{\mu\sin\theta}{r^3}$.

1.70. Equation of continuity or conservation of mass. Let ρ denote the density of the fluid at a point $P(\mathbf{r})$ of the mass of the fluid contained in any closed surface S *fixed* in space and containing a volume V. The continuity equation is based upon the following maxim.

The rate at which the mass of fluid inside any volume is increasing is equal to the source rate of mass within the volume minus the rate at which mass flows out through the surface of the volume.



Now, if mass of the fluid within this fixed surface is m, then

$$\frac{\partial m}{\partial t}=\frac{\partial}{\partial t}\int_V \rho dv=\int_V \frac{\partial\rho}{\partial t}dv \tag{1}$$

since volume does not vary with time. Further, if R is the source rate of mass per unit volume, then the fluid mass generated is

$$\int_V R dv \tag{2}$$

Also the rate at which mass flows *out* $=\displaystyle\int_S \rho\mathbf{q}.d\mathbf{S}$ (3)

The above maxim now provides the mathematical formulation

$$\int_V \frac{\partial \rho}{\partial t}\,dv = \int_V R\,dv - \int_S \rho\mathbf{q}\,.\,d\mathbf{S} \tag{3'}$$

$$= \int_V R\,dv - \int_V \operatorname{div}(\rho\mathbf{q})\,dv \quad \text{[by Div. theorem]}$$

or

$$\int_V \left[\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\mathbf{q}) - R\right] dv = 0. \tag{4}$$

The result (4) will hold for any arbitrarily chosen volume V. Hence the integrand itself must vanish and the continuity equation can be written as

$$\partial\rho/\partial t + \operatorname{div}(\rho\mathbf{q}) = R. \tag{5}$$

For the very special but important case, when $R=0$, the source-free equation of continuity is

$$\partial\rho/\partial t + \operatorname{div}(\rho\mathbf{q}) = 0. \tag{6}$$

NOTE 1. In the absence of sources within the surface, i.e. when $R=0$, the continuity maxim simply reads thus :

'The increase in the mass of the fluid within the fixed surface during the time δt must be equal to the excess of the mass that flows in over the mass that flows out in the same interval δt.'

NOTE 2. The forms (3′) and (5) or (6) are known as the integral and differential forms of the equation of continuity.

Cor. 1. Since $\nabla\,.\,(\rho\mathbf{q}) = \rho\,(\nabla\,.\,\mathbf{q}) + \mathbf{q}\,.\,(\nabla\rho)$, the equation of continuity may be written as

$$\frac{\partial \rho}{\partial t} + (\mathbf{q}\,.\,\nabla)\rho + \rho(\nabla\,.\,\mathbf{q}) = 0 \quad \text{or} \quad \frac{d\rho}{dt} + \rho(\nabla\,.\,\mathbf{q}) = 0. \tag{7}$$

Since $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$; $\nabla\,.\,(\rho\,\mathbf{q}) = \dfrac{\partial(u\rho)}{\partial x} + \dfrac{\partial(v\rho)}{\partial y} + \dfrac{\partial(w\rho)}{\partial z}$

the equation of continuity (7) can be put in the Cartesian form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(u\rho) + \frac{\partial}{\partial y}(v\rho) + \frac{\partial}{\partial z}(w\rho) = 0$$

or

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0.$$

Cor. 2. If the fluid be incompressible, $d\rho/dt=0$, so that the equation of continuity reduces to

$$\nabla\,.\,\mathbf{q} = 0, \quad \text{or} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

Thus the velocity vector $\mathbf{q}$ is *Solenoidal*. Obviously $(\nabla\,.\,\mathbf{q})$ gives the rate of volume expansion of a fluid element. For this reason, it may be called *dilatation* or *expansion*.

Cor. 3. If the motion is irrotational, $q = -\text{grad } \phi$, and hence

$$\text{div } q = 0, \Rightarrow \nabla^2\phi = 0 \quad \text{by (5)} \qquad \textit{Laplaces' equation}$$

Cor. 4. For the incompressible irrotational flow ($d\rho/dt = 0$ and $q = -\nabla\phi$) we obtain from (5)

$$\nabla^2\phi = -R \qquad \text{(Poisson's equation)}$$

NOTE. Since $\text{div } q = -(d\rho/dt)/\rho$, we can interpret div q as the relative rate at which the density is decreasing. Thus, $\text{div } q > 0 \Rightarrow (d\rho/dt) < 0$ and consequently an attenuation of the fluid at the point considered ; hence the term *divergence*.

Ex. 1 What physical significance is implied in the equation of continuity ? Express the equation of continuity in the form

$(\partial\rho/\partial t) + \text{div } (\rho q) = 0$. [q denotes the velocity vector] [*Del 1960*]

Ex. 2. By taking any region in a fluid bounded by a closed surface, show that the equation of continuity is

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho V) = 0.$$

What is the physical meaning of $\nabla \times V = 0$?

When this condition is satisfied and the motion is steady, show that the equation of continuity can be put in the form

$$\frac{\nabla^2\phi_1}{\phi_1} = \frac{\nabla^2\rho_1}{\rho_1}$$

where $\rho_1 = \rho^{\frac{1}{2}}$, $\phi_1 = \rho^{\frac{1}{2}}\phi$, $V = -\nabla\phi$. (*Mad* 1957)

Exp. 1. *A pulse travelling along a fine straight uniform tube filled with gas causes the density at time t and distance x from the origin where the velocity is u_0 to become $\rho_0\,\phi(vt-x)$. Prove that the velocity u (at time t and distance x from the origin) is given by*

$$v + \frac{(u_0 - v)\,\phi(vt)}{\phi(vt-x)}. \qquad [\textit{Del 1956}; \textit{Pna 1964}]$$

Sol. The equation of continuity in the present case is

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \quad \text{or} \quad \frac{\partial\rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial\rho}{\partial x} = 0. \qquad (1)$$

Since $\rho = \rho_0\,\phi(vt-x) = \rho_0\,\phi(z)$ say, [by hypothesis]

$$\therefore \quad \frac{\partial\rho}{\partial x} = \rho_0\frac{d\phi}{dz}\frac{\partial z}{\partial x} = -\rho_0\frac{d\phi}{dz} = -\rho_0\phi'(z), \quad \text{Say}$$

$$\frac{\partial\rho}{\partial t} = \rho_0\frac{d\phi}{dz}\frac{\partial z}{\partial t} = v\rho_0\phi'(z).$$

With these values (1) reduces to

$$v\phi'(z) - \phi(z)\,\partial u/\partial z - u\,\phi'(z) = 0 \qquad (\partial x = -\partial z)$$

or

$$\frac{du}{v-u} - \frac{d\phi}{\phi} = 0.$$

Integrating this equation, we get

$$\log(v-u) + \log\phi = \log A$$

or

$$(v-u)\,\phi = A.$$

At any time t, when $x=0$, $u=u_0$, so that $\phi(vt-x) = \phi(vt)$. Hence

$$A = (v-u_0)\,\phi(vt)$$

$$\therefore \qquad (v-u)\,\phi=(v-u_0)\,\phi(vt)$$

or

$$u=v+\frac{(u_0-v)\,\phi(vt)}{\phi(vt-x)}.$$

Ex. For a fluid in motion, explain the process of differentiation following the stream lines and derive the equation of continuity.

A gas is moving in a uniform straight tube. Prove that if the density be $f(at-x)$ at a point, where t is the time and x is the distance of the point from an end of the tube, its velocity is

$$\frac{af(at-x)+(v-a)\,f(at)}{f(at-x)}$$

where v is the velocity at that end of the tube and a is a constant [*Bom* 1952]

Exp. 2. *If q is the resultant velocity at any point of a fluid which is moving irrotationally in two dimensions, prove that*

$$\left(\frac{\partial q}{\partial x}\right)^2+\left(\frac{\partial q}{\partial y}\right)^2=q\nabla^2 q.$$

[*Ban 65 ; Bom 56 ; 52 ; Del 65 ; Gat 59, 53 ; Gor 60 ; Jad 58 ; Mad 59, 57 ; Osm 63 ; Pb 50 ; Pna 60*]

Sol. Since the motion is irrotational and density is constant, we have

$$q^2=\phi_x^2+\phi_y^2\ ;\ \phi_{xx}+\phi_{yy}=0 \tag{1}$$

where suffixes indicate partial derivatives.

To get the form of $q\nabla^2 q$, we have to appeal to Laplacian expansion

$$\nabla^2(ab)=a\nabla^2 b+b\nabla^2 a+2(\nabla a)\cdot(\nabla b)$$

where we put $a=b=q$ to obtain

$$\nabla^2 q^2=2[q\nabla^2 q+(\nabla q)^2] \tag{2}$$

Now

$$\begin{aligned}\nabla^2 q^2&=\nabla^2(\phi_x^2+\phi_y^2)=\nabla^2\phi_x^2+\nabla^2\phi_y^2\\&=2[\phi_x\nabla^2\phi_x+(\nabla\phi_x)^2]+2[\phi_y\nabla^2\phi_y+(\nabla\phi_y)^2],\text{ as in (2)}\\&=2[(\nabla\phi_x)^2+(\nabla\phi_y)^2] \qquad [\because\ \nabla^2\phi_x=\partial(\nabla^2\phi)/\partial x=0,\text{ etc.}]\\&=2[(\mathbf{i}\phi_{xx}+\mathbf{j}\phi_{yx})^2+(\mathbf{i}\,\phi_{xy}+\mathbf{j}\,\phi_{yy})^2]\\&=4[\phi^2_{xx}+\phi^2_{xy}] \qquad [\because\ \phi_{yy}=-\phi_{xx}]\end{aligned} \tag{3}$$

Further, taking the gradient of first of (1) we get

$$\begin{aligned}q\nabla q&=\phi_x\,\nabla\phi_x+\phi_y\nabla\phi_y\\&=\mathbf{i}(\phi_x\,\phi_{xx}+\phi_y\,\phi_{xy})+\mathbf{j}(\phi_x\,\phi_{yx}+\phi_y\,\phi_{yy}).\end{aligned}$$

$$\therefore \qquad q^2(\nabla q)^2=(\phi^2_x+\phi^2_y)\,(\phi^2_{xx}+\phi^2_{xy}) \qquad [\because\ \phi_{yy}=-\phi_{xx}]$$

or

$$(\nabla q)^2=\phi^2_{xx}+\phi^2_{xy}\quad\text{by (1)} \tag{4}$$

From (3) and (4) we get $\nabla^2 q^2=4(\nabla q)^2$; whence (2) yields

$$4(\nabla q)^2=2[q\,\nabla^2 q+(\nabla q)^2]\Rightarrow(\nabla q)^2=q\,\nabla^2 q\ ;$$

which is the required result.

1.71. Equation of continuity in the Lagrangian method. Let us consider a fluid particle of infinitesimal volume dv and density ρ at time t. Since the mass of the fluid-particle is invariant as it moves about, we must have

$$\frac{d}{dt}(\rho\ dv)=0$$

$$\therefore \qquad \rho\ dv=\text{const.}=\rho_0\ dv_0\ \text{(say)}, \tag{1}$$

where $\rho_0\ dv_0$ refers to the mass of the particle in its initial position.

In cartesian rectangular co-ordinates

$$dv=dx\,dy\,dz\;;\quad dv_0=da\,db\,dc\;;$$

$$dx\,dy\,dz=J\,da\,db\,dc$$

where $$J=\partial(x,y,z)/\partial(a,b,c).$$

Thus in the present notation, the equation of continuity (1) becomes

$$\rho J=\rho_0.$$

NOTE. It is not necessary that $\mathbf{r}_0=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ should be the initial position vector. Any variable vector which can serve to identify a particle and which changes continuously from one particle to another will serve the purpose.

Ex. Explain the difference between Eulerian and Lagrangian methods in Hydrodynamics, and deduce the equation of continuity in each case. *(Ban 1947)*

Exp. *Show that in the motion of a fluid in two dimensions if the co-ordinates* (x,y) *of an element at any time be expressed in terms of the initial co-ordinates* (a, b) *and the time, the motion is irrotational, if*

$$\frac{\partial(\dot{x},x)}{\partial(a,b)}+\frac{\partial(\dot{y},y)}{\partial(a,b)}=0 \qquad [Jab\ 1960]$$

Sol. If u, are v the components of velocity parallel to co-ordinate axes, then $\dot{x}=u\,;\ \dot{y}=v$.

Also $$\frac{\partial u}{\partial a}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial a}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial a},\quad \frac{\partial u}{\partial b}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial b}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial b}$$

and two more expressions for $\partial v/\partial a$ and $\partial v/\partial b$. Now

$$\frac{\partial(\dot{x},x)}{\partial(a,b)}+\frac{\partial(\dot{y},y)}{\partial(a,b)}=\left(\frac{\partial u}{\partial a}\frac{\partial x}{\partial b}-\frac{\partial u}{\partial b}\frac{\partial x}{\partial a}\right)+\left(\frac{\partial v}{\partial a}\frac{\partial y}{\partial b}-\frac{\partial v}{\partial b}\frac{\partial y}{\partial a}\right)$$

$$=\frac{\partial x}{\partial b}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial a}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial a}\right)-\frac{\partial x}{\partial a}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial b}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial b}\right)+$$

$$\frac{\partial y}{\partial b}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial a}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial a}\right)-\frac{\partial y}{\partial a}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial b}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial b}\right)$$

$$=\frac{\partial u}{\partial y}\left(\frac{\partial x}{\partial b}\frac{\partial y}{\partial a}-\frac{\partial x}{\partial a}\frac{\partial y}{\partial b}\right)+\frac{\partial v}{\partial x}\left(\frac{\partial x}{\partial a}\frac{\partial y}{\partial b}-\frac{\partial y}{\partial a}\frac{\partial x}{\partial b}\right)$$

$$=\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\left(\frac{\partial x}{\partial a}\frac{\partial y}{\partial b}-\frac{\partial y}{\partial a}\frac{\partial x}{\partial b}\right)=\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\frac{\partial(x,y)}{\partial(a,b)}.$$

Now $\partial(x,y)/\partial(a,b)\neq 0$ by virtue of Lagrangian equation of continuity, therefore

$$\frac{\partial(\dot{x},x)}{\partial(a,b)}+\frac{\partial(\dot{y},y)}{\partial(a,b)}=0$$

if and only if $\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}=0$, i.e. *only if the motion is irrotational.*

1.72. Equivalence of the Eulerian and the Lagrangian forms of the equation of continuity. For the sake of simplicity, we transform equation of continuity in Lagrangian form to the Eulerian form. As such, we differentiate the form $\rho J=\rho_0$ and get

$$J(d\rho/dt)+\rho(dJ/dt)=0. \qquad (1)$$

Since $dJ/dt = J \operatorname{div} \mathbf{q}$ (§0.50, p 7) ; equation (1) reduces to

$$(d\rho/dt) + \rho \operatorname{div} \mathbf{q} = 0.$$

Conversely, let $\dfrac{d\rho}{dt} + \rho \operatorname{div} \mathbf{q} = 0$ be the Eulerian's continuity-condition so that, setting $J \operatorname{div} \mathbf{q} = (dJ/dt)$, we have

$$J\frac{d\rho}{dt} + \rho\frac{dJ}{dt} = 0, \text{ i.e. } \frac{d}{dt}(\rho J) = 0.$$

This leads to $\rho J = \text{constant} = \rho_0$ (say), which is the Lagrangian form of continuity condition.

Hence the two forms are essentially equivalent.

Ex. Find the equation of continuity in the Lagrangian method and show that it is equivalent to

$$(d\rho/dt) + \rho \operatorname{div} \mathbf{q} = 0. \qquad (Ag\ 1953)$$

1.720. Eulerian form of continuity condition via Lagrangian system. Let $\rho(\mathbf{r}, t)$ be the mass per unit volume of a homogeneous fluid at any point $\mathbf{r}$ and time t.

Then the mass of any finite volume V is

$$m = \int_V \rho(\mathbf{r}, t)\, dv. \tag{1}$$

If V is a *material* volume having no sources or sinks within its medium, then putting $F = \rho$ in § 1.301, p. 22, (i.e. Reynold's transport theorem), we get

$$\frac{dm}{dt} = \int_V \left[\frac{d\rho}{dt} + \rho(\operatorname{div} \mathbf{q})\right] dv.$$

We take it as a principle that the mass of a material volume does not change so that $dm/dt = 0$. Now this is true for an arbitrary volume, and hence the integrand itself must vanish *everywhere*, it being a continuous function. Hence

$$d\rho/dt + \rho \operatorname{div} \mathbf{q} = 0. \tag{1}$$

NOTES. 1. If we put $F = \rho f$ in the Reynold's transport theorem, we have

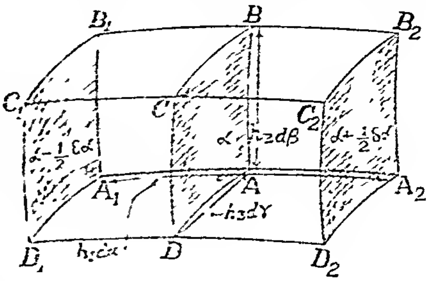
$$\frac{d}{dt}\int_V \rho f\, dv = \int_V \left[\frac{d}{dt}(\rho f) + \rho f \operatorname{div} \mathbf{q}\right] dv$$

$$= \int_V \left[\rho\frac{df}{dt} + f\left(\frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{q}\right)\right] dv$$

$$= \int_V \rho\frac{df}{dt}\, dv \qquad \text{by (1)}$$

2. In the above derivation, the material volume has been permitted to *move* in the Lagrangian fashion. But the result obtained is identical with Euler's approach in which the surface is held *fixed*. In view of this fact, § 1.72 is indeed *superfluous*.

§1.73. Equation of continuity in orthogonal curvilinear coordinates. Let ρ denote the density of the fluid at $P(\alpha, \beta, \gamma)$ and with this point as centre construct a small curvilinear parallelopiped with edges of length $h_1 d\alpha$, $h_2 d\beta$, $h_3 d\gamma$ (see adjacent Fig.). Let the components of the velocity vector $\mathbf{q}$ in the (α, β, γ) directions be (q_1, q_2, q_3). The continuity equation is based upon the following maxim :



The rate at which the mass of fluid inside any volume is increasing is equal to the source rate of mass within the volume minus the rate at which mass flows out through the surface of the volume.

Now, if mass of the fluid within this fixed surface is m, then

$$\frac{\partial m}{\partial t}=\frac{\partial}{\partial t}\left[\rho h_1 h_2 h_3\, d\alpha d\beta d\gamma\right]=h_1 h_2 h_3\, d\alpha d\beta d\gamma \frac{\partial \rho}{\partial t} \tag{1}$$

because volume does not vary with time. Further, if R is the source rate of mass per unit volume, then the fluid mass generated is

$$R dv=R h_1 h_2 h_3 d\alpha d\beta d\gamma. \tag{2}$$

Also, the rate at which mass flows $out=+\int_S \rho\mathbf{q}\,.\,d\mathbf{S}$ (3)

The above maxim now provides the mathematical formulation

$$\frac{\partial m}{\partial t}=R dv-\int_S \rho\mathbf{q}\,.\,ds \tag{4}$$

To calculate the value of the integral in (4), we need find the flux of vector $\rho\mathbf{q}$ across all the six faces of the surface. Now flux across the curvilinear area $ABCD$ is $\rho q_1 h_2 d\beta h_3 d\gamma=f(\alpha,\beta,\gamma)$ say ; hence the flux across the faces $A_2B_2C_2D_2$ and $A_1B_1C_1D_1$ shall be $f(\alpha+\frac{1}{2}d\alpha, \beta.\gamma)$ and $f(\alpha-\frac{1}{2}d\alpha, \beta, \gamma)$ respectively. Thus, the net flux across these two faces is

$$f(\alpha+\tfrac{1}{2}d\alpha, \beta, \gamma)-f(\alpha-\tfrac{1}{2}d\alpha, \beta, \gamma)\doteq d\alpha\frac{\partial f}{\partial \alpha}= d\alpha d\beta d\gamma \frac{\partial}{\partial \alpha}(\rho q_1 h_2 h_3)\cdot$$

Similarly, the net flux across the faces $A_2A_1D_1D_2$ and $B_2B_1C_1C_2$ is

$$d\alpha d\beta d\gamma\ \partial(\rho q_2 h_3 h_1)/\partial\beta\ ;$$

and that the net flux across the faces $A_1A_2B_2B_1$ and $C_1D_1D_2C_2$ is

$$d\alpha d\beta d\gamma\ \partial(\rho q_3 h_1 h_2)/\partial\gamma.$$

Summing now over all the six faces, the total flux of $\rho\mathbf{q}$ is

$$\left[\frac{\partial}{\partial \alpha}(\rho q_1 h_2 h_3)+\frac{\partial}{\partial \beta}(\rho q_2 h_3 h_1)+\frac{\partial}{\partial \gamma}(\rho q_3 h_1 h_2)\right]d\alpha d\beta d\gamma=\int_S \rho\mathbf{q}\,.\,d\mathbf{S} \tag{5}$$

Substitutions from (1), (2), (5) into (4) yield

$$\frac{\partial \rho}{\partial t}+\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial \alpha}(\rho q_1 h_2 h_3)+\frac{\partial}{\partial \beta}(\rho q_2 h_3 h_1)+\frac{\partial}{\partial \gamma}(\rho q_3 h_1 h_2)\right]=R. \qquad (6)$$

NOTE. If there are no sources within the surface, $R=0$; which implies that the increase in the mass of fluid within the surface in any time δt must be equal to the excess of the mass that flows in over the mass that flows out. And the equation of continuity in that case reduces to

$$\frac{\partial \rho}{\partial t}+\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial \alpha}(\rho q_1 h_2 h_3)+\frac{\partial}{\partial \beta}(\rho q_2 h_3 h_1)+\frac{\partial}{\partial \gamma}(\rho q_3 h_1 h_2)\right]=0. \qquad (7)$$

Exp. *Obtain equation of continuity of an inviscid fluid in orthogonal curvilinear coordinates by using the expression for div of a vector. Hence deduce the same in cylindrical and spherical polar coordinates.*

Sol. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ denote unit vectors tangent to the coordinate curves at any point.

Let $$\mathbf{q}=q_1\mathbf{e}_1+q_2\mathbf{e}_2+q_3\mathbf{e}_3$$

so that (q_1, q_2, q_3) are velocity components of $\mathbf{q}$ along $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ respectively.

The equation of continuity is

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho \mathbf{q})=0 \qquad (1)$$

We know from vector calculus that for any vector point function $\mathbf{f}$, with usual notation [Vide §0.20(2), p. 4].

$$\nabla\cdot\mathbf{f}=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial \alpha}(f_1 h_2 h_3)+\frac{\partial}{\partial \beta}(f_2 h_3 h_1)+\frac{\partial}{\partial \gamma}(f_3 h_1 h_2)\right];$$

where $\mathbf{f}=f_1\mathbf{e}_1+f_2\mathbf{e}_2+f_3\mathbf{e}_3$.

Replacing f by $\rho\mathbf{q}$ and f_1 by ρq_1, etc. we get

$$\nabla\cdot(\rho\mathbf{q})=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial \alpha}(\rho q_1 h_2 h_3)+\frac{\partial}{\partial \beta}(\rho q_2 h_3 h_1)+\frac{\partial}{\partial \gamma}(\rho q_3 h_1 h_2)\right] \qquad (2)$$

Hence the equation of continuity is, by (1) and (2)

$$\frac{\partial \rho}{\partial t}+\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial \alpha}(\rho q_1 h_2 h_3)+\frac{\partial}{\partial \beta}(\rho q_2 h_3 h_1)+\frac{\partial}{\partial \gamma}(\rho q_3 h_1 h_2)\right]=0. \qquad (3)$$

(i) *For cylindrical coordinates*; $h_1=1$, $h_2=r$, $h_3=1$; $(\alpha, \beta, \gamma)\equiv(r, \theta, z)$. With (q_1, q_2, q_3) in the (r, θ, z)-directions, (3) provides

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial r}\left[\frac{\partial}{\partial r}(\rho q_1 r)+\frac{\partial}{\partial \theta}(\rho q_2)+\frac{\partial}{\partial z}(\rho q_3 r)\right]=0$$

or $$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho q_1 r)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho q_2)+\frac{\partial}{\partial z}(\rho q_3)=0.$$

(ii) *For spherical coordinates*: $h_1=1, h_2=r, h_3=r\sin\theta$, $(\alpha, \beta, \gamma)\equiv(r, \theta, \varphi)$. With (q_1, q_2, q_3) in the (r, θ, φ)-directions, (3) provides

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(\rho q_1 r^2\sin\theta)+\frac{\partial}{\partial \theta}(\rho q_2 r\sin\theta)+\frac{\partial}{\partial \varphi}(\rho q_3 r)\right]=0$$

or $$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho q_1 r^2)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho q_2\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}(\rho q_3)=0.$$

Ex. 1. Obtain the equation of continuity in generalized orthogonal system of coordinates and deduce the equation of continuity (i) in spherical polar coordinates, and (ii) in cylindrical coordinates. *(Del 1961, 57)*

Ex. 2. Show that if ξ, η, ζ be the orthogonal coordinates and if (u, v, w) be the corresponding component velocities, the equation of continuity is

$$\frac{\partial \rho}{\partial t}+\rho(us_1+vs_2+ws_3)+h_1\frac{\partial}{\partial \xi}(\rho u \quad h_2\frac{\partial}{\partial \eta}(\rho v)+h_3\frac{\partial}{\partial \zeta}(\rho w)=0,$$

where $\dfrac{1}{h_1{}^2}=\left(\dfrac{\partial x}{\partial \xi}\right)^2+\left(\dfrac{\partial y}{\partial \eta}\right)^2+\left(\dfrac{\partial z}{\partial \zeta}\right)^2$; etc. and s_1, s_2, s_3 are respectively the sums of the principal curvatures of the three orthogonal surfaces.

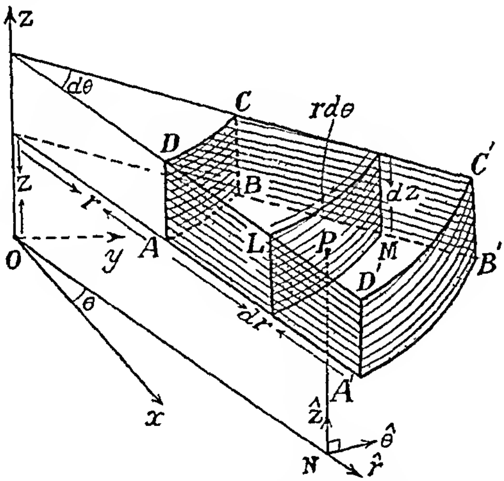
If the motion is steady, deduce the equation in spherical polar coordinates for a liquid and show that for irrotational flow, the velocity potential ϕ satisfies Laplace's equation. [*Lkn 1959*]

Ex. 3. If a thin stratum of liquid moves irrotationally on the surface of a sphere, prove that the velocity potential ϕ satisfies

$$\sin\theta\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \phi}{\partial \theta}\right)+\frac{\partial^2 \phi}{\partial \omega^2}=0.$$

Obtain the general solution of this differential equation. [*Bom 1955*]

1·74. Equation of continuity in cylindrical coordinates. Let ρ denote the density of the fluid at (r, θ, z) and with this point as centre construct a small curvilinear parallelopiped with edges of length dr, $rd\theta$, dz (See Fig.). Let the components of the velocity vector $\mathbf{q}$ in the (r, θ, z)-directions be (q_1, q_2, q_3) respectively. The continuity equation is based upon the following maxim :



The rate at which the mass of fluid inside any volume is increasing is equal to the source rate of mass within the volume minus the rate at which mass flows out through the surface of the volume.

Now, if mass of the fluid within this *fixed* surface is m, then

$$\frac{\partial m}{\partial t}=\frac{\partial}{\partial t}(\rho r\,d\theta\,dr\,dz)=rd\theta\,dr\,dz\frac{\partial \rho}{\partial t} \tag{1}$$

because volume does not vary with time. Further if R is the source rate of mass per unit volume, then the fluid mass generated is

$$R\,dv=R\,rd\theta dr dz \tag{2}$$

Also, the rate at which mass flows out $=+\int_S \rho\mathbf{q}\cdot d\mathbf{S}$ (3)

The above maxim now provides the mathematical formulation

$$\frac{\partial m}{\partial t}=R\,dv-\int_S \rho\mathbf{q}\cdot d\mathbf{S} \tag{4}$$

To calculate the value of the integral in (4), we need find the flux of vector $\rho\mathbf{q}$ across all the six faces of the surface. Now flux across the curvilinear area LM through P is $\rho q_1\, r d\theta dz=f(r, \theta, z)$ say ; hence the flux across the faces $ABCD$ and $A'B'C'D'$ shall be $f(r-\frac{1}{2}dr, \theta, z)$ and $f(r+\frac{1}{2}dr, \theta, z)$ respectively. Thus, the net flux across these two faces is

$$f(r+\tfrac{1}{2}\,dr, \theta, z)-f(r-\tfrac{1}{2}dr, \theta, z)\doteq dr\,\frac{\partial f}{\partial r}=dr d\theta dz\,\frac{\partial}{\partial r}(\rho q_1\, r).$$

Similarly, the net flux across the faces $AA'D'DA$ and $BB'C'CB$ is

$$dr\, d\theta\, dz\ \partial(\rho q_2)/\partial\theta\ ;$$

and the net flux across the faces $AA'B'BA$ and $D'C'CDD'$ is

$$dr\, d\theta\, dz\ \partial(\rho q_3 r)/\partial z.$$

Summing now over all the six faces, the total flux of $\rho\mathbf{q}$ is

$$\left[\frac{\partial}{\partial r}(\rho q_1 r)+\frac{\partial}{\partial\theta}(\rho q_2)+\frac{\partial}{\partial z}(\rho q_3 r)\right]dr\, d\theta\, dz=\int_S \rho\mathbf{q}\cdot d\mathbf{S} \tag{5}$$

Substitutions from (1), (2) and (5) into (4) yield

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\left[\frac{\partial}{\partial r}(\rho q_1 r)+\frac{\partial}{\partial\theta}(\rho q_2)+\frac{\partial}{\partial z}(\rho q_3 r)\right]=R \tag{6}$$

For the very special but important case, when $R=0$ we get

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\left[\frac{\partial}{\partial r}(\rho q_1 r)+\frac{\partial}{\partial\theta}(\rho q_2)+\frac{\partial}{\partial z}(\rho q_3 r)\right]=0. \tag{7}$$

Ex. Develop from first principles the polar form of the equation of continuity for two-dimensional incompressible flow

$$\partial(ur)/\partial r+\partial v/\partial\theta=0.$$

If, in a particular flow, $u=-\mu\cos\theta/r^2$, determine the value of v and find the magnitude of the resulting velocity q. [*Ans* : $v=-\mu\sin\theta/r^2$, $q=\mu/r^2$]

1·75. Equation of continuity in spherical polar coordinates. Let ρ denote the density of the fluid at $P(r, \theta, \phi)$ and with this point as centre construct a small curvilinear parallelopiped with edges of length dr, $rd\theta$, $r\sin\theta\, d\phi$ (See Fig.). Let the components of the velocity vector $\mathbf{q}$ in the (r, θ, ϕ) directions be (q_1, q_2, q_3) respectively. The continuity equation is based upon the following maxim :

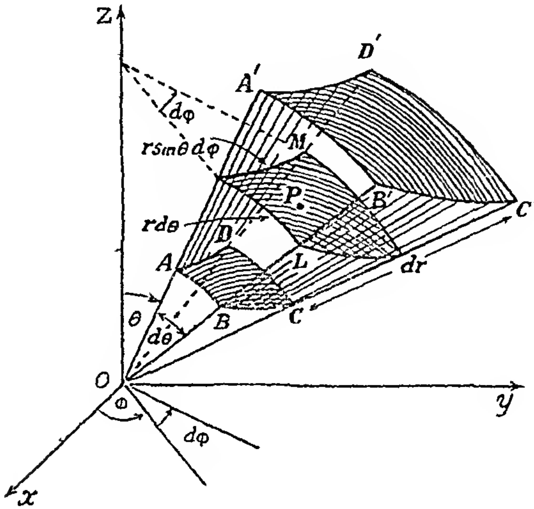
The rate at which the mass of fluid inside any volume is increasing is equal to the source rate of mass within the volume minus the rate at which mass flows out through the surface of the volume.

Now, if mass of the fluid within this fixed surface is m, then

$$\frac{\partial m}{\partial t}=\frac{\partial}{\partial t}(\rho r^2\sin\theta\, dr\, d\theta\, d\phi)=r^2\sin\theta\, dr\, d\theta\, d\phi\,\frac{\partial\rho}{\partial t} \tag{1}$$

because volume does not vary with time. Further if R is the source rate of mass per unit volume, the fluid mass generated is

$$R\,dv = R\,r^2 \sin\theta\,dr\,d\theta\,d\varphi \tag{2}$$



Also, the rate at which mass flows *out* of the surface $= \int_S \rho\mathbf{q}\cdot d\mathbf{S}$ (3)

The above maxim now provides the mathematical formulation

Substitutions from (1), (2), (5) into (4) yield

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2 \sin \theta}\left[\sin \theta \frac{\partial}{\partial r}(\rho q_1 r^2)+r \frac{\partial}{\partial \theta}(\rho q_2 \sin \theta)+\frac{\partial}{\partial \varphi}(\rho q_3 r)\right]=R. \quad (6)$$

For the very special but important case, when $R=0$, we get

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho q_1 r^2)+\frac{1}{r \sin \theta}\frac{\partial}{\partial \theta}(\rho q_2 \sin \theta)+\frac{1}{r \sin \theta}\frac{\partial}{\partial \varphi}(\rho q_3)=0. \quad (7)$$

Ex. *Find the source-free equation of continuity of an inviscid fluid in cylindrical and spherical polar coordinates, by using the expression for div of a vector.*

Sol. (i) Let the components of the velocity vector $\mathbf{q}$ in the (r, θ, z)-directions be (q_1, q_2, q_3). The source-free equation of continuity is

$$\partial\rho/\partial t+\text{div}\,(\rho\mathbf{q})=0 \quad (1)$$

Setting $\mathbf{f}=\rho\mathbf{q}$ in the expression for $\text{div}\,\mathbf{f}=\frac{1}{r}\left[\frac{\partial}{\partial r}(rf_1)+\frac{\partial f_2}{\partial \theta}+\frac{\partial f_3}{\partial z}\right]$ and using (1) we obtain

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\left[\frac{\partial}{\partial r}(r\rho q_1)+\frac{\partial(\rho q_2)}{\partial \theta}+\frac{\partial}{\partial z}(\rho q_3)\right]=0. \quad (2)$$

(ii) Let the components of the velocity vector $\mathbf{q}$ in the (r, θ, φ)-directions be (q_1, q_2, q_3). Setting $\mathbf{f}=\rho\mathbf{q}$ in the expression (in spherical polar coordinates) for

$$\text{div}\,\mathbf{f}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 f_1)+\frac{1}{r \sin \theta}\frac{\partial}{\partial \theta}(\sin \theta f_2)+\frac{1}{r \sin \theta}\frac{\partial f_3}{\partial \varphi}$$

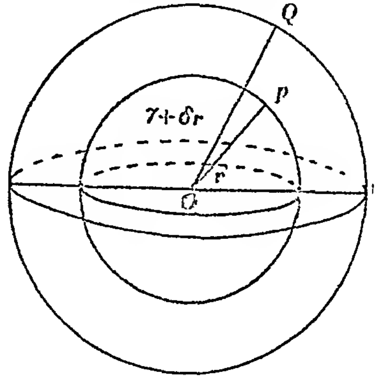
and using (1) we get

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho q_1)+\frac{1}{r \sin \theta}\frac{\partial}{\partial \theta}(\sin \theta \rho q_2)+\frac{1}{r \sin \theta}\frac{\partial \rho q_3}{\partial \varphi}. \quad (3)$$

1·76. Some symmetrical forms of the equation of continuity. When the motion possesses certain symmetrical properties, the equation of continuity can always be simplified. We shall consider three such useful cases.

(i) **Spherical symmetry.** Here motion is symmetrical about the centre and velocity $\mathbf{q}$ in the direction OP is a function of r and t only. Let us consider two concentric spheres. Mass gained by the flow through the inner surface is

$$4\pi r^2 q_r\, \rho=f(r, t) \quad \text{(say)}$$



Mass lost by the flow through the outer surface

$$=f(r+\delta r, t)$$

Mass between the two spheres at time t

$$=4\pi r^2 \rho.\delta r$$

Hence, by continuity-principle,

$$\frac{\partial \rho}{\partial t}\cdot 4\pi\, r^2.\delta r=f(r, t)-f(r+\delta r, t)=-\frac{\partial f}{\partial r}\delta r=-\delta r.\frac{\partial}{\partial r}(4\pi\, r^2\, q_r.\rho)$$

$$\therefore \quad \frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho\, q_r r^2)=0 \quad (1)$$

If ρ is constant, $\partial\rho/\partial t=0$, hence

$$r^2 q_r = F(t) = R^2 v^2 \quad \text{(say)}. \tag{2}$$

In a steady flow, $F(t)$ shall be an absolute constant.
NOTE. The relation (2) at once follows from §1.75(7), p. 50 by deleting the terms containing references to θ and φ. The above derivation is from first principles.

(ii) **Cylindrical symmetry.** Here the velocity **q** at any point P is perpendicular to a fixed axis OZ and is a function of t and r, where t is the time and r is the perpendicular distance of P from OZ.

Consider two cylinders of radii r, $r+\delta r$ with OZ as axis, bounded by barrier planes at a unit distance apart.

Rate of flow across the inner surface$=\rho q.\, 2\pi r = f(t, r)$ (say) (1)

Rate of flow across the outer surface$=f(t, r+\delta r)$ (2)

Rate of change of mass $=\dfrac{\partial}{\partial t}(\rho.2\pi r.\, \delta r)$ (3)

Hence, by the principle of continuity

$$\begin{aligned}(\partial\rho/\partial t)\, 2\pi r.\, \delta r &= f(t, r) - f(t, r+\delta r)\\ &= -\delta r\, \partial f/\partial r\\ &= -\delta r\, \partial(\rho q\, 2\pi r)/\partial r.\end{aligned}$$

Thus,
$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho q r)=0 \tag{4}$$

If ρ is constant, $\partial\rho/\partial t=0$ and hence

$$rq = F(t) = RV \qquad \text{(say)}. \tag{5}$$

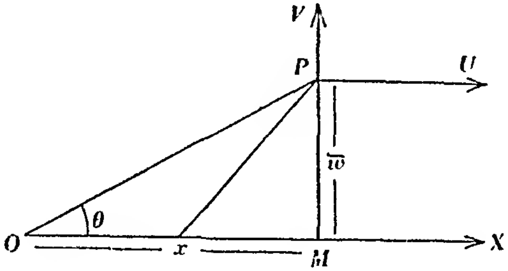
In a steady flow, $F(t)$ is an absolute constant.

NOTE. The relation (4) at once follows from §1·74(7), p. 48, by deleting the terms containing references to θ and z. The above derivation is from first principles.

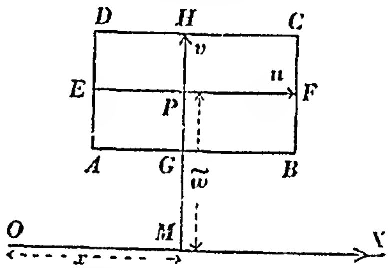
(iii) **Axial symmetry.*** *If the motion is such, that velocity at P is in the plane XOP and depends only on x, ϖ, t (Fig.) and motion is the*

* Axially symmetric flow is 3-dim. flow such that with a suitable choice of cylindrical polar coordinates (r, θ, z) every physical variable is independent of θ; or with a suitable choice of space polar coodinates (r, θ, φ), every physical variable is independent of φ.

same in all the planes passing through a fixed axis OX, it is said to possess an axial symmetry.



Consider a rectangle $ABCD$ in the plane XOP with P as centre and with $AB=\delta x$ parallel to OX and $AD=\delta\varpi$ perpendicular to OX. Let (u, v) be the components of velocity at P parallel and perpendicular to OX. Let EF, GH be the centre-lines of the rectangle as shown in the Fig. Then the area generated by EF on rotating about OX is



$$2\pi\,\varpi\,\delta x.$$

Area generated by GH on rotating about OX is

$$\pi\,(\varpi+\tfrac{1}{2}\,\delta\varpi)^2-\pi\,(\varpi-\tfrac{1}{2}\,\delta\varpi)^2=2\pi\,\varpi\,\delta\varpi$$

Rate at which mass crosses the area $2\pi\,\varpi\,\delta x$ from below to above

$$\rho v.\,2\pi\,\varpi\,\delta x=f_1\,(t,\,x,\,\varpi).\,\delta x \qquad \text{(say)}.$$

Rate at which mass crosses the area $2\pi\,\varpi\,\delta\varpi$ from left to right

$$\rho u\,2\pi\,\varpi\,\delta\varpi=f_2\,(t,x,\,\varpi).\,\delta\varpi \qquad \text{(say)}.$$

Rate of gain of mass through the curved surfaces of the ring formed by the rotation of the rectangle

$$=f_1\,(t,\,x,\,\varpi-\tfrac{1}{2}\,\delta\varpi)\,\delta x-f_1\,(t,\,x,\,\varpi+\tfrac{1}{2}\,\delta\varpi)\,\delta x$$

$$=-\delta x.\,\delta\varpi.\,\frac{\partial f_1}{\partial\varpi}$$

$$=-\delta x.\,\delta\varpi\,\frac{\partial}{\partial\varpi}\,(\rho v\;2\pi\,\varpi).$$

Rate of gain of mass through the plane faces of the ring

$$=f_2\,(t,\,x-\tfrac{1}{2}\,\delta x,\,\varpi)\,\delta\varpi-f_2\,(t,\,x+\tfrac{1}{2}\,\delta x,\,\varpi)\,\delta\varpi$$

$$=-\delta x.\,\delta\varpi\frac{\partial f_2}{\partial x}$$

$$=-\delta x.\,\delta\varpi\frac{\partial}{\partial x}(\rho u 2\pi\varpi)$$

The mass within the ring is $\rho.\,2\pi\varpi.\,\delta x.\,\delta\varpi$.

Hence *by conserving the mass* we get

$$\frac{\partial\rho}{\partial t}.\,2\pi\varpi\,\delta x.\,\delta\varpi+\delta x\,\delta\varpi\,\frac{\partial}{\partial x}(\rho\,u\,2\pi\varpi)+\delta x.\,\delta\varpi\,\frac{\partial}{\partial\varpi}(\rho v.\,2\pi\varpi)=0$$

or
$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\rho\,v\varpi)=0.$$

If ρ is constant, $\partial\rho/\partial t=0$, then the last equation can be written as

$$\varpi\frac{\partial u}{\partial x}+\frac{\partial}{\partial\varpi}(v\varpi)=0 \quad\text{or}\quad \frac{\partial}{\partial x}(u\varpi)+\frac{\partial}{\partial\varpi}(v\varpi)=0.$$

Ex. Homogeneous liquid moves so that the path of any particle P lies in the plane POX, where OX is fixed axis. Prove that if $OP=r$ and $\angle XOP=\theta$, the equation of continuity may be written as

$$\frac{\partial}{\partial r}(ur^2)-\frac{\partial}{\partial\mu}(vr\sin\theta)=0,$$

where u, v are the component velccities along and perpendicular to OP in the plane POX and $\mu=\cos\theta$. *(Del 1956)*

1·77. Method of writing the continuity-conditions. Let ρ be the density of the fluid and construct a parallelopiped whose edges are $\lambda d\alpha$, $\mu d\beta$, $\nu d\gamma$ in any coordinate system. Write

lengths of elements :	$\lambda d\alpha$	$\mu d\beta$	$\nu d\gamma$
components of velocity :	u	v	w

To calculate the flux along the first length, take the negative derivative with regard to the first length of the product

density × velocity in the first direction × product of remaining lengths

Then multiply this by the first length itself. Thus, the first flux is

$$-\frac{1}{\lambda}\,\frac{\partial}{\partial\alpha}(\rho u\,\mu d\beta.\,\nu d\gamma)$$

Now calculate the remaining fluxes similarly; add them up and equate this sum to the time derivative of the mass (i.e. $\rho\lambda d\alpha\,\mu d\beta\,\nu d\gamma$).

To be exact, let us write the continuity equations in the various systems already dealt with.

Cartesians :
$$\begin{cases} dx & dy & dz \\ u & v & w \end{cases}$$

$$-\frac{\partial}{\partial x}(\rho u dydz)\,dx-\frac{\partial}{\partial y}(\rho v dzdx)\,dy-\frac{\partial}{\partial z}(\rho w dxdy)\,dz=\frac{\partial}{\partial t}(\rho dxdydz)$$

or
$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)+\frac{\partial\rho}{\partial t}=0.$$

Cylindricals : $\begin{cases} dr & 0 & dz \\ u & v & w \end{cases}$

$$-\frac{\partial}{\partial r}(\rho u r d\theta dz)\,dr-\frac{\partial}{r\partial\theta}(\rho v dz dr)\,r d\theta-\frac{\partial}{\partial z}(\rho w dr r d\theta)\,dz=\frac{\partial}{\partial t}(\rho dr.\,r d\theta.\,dz),$$

or
$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho u r)+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0.$$

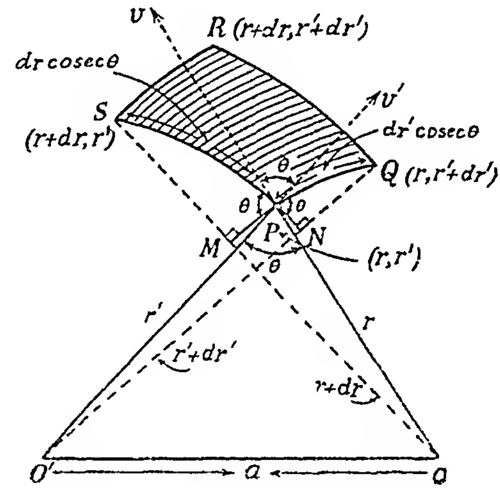
We can similarly write the continuity- condition in space polar coordinates.

1.78. Problems with solutions

(1) Equation of continuity in dipolar coordinates. *In the motion of a homogeneous liquid in two dimensions, the velocity at any point is given by two components v, v' along the directions which pass through the fixed points distant a from one another. Show that the equation of continuity is*

$$\frac{\partial v}{\partial r}+\frac{\partial v'}{\partial r'}+\frac{r^2+r'^2-a^2}{2r\,r'}\left(\frac{\partial v}{\partial r'}+\frac{\partial v'}{\partial r}\right)+\frac{v}{r}+\frac{v'}{r'}=0,$$

where r, r' are the distances of any point of the liquid from the fixed points. (*Del 1958, 36*)



Sol. The dipolar coordinates of P are (r, r') and we need construct an elementary parallelogram $PQRS$ with extremities at (r, r'), $(r, r'+dr')$, $(r+dr\ r'+dr')$ and $(r+dr, r')$ respectively. For this purpose, draw arcs of radii r, $r+dr$ and r', $r'+dr'$ with centres at O and O' $(OO'=a)$, their intersections giving the required figure.

We thus have the usual parallelogram with edges, and the velocities in these directions as :

edges :	dr' cosec θ	dr cosec θ
directions :	$v'+v\cos\theta$	$v+v'\cos\theta$

Then, the fluxes across the edges dr' cosec θ and dr cosec θ are respectively

$$-\frac{1}{\operatorname{cosec}\theta}\frac{\partial}{\partial r'}[\rho(v'+v\cos\theta)\,dr\operatorname{cosec}\theta]\,dr'\operatorname{cosec}\theta\ ;$$
$$-\frac{1}{\operatorname{cosec}\theta}\frac{\partial}{\partial r}[\rho(v+v'\cos\theta)\,dr'\operatorname{cosec}\theta]\,dr\operatorname{cosec}\theta.$$

Since the time-rate of change of mass is $\partial[\rho\, dr'\operatorname{cosec}\theta.\, dr\operatorname{cosec}\theta]/\partial t$, the continuity condition obtained by equating the sum of fluxes to time rate of mass, after some reduction, is

$$\operatorname{cosec}\theta\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial r}\left[\frac{\rho(v+v'\cos\theta)}{\sin\theta}\right]+\frac{\partial}{\partial r'}\left[\frac{\rho(v'+v\cos\theta)}{\sin\theta}\right]=0.$$

For incompressible fluids, $\partial\rho/\partial t=0$; hence this reduces to

$$\frac{\partial}{\partial r}\left(\frac{v+v'\cos\theta}{\sin\theta}\right)+\frac{\partial}{\partial r'}\left(\frac{v'+v\cos\theta}{\sin\theta}\right)=0.$$

Carrying out the indicated differentiations we obtain

$$\frac{1}{\sin\theta}\left[\frac{\partial v}{\partial r}+\frac{\partial v'}{\partial r'}+\left(\frac{\partial v'}{\partial r}+\frac{\partial v}{\partial r'}\right)\cos\theta+v'\,\frac{\partial\cos\theta}{\partial r}+v\,\frac{\partial\cos\theta}{\partial r'}\right]-$$

$$\cos\theta\left[\frac{v+v'\cos\theta}{\sin^2\theta}\,\frac{\partial\theta}{\partial r}+\frac{v'+v\cos\theta}{\sin^2\theta}\,\frac{\partial\theta}{\partial r'}\right]=0 \qquad (1)$$

Now $$\frac{\partial}{\partial r}(\cos\theta)=\frac{\partial}{\partial r}\left(\frac{r^2+r'^2-a^2}{2rr'}\right)=\frac{1}{2}\frac{\partial}{\partial r}\left(\frac{r}{r'}+\frac{r'}{r}-\frac{a^2}{rr'}\right)$$

$$=\tfrac{1}{2}\left(\frac{1}{r'}-\frac{r'}{r^2}+\frac{a^2}{r^2r'}\right)=\frac{r^2-r'^2+a^2}{2r'r^2}$$

$$=\frac{2r^2-(r^2+r'^2-a^2)}{2r^2r'}=\frac{1}{r'}-\frac{\cos\theta}{r}.$$

Similarly $\partial(\cos\theta)/\partial r'=(1/r)-(\cos\theta/r')$.

Further, $\partial(\cos\theta)/\partial r=-\sin\theta\,\partial\theta/\partial r$; $\partial(\cos\theta)/\partial r'=-\sin\theta\,\partial\theta/\partial r'$

Substitutions in (1) provide the result

$$\frac{\partial v}{\partial r}+\frac{\partial v'}{\partial r'}+\left(\frac{\partial v'}{\partial r}+\frac{\partial v}{\partial r'}\right)\cos\theta+v'\left(\frac{1}{r'}-\frac{\cos\theta}{r}\right)+v\left(\frac{1}{r}-\frac{\cos\theta}{r'}\right)+$$

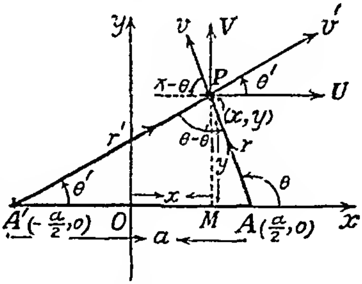
$$\left[\frac{v+v'\cos\theta}{\sin^2\theta}\left(\frac{1}{r'}-\frac{\cos\theta}{r}\right)+\frac{v'+v\cos\theta}{\sin^2\theta}\left(\frac{1}{r}-\frac{\cos\theta}{r'}\right)\right]\cos\theta=0$$

or $$\frac{\partial v}{\partial r}+\frac{\partial v'}{\partial r'}+\cos\theta\left(\frac{\partial v'}{\partial r'}+\frac{\partial v}{\partial r}\right)+\frac{v'}{r'}+\frac{v}{r}=0 \qquad (2)$$

Putting for $\cos\theta=(r^2+r'^2-a^2)/2rr'$, in (2), the result follows.

Aliter. The equation of continuity for a liquid in two dimensions is

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0 \qquad (1)$$



and we need transform it to the dipolar coordinates.

Now for any function $f=f(r,\ r',\ \theta,\ \theta')$ we always have

$$\frac{\partial f}{\partial h}=\frac{\partial f}{\partial r}\,\frac{\partial r}{\partial h}+\frac{\partial f}{\partial r'}\,\frac{\partial r'}{\partial h}+\frac{\partial f}{\partial\theta}\,\frac{\partial\theta}{\partial h}+\frac{\partial f}{\partial\theta'}\,\frac{\partial\theta'}{\partial h}$$

By setting $h=x$ and $h=y$ in turn, we get the operators

$$\left.\begin{aligned}\frac{\partial}{\partial x}&=\frac{\partial r}{\partial x}\,\frac{\partial}{\partial r}+\frac{\partial r'}{\partial x}\,\frac{\partial}{\partial r'}+\frac{\partial\theta}{\partial x}\,\frac{\partial}{\partial\theta}+\frac{\partial\theta'}{\partial x}\,\frac{\partial}{\partial\theta'}\\ \frac{\partial}{\partial y}&=\frac{\partial r}{\partial y}\,\frac{\partial}{\partial r}+\frac{\partial r'}{\partial y}\,\frac{\partial}{\partial r'}+\frac{\partial\theta}{\partial y}\,\frac{\partial}{\partial\theta}+\frac{\partial\theta'}{\partial y}\,\frac{\partial}{\partial\theta'}\end{aligned}\right\} \qquad (\alpha)$$

Now with reference to the adjacent figure

$r^2=(x-\tfrac{1}{2}a)^2+y^2$, $r'^2=(x+\tfrac{1}{2}a)^2+y^2$, $\theta=\tan^{-1}[y/(x-\tfrac{1}{2}a)]$, $\theta'=\tan^{-1}[y/(x+\tfrac{1}{2}a)$

$$\therefore\quad \frac{\partial r}{\partial x}=\frac{x-\frac{1}{2}a}{r},\quad \frac{\partial r'}{\partial x}=\frac{x+\frac{1}{2}a}{r'},\quad \frac{\partial r}{\partial y}=\frac{y}{r},\quad \frac{\partial r'}{\partial y}=\frac{y}{r'}$$

$$\frac{\partial\theta}{\partial x}=\frac{-y}{y^2+(x-\frac{1}{2}a)^2}=\frac{-y}{r^2},\quad \frac{\partial\theta'}{\partial x}=\frac{-y}{r'^2}\ ;\ \frac{\partial\theta}{\partial y}=\frac{x-\frac{1}{2}a}{r^2},\quad \frac{\partial\theta'}{\partial y}=\frac{x+\frac{1}{2}a}{r^2}.$$

Substitutions in (α) yield the results

$$\left.\begin{aligned}\frac{\partial}{\partial x}&=\cos\theta\,\frac{\partial}{\partial r}+\cos\theta'\frac{\partial}{\partial r'}-\frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta}-\frac{\sin\theta'}{r'}\,\frac{\partial}{\partial\theta'}\\ \frac{\partial}{\partial y}&=\sin\theta\,\frac{\partial}{\partial r}+\sin\theta'\,\frac{\partial}{\partial r'}+\frac{\cos\theta}{r}\,\frac{\partial}{\partial\theta}+\frac{\cos\theta'}{r'}\,\frac{\partial}{\partial\theta'}\end{aligned}\right\}\qquad(\beta)$$

Putting $U=v\cos\theta+v'\cos\theta'$, $V=v\sin\theta+v'\sin\theta'$ in (1) and applying the operators in (β) to (1) we get

$$\left(\cos\theta\,\frac{\partial}{\partial r}+\cos\theta'\frac{\partial}{\partial r'}-\frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta}-\frac{\sin\theta'}{r'}\,\frac{\partial}{\partial\theta'}\right)(v\cos\theta+v'\cos\theta')+$$
$$\left(\sin\theta\,\frac{\partial}{\partial r}+\sin\theta'\,\frac{\partial}{\partial r'}+\frac{\cos\theta}{r}\,\frac{\partial}{\partial\theta}+\frac{\cos\theta'}{r'}\,\frac{\partial}{\partial\theta'}\right)(v\sin\theta+v'\sin\theta')=0\qquad(2)$$

These give, after collecting terms

$$\frac{\partial v}{\partial r}+\frac{\partial v'}{\partial r'}+(\cos\theta\cos\theta'+\sin\theta\,\sin\theta')\left(\frac{\partial v'}{\partial r}+\frac{\partial v}{\partial r'}\right)+\frac{v}{r}+\frac{v'}{r'}=0\qquad(3)$$

because $\partial v'/\partial\theta'$, $\partial v/\partial\theta$, $\partial v/\partial\theta'$, $\partial v'/\partial\theta$ all vanish separately; as v and v' are functions of r and r' only. Also,

$$\cos\theta\cos\theta'+\sin\theta\sin\theta'=\cos(\theta-\theta')=(r^2+r'^2-a^2)/2rr'.$$

Substitution in (3) yields

$$\frac{\partial v}{\partial r}+\frac{\partial v'}{\partial r'}+\frac{r^2+r'^2-a^2}{2rr'}\left(\frac{\partial v'}{\partial r}+\frac{\partial v}{\partial r'}\right)+\frac{v}{r}+\frac{v'}{r'}=0.$$

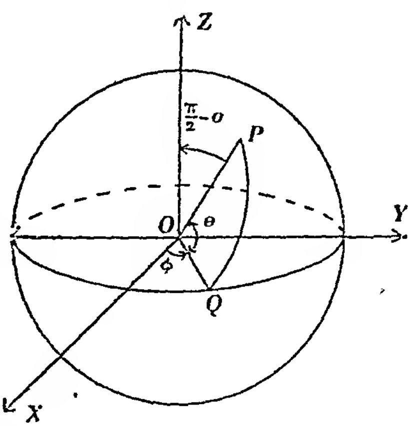
(2) *If every particle moves on the surface of a sphere, prove that the equation of continuity is*

$$\frac{\partial\rho}{\partial t}\cos\theta+\frac{\partial}{\partial\theta}(\rho\omega\cos\theta)+\frac{\partial}{\partial\phi}(\rho\omega'\cos\theta)=0$$

ρ being the density θ, ϕ the latitude and longitude of any element ond ω, ω' the angular velocities of the element in latitude and longitude respectively.

[Ag 1965, 56, 53 ; Ban 47 ; Bom 57 ; Del 64, 55, 42, 34 ; Jab 59 ; Jad 58 ; Kr 59 ; Mad 59 ; Osm 63, 61 ; Pb 56 ; Sag 55]

Sol. By definitions, $\angle QOP=\theta$ (latitute), $\angle XOQ=\phi$ (longitude). In order to confirm with the space polar coordinates, we write $90-\theta=\theta'$ and hence, in this system, the coordinates of any point P are (r,θ',ϕ). Obviously, when θ' increases, θ decreases. Since r is constant for P, it being a point on the surface of sphere, $q_r=0=u$. Let us construct the usual arallelopiped with edges, and the velocities in those directions as:



$$\left.\begin{array}{llll}\text{edges}: & dr & r\,d\theta' & r\sin\theta'd\phi\\ \text{velocities}: & 0 & r\omega & r\sin\theta'\omega'\end{array}\right\}(1)$$

Then the flux across the face of area $rd\theta'.r\sin\theta'd\phi$ is zero; those of the faces of areas $dr\,r\sin\theta'd\phi$ and $r\,drd\theta'$ are respectively

$$\frac{1}{r}\,\frac{\partial}{\partial\theta'}(\rho r\omega\,dr.r\sin\theta'd\phi)\,rd\theta',$$

$$\frac{1}{r\sin\theta'}\,\frac{\partial}{\partial\phi}(\rho\,r\sin\theta'\omega'.dr.r\,d\theta')\times r\sin\theta'd\phi.$$

Since the time-rate of change of mass is $\partial(\rho\,dr.\,rd\theta'.\,r\sin\theta'\,d\phi)/\partial t$, the

continuity-condition obtained by equating the sum of fluxes to time-rate of mass, after some reduction, is

$$\frac{\partial \rho}{\partial t} \sin \theta' + \frac{\partial}{\partial \theta'} (\rho\omega \sin \theta') + \frac{\partial}{\partial \varphi} (\rho\omega' \sin \theta') = 0 \tag{2}$$

Since $\theta' = 90 - \theta$; $\partial\theta' = -\partial\theta$ and also velocity shall be $-r\omega$ in the direction of θ, contrary to $+r\omega$ in the θ'-direction since θ and θ' are oppositely oriented. Making the changes in (2) we get

$$\frac{\partial \rho}{\partial t} \cos \theta + \frac{\partial}{\partial \theta} (\rho\omega \cos \theta) + \frac{\partial}{\partial \varphi} (\rho\omega' \cos \theta) = 0.$$

Note that $\mathbf{q} = \boldsymbol{\omega} \times \mathbf{r}$ is utilized in (1), in order to write the linear velocities from those of angular velocities.

(3) *If the lines of motion are curves on the surfaces of spheres all touching the plane of xy at the origin O, the equation of continuity is*

$$r \sin \theta \frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial \varphi} + \sin \theta \frac{\partial (\rho u)}{\partial \theta} + \rho u (1 + 2 \cos \theta) = 0$$

where r is the radius CP of one of the spheres, θ the angle PCO, u the velocity in the plane PCO, v the perpendicular velocity, and φ the inclination of the plane PCO to a fixed plane through the axis of z.

[*Ag 1961, 57 ; Ban 64, 54 ; Bom 64 (Old) ; Del 62, 53, 45, 38 ; Pb 50 ; Raj 64*]

Sol. Let us consider any two consecutive spheres, with their centres C, C' on z-axis, distant δr apart. If the join of C to P (on one sphere) meets the second sphere in Q, then if

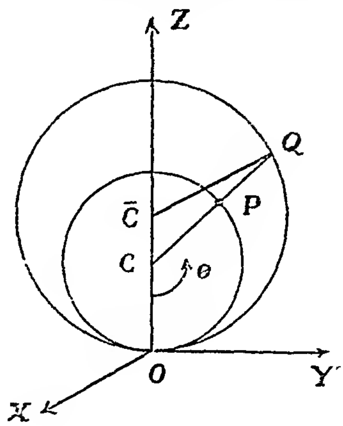
$$CP = r,\ C'Q = r + \delta r,\ CC' = \delta r,\ \underline{|OCP} = \theta,\ PQ = l\ ;$$

then $$(r + \delta r)^2 = (\delta r)^2 + (r + l)^2 + 2\delta r (r + l) \cos \theta \quad \text{(by Cosine formula).}$$

This gives $$r(1 - \cos \theta)\, \delta r = (r + \delta r \cos \theta)\, l \doteq lr$$

or $$l \doteq r(1 - \cos \theta)\, \delta r.$$

Since the lines of motion lies on the surface of spheres there will be no component of velocity along PQ. Further, the angle between the planes PCO and XOZ (fixed plane) is φ, and this is the usual coordinate in space polar coordinates. We can therefore construct an elementary parallelopiped with edges, and velocities in those directions as :



$$\left.\begin{array}{llll} \text{edges :} & (1 - \cos \theta)\, dr & r d\theta & r \sin \theta d\varphi \\ \text{velocities :} & 0 & u & v \end{array}\right\} \quad (1)$$

The flux across the face of area $r d\theta\ r \sin \theta\, d\varphi$ is zero ; those of the faces of areas $r \sin \theta\, d\varphi.\ (1 - \cos \theta)\, dr$ and $(1 - \cos \theta)\ dr . r d\theta$ are respectively

$$-\frac{1}{r} \frac{\partial}{\partial \theta} [\rho u\, r \sin \theta d\varphi . (1 - \cos \theta)\, dr]\, r d\theta\ ;$$

$$-\frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} [\rho\, v\, r d\theta . (1 - \cos \theta)\, dr]\, r \sin \theta d\varphi.$$

Since the time-rate of change of mass is $\partial[\rho(1 - \cos \theta)\, dr . r d\theta . r \sin \theta d\varphi]/\partial t$. the continuity condition obtained by equating the sum of fluxes to time-rate of mass, after some reduction, is

$$r \sin \theta\, (1 - \cos \theta) \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \theta} [\rho u \sin \theta (1 - \cos \theta)] + (1 - \cos \theta) \frac{\partial}{\partial \varphi} (\rho v) = 0$$

or $$r \sin\theta \frac{\partial \rho}{\partial t} + \sin\theta \frac{\partial(\rho u)}{\partial \theta} + \rho u\,(1+2\cos\theta) + \frac{\partial}{\partial \varphi}(\rho v) = 0.$$

(4) *A thin stratum of incompressible fluid is contained between two concentric spheres. Show that the velocity at any point is equivalent to the components*

$$-\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\omega};\ \frac{\partial\psi}{\partial\theta}$$

along the meridian and parallel of latitude respectively. Also if the fluid be homogeneous, and the motion irrotational, prove that

$$\frac{\partial\phi}{\partial\theta} = \frac{1}{\sin\theta}\frac{\partial\psi}{\partial\omega};\ \frac{\partial\psi}{\partial\theta} = -\frac{1}{\sin\theta}\frac{\partial\phi}{\partial\omega}$$

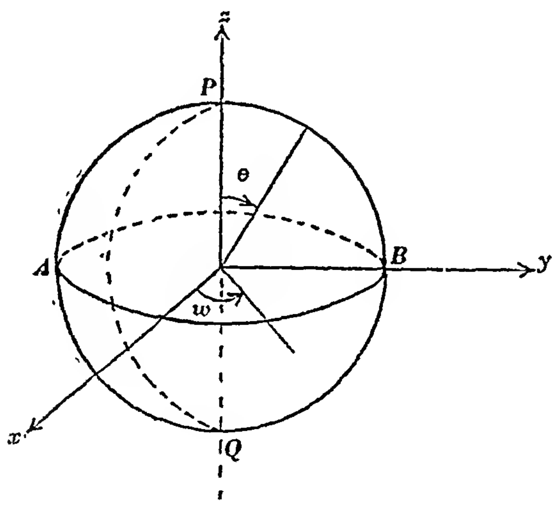
and deduce that

$$\phi + i\psi = F\,(e^{i\omega}\tan\theta/2)$$

[*Jad 1959*; *Del 56*]

Sol. PQ (Fig.) is the vertical axis of rotation and *any great circle drawn with PQ as diameter is known as a meridian.* Obviously the velocity increasing in the direction of θ is that which corresponds to that along the meridian. Similarly the velocity in the direction of w increasing corresponds to that along the parallel AB.

Since the fluid stratum is thin, it follows that the liquid ultimately is on the sphere $r=a$ (say), so that $q_r=0$; $q_\theta=u$; $q_\omega=v$ (say).



The equation of continuity in the present case, taking ρ as constant, is given by

$$\frac{\partial}{\partial\theta}(u\sin\theta) + \frac{\partial v}{\partial\omega} = 0. \tag{1}$$

The stream lines are given by

$$\frac{a\,d\theta}{u} = \frac{a\sin\theta\, d\omega}{v}; \quad \text{or} \quad -u\sin\theta\, d\omega + v\,d\theta = 0.$$

This equation is exact by virtue of (1), so that it forms an exact differential $d\psi$ (say).

$$\therefore \qquad -u \sin \theta \, d\omega + v d\theta = d\psi = \frac{\partial \psi}{\partial \theta} d\theta + \frac{\partial \psi}{\partial \omega} d\omega$$

whence
$$u = -\frac{1}{\sin \theta} \frac{\partial \psi}{\partial \omega}; \; v = \frac{\partial \psi}{\partial \theta}.$$

Now, when the motion is irrotational, the velocity potential ϕ necessarily exists. Thus if the sphere is of unit radius (or includes its radius a in ϕ), then

$$-\frac{\partial \phi}{\partial \theta} = u = -\frac{1}{\sin \theta} \frac{\partial \psi}{\partial \omega}; \; \frac{1}{\sin \theta} \frac{\partial \phi}{\partial \omega} = -\frac{\partial \psi}{\partial \theta} = -v$$

which lead to

$$\frac{\partial \phi}{\partial \theta} = \frac{1}{\sin \theta} \frac{\partial \psi}{\partial \omega} \text{ and } \frac{\partial \psi}{\partial \theta} = -\frac{1}{\sin \theta} \frac{\partial \phi}{\partial \omega}. \tag{2}$$

From (2) we get

$$\frac{\partial}{\partial \theta}(\phi + i\psi) = \frac{1}{\sin \theta} \frac{\partial}{\partial \omega}(\psi - i\phi) = \frac{-i}{\sin \theta} \frac{\partial(\phi + i\psi)}{\partial \omega}$$

Setting $\phi + i\psi = W$, the above reduces to

$$\sin \theta \frac{\partial W}{\partial \theta} + i \frac{\partial W}{\partial \omega} = 0. \tag{3}$$

This may be compared with Lagrange's partial differential equation $P(\partial z/\partial x) + Q(\partial z/\partial y) = R$ whose solutions are $dx/P = dy/Q = dz/R$. Hence (3) yields

$$\text{cosec } \theta d\theta = -i d\omega = dW/0.$$

The first two members give $e^{i\omega} \tan \theta/2 = \text{const.} = F(A)$ say, and the last gives $W = A$. Combining we get

$$W = \phi + i\psi = F(e^{i\omega} \tan \theta/2).$$

Problems For Solutions

1. Each particle of a mass of liquid moves in a plane through the axis of z. Find the equation of continuity. *[Sag 1954]*

2. Fluid is moving in a tube of variable cross-section A; prove that, if v is the velocity at a point s and ρ is the density, the equation of continuity is

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial s}(\rho A v) = 0.$$ *[Osm 1960]*

3. A mass of fluid moves in such a way that each particle describes a circle in one plane about a fixed axis. Show that the equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho \omega)}{\partial \theta} = 0,$$

where ω is the angular velocity of a particle whose azimuthal angle is θ at time t.

4. The particles of a fluid move symmetrically in space with regard to a fixed centre (i.e. radially). Prove that the equation of continuity is

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + \frac{\rho}{r^2} \frac{\partial}{\partial r}(r^2 u) = 0,$$

where u is the velocity at distance r. *[Jab 1960; Osm 62]*

5. A mass of fluid is in motion so that the lines of motion lie on the surface

of coaxial cylinders. Show that the equation of continuity is

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho u)+\frac{\partial}{\partial z}(\rho v)=0,$$

where u, v are the velocities perpendicular and parallel to z.

6. If the lines of motion are curves on the surface of cones having their vertices at the origin, and the z-axis as the common axis, show that the equation of continuity is

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial r}(\rho u)+\frac{2\rho u}{r}+\frac{\operatorname{cosec}\theta}{r}\frac{\partial}{\partial \varphi}(\rho v)=0$$

where u and v denote the velocity components in directions of r and φ respectively. [*Ag 1960, 58, 54, 45 ; Del 49 ; Mar 62*]

[The result follows from continuity-condition in space polar coordinates by putting the velocity component in the θ-direction equal to zero ; because lines of motion lie on cone θ=const.]

7. If the motion of a liquid be in two dimensions, prove that, if at any instant the velocity be everywhere the same in magnitude, it is so in direction.

8. In the steady motion of homogeneous liquid if the surfaces $f_1=a_1$, $f_2=a_2$ define the stream lines, prove that the most general values of the velocity components u, v, w are

$$F(f_1,f_2)\frac{\partial(f_1,f_2)}{\partial(y,z)},\ F(f_1,f_2)\frac{\partial(f_1,f_2)}{\partial(z,x)},\ F(f_1,f_2)\frac{\partial(f_1,f_2)}{\partial(x,y)},$$

where F is any arbitrary function. [*Cal 1955 ; Del 60 ; Gu 58, 53 ; Pb 61, 54*]

9. A pipe of circular cross-section and length l has a diameter which decreases uniformly with the distance from either end, being a minimum at its mid-point. Homogeneous liquid flows steadily through the pipe, v being its velocity of influx into one end. If λ be the ratio of the smallest to the largest cross-sectional diameter respectively, show that the time taken for a liquid particle to traverse the pipe is

$$l(1+\lambda+\lambda^2)/3v.$$

10. Show that $\phi=(x-t)(y-t)$ represents the velocity potential of an incompressible two-dimensional fluid. Show that the stream lines at time t are the curves

$$(x-t)^2-(y-t)^2=\text{const.},$$

and that the paths of the fluid particles have the equations

$$\log|x-y|=\tfrac{1}{2}\{(x+y)-a(x-y)^{-1}\}+b$$

where a, b are constants. [*Kuru 1965*]

11. The position of a point in a plane is determined by the length r of the tangent from it to a fixed circle of radius a, and the inclination θ of the tangent to a fixed line. Show that the continuity-condition for a liquid moving irrotationally in the plane will be

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{a^2}{r^2}\left(\frac{\partial^2\phi}{\partial r^2}-\frac{1}{r}\frac{\partial\phi}{\partial r}\right)+\frac{a}{r^2}\left(2\frac{\partial^2\phi}{\partial r\partial\theta}-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)=0.$$

12. If the velocity potential of a liquid is of the form $\phi=f(r)\,g(\theta)\,h(z)$, where (r,θ,z) are the cylindrical coordinates, prove that the equation of continuity is satisfied if f, g, h satisfy the three equations

$$r^2\frac{\partial^2 f}{\partial r^2}+r\frac{\partial f}{\partial r}+(a^2r^2-n^2)f=0,\ \frac{\partial^2 g}{\partial\theta^2}+n^2g=0;\ \frac{\partial^2 h}{\partial z^2}-a^2h=0$$

where n and a are constants, and hence show that

$$\phi=\Sigma A\cosh a(z-c)\cos n(\theta-\alpha)\int_0^{\pi}\cos(aR\sin\omega-n\omega)\,d\omega.$$

1·79. Kinematically possible incompressible fluid motion

If the velocity vector $\mathbf{q}=(u, v, w)$ is kinematically possible for an incompressible fluid motion, then the equation of continuity must be satisfied. If in addition the motion is irrotational, then curl $\mathbf{q}=0$, or in cartesian form

$$\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}=0\,;\ \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}=0\,;\ \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0. \tag{1}$$

Evidently, in such a case the velocity potential ϕ necessarily exists and is given by $\mathbf{q}=-\text{grad}\,\phi$,

or
$$u=-\frac{\partial \phi}{\partial x},\ v=-\frac{\partial \phi}{\partial y},\ w=-\frac{\partial \phi}{\partial z}.$$

In case the equations (1) are not satisfied, i.e. curl $\mathbf{q}\neq 0$, then the motion is vortical (rotational) and velocity potential cannot exist.

The stream lines, if needed, are easily obtained by solving the differential equations

$$\mathbf{q}\times d\mathbf{r}=0 \ \text{ or } \ \frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}.$$

The following examples will make the procedures very clear.

1. *Show that* $$u=-\frac{2xyz}{(x^2+y^2)^2}\,;\ v=\frac{(x^2-y^2)z}{(x^2+y^2)^2}\,;\ w=\frac{y}{(x^2+y^2)}$$

are the velocity components of a possible liquid motion. Is this motion irrotational?
[*Alig 1957 ; Del 57 ; Jab 62 ; Lkn 45, Osm 61 ; Pb 63*]

Sol. $$\frac{\partial u}{\partial x}=2yz\,\frac{(3x^2-y^2)}{(x^2+y^2)^3}\,;\ \frac{\partial v}{\partial y}=2yz\,\frac{(y^2-3x^2)}{(x^2+y^2)^3}\,;\ \frac{\partial w}{\partial z}=0.$$

The equation of continuity for the incompressible fluid is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.$$

Substitutions lead to

$$\frac{2yz(3x^2-y^2)}{(x^2+y^2)^3}+\frac{2yz(y^2-3x^2)}{(x^2+y^2)^3}+0=0$$

which is satisfied.

For the motion to be possible it is evidently necessary that the equation of continuity should be satisfied.

For irrotational motion

$$\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}=0\,;\ \frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}=0\,;\ \frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=0.$$

Here $$\frac{\partial v}{\partial z}=\frac{x^2-y^2}{(x^2+y^2)^2}\,;\ \frac{\partial w}{\partial y}=\frac{x^2-y^2}{(x^2+y^2)^2},\ \text{etc.}$$

Thus all the three equations referred to above are satisfied. Hence the motion is irrotational.

2. *Prove that the liquid motion is possible when velocity at (x, y, z) is given by*

$$u=(3x^2-r^2)/r^5,\ v=3xy/r^5,\ w=3xz/r^5,$$

where $r^2=x^2+y^2+z^2$, and the stream lines are the intersection of the surfaces

$$(x^2+y^2+z^2)^3=c(y^2+z^2)^2$$

by the planes passing through OX. State if the motion is irrotational giving reasons for your answer. [*Alig 1961 ; Ban 61 ; Bom 61 ; Del 53 ; Gor 61*]

Sol. For the motion to be possible, it is evidently necessary that the equation of continuity should be satisfied. The equation of continuity for the incompressible fluid is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \tag{1}$$

Here $\dfrac{\partial u}{\partial x}=\dfrac{3x(3r^2-5x^2)}{r^7}$; $\dfrac{\partial v}{\partial y}=\dfrac{3x(r^2-5y^2)}{r^7}$; $\dfrac{\partial w}{\partial z}=\dfrac{3x(r^2-5z^2)}{r^7}$.

Substitutions in (1) show that continuity equation is definitely satisfied.

For the equation of stream lines,

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w},$$

we have

$$\frac{dx}{3x^2-r^2}=\frac{dy}{3xy}=\frac{dz}{3zx}. \tag{2}$$

From the last two members, we get

$$y=az, \tag{3}$$

which is a plane passing through OX. We write (2) as

$$\frac{xdx+ydy+zdz}{x(3r^2-r^2)}=\frac{ydy+zdz}{3x(y^2+z^2)}$$

or

$$\frac{3\Sigma(2xdx)}{\Sigma x^2}=\frac{2(2y\,dy+2z\,dz)}{(y^2+z^2)}$$

Integration yields

$$3\log(x^2+y^2+z^2)=2\log(y^2+z^2)+\log c$$

or

$$(x^2+y^2+z^2)^3=c(y^2+z^2)^2,$$

the stream lines are intersections of (3) and (4).

We shall now prove that curl $\mathbf{q}=0$. Easily we find

$$\frac{\partial u}{\partial y}=\frac{-3y(5x^2-r^2)}{r^7},\quad \frac{\partial u}{\partial z}=\frac{-3z(5x^2-r^2)}{r^7},\quad \frac{\partial v}{\partial x}=\frac{3y(r^2-5x^2)}{r^7}$$

$$\frac{\partial v}{\partial z}=\frac{-15xyz}{r^7},\quad \frac{\partial w}{\partial x}=\frac{3z(r^2-5x^2)}{r^7},\quad \frac{\partial w}{\partial y}=\frac{-15xyz}{r^7}.$$

Evidently the conditions of irrotational motion :

$$\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}=0\ ;\ \frac{\partial v}{\partial z}-\frac{\partial w}{\partial x}=0\ ;\ \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0 \text{ are all satisfied.}$$

NOTE. Another expression for stream lines is :

$$\frac{dx}{3x^2-r^2}=\frac{dy}{3xy}=\frac{dz}{3zx}=\frac{xdx+ydy+zdz}{3xr^2-xr^2}=\frac{rdr}{2xr^2}=\frac{dr}{2xr}$$

Equating the second and third expressions gives $y=az$ while second and last gives

$$r^3=by^2 \quad\text{or}\quad (x^2+y^2+z^2)^3=b^2y^4.$$

Ex. 1. Show that the following motions are kinematically possible for an incompressible fluid (c is constant).

(i) $u=cx,\ v=-cy,\ w=0$; (ii) $u=cx,\ v=-cy,\ w=cx$

(iii) $u=cx,\ v=cy,\ w=-2cz$; (iv) $u=cx/r^3,\ v=cy/r^3,\ w=cz/r^3\ [r^2=x^2+y^2+z^2]$

(v) $u=\dfrac{cx}{(x^2+y^2)},\ v=\dfrac{cy}{(x^2+y^2)},\ w=0.$ (vi) $u=yzt,\ v=zxt,\ w=xyt.$

Ex. 2. Show that the following motions are kinematically impossible for an incompressible fluid (c is constant).

(i) $u=cz,\ v=-cy,\ w=-2cz,$ (ii) $u=cx,\ v=cy,\ w=cz$

Ex. 3. If the velocity of an incompressible fluid at the point (x, y, z) is given by

$$[3xz/r^5,\ 3yz/r^5,\ (3z^2-r^2)/r^5]$$

prove that the liquid motion is possible and that the velocity potential is $\cos\theta/r^2$. Find also the stream lines. [*Ban 1965*; *Del 65*; *Pb 59*].

HINT:
$$-d\phi=\mathbf{q}.d\mathbf{r}=-\frac{3z(\Sigma x dx)}{(\Sigma x^2)^{5/2}}=\frac{dz}{(\Sigma x^2)^{3/2}}$$

Integrating we get

$$-\phi=-\frac{z}{(\Sigma x^2)^{3/2}}+\int\frac{dz}{(x^2+y^2+z^2)^{3/2}}-\int\frac{dz}{(\Sigma x^2)^{3/2}}$$

$$\phi=z/(x^2+y^2+z^2)^{3/2}=\cos\theta/r^2.$$

Ex. 4₁. A two-dimensional fluid motion is specified in the Lagrangean manner by the equations

$$x=ae^{kt},\ y=be^{-kt}.$$

Show that the path of a particle is $xy=ab$; the velocities in the Eulerian form are $u=kx,\ v=-ky$; the motion is steady as well as kinematically possible for an incompressible fluid.

Ex. 4₂. A two-dimensional fluid motion is specified in the Lagrangean manner by the equations

$$x=ae^{kt},\ \ y=be^{-kt}+a(1-e^{-kt}).$$

Show that the path of a particle is $xy=ab+ax-a^2$; the velocities in the Eulerian form are $u=kx,\ v=-ky+kx\,e^{-kt}$; the motion is non-steady and is kinematically possible for an incompressible fluid.

[HINT: Equation of continuity in the Lagrangean form, viz $\rho J=\rho_0$ may be made use of, for proving that displacements are kinematically possible for an incompressible fluid.]

5. Show that the fluid motion given by the velocity fluid

$$\mathbf{q}=\{b(a^3r^{-3}-1)\cos\theta,\ b(1+\tfrac{1}{2}a^3r^{-3})\sin\theta\}$$

is irrotational, where $(r,\ \theta)$ are polar coordinates and $a,\ b$ are constants. Determine the velocity potential ϕ for this motion.

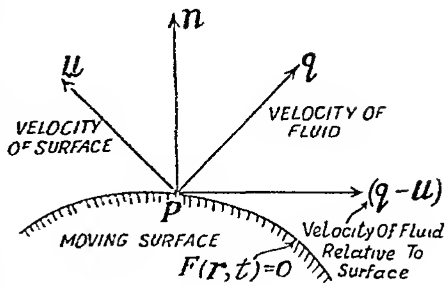
[curl $\mathbf{q}=0$, $\mathbf{q}=-\nabla\phi=-(\partial\phi/\partial r,\ r^{-1}\,\partial\phi/\partial\theta,\ \partial\phi/\partial z)$. Integrate and observe that arbitrary functions $f_1(\theta), f_2(r)$ vanish by comparison.]

1·80. Boundary surface. We now derive the *differential equation satisfied by a boundary surface of a fluid.* This is precisely to find the *condition that the surface* $F(x,\ y,\ z,\ t)=0$ *may be a boundary surface.*

When the fluid is in contact with a bounding surface,

$$F(x, y, z, t)=0 \text{ or } F(\mathbf{r}, t)=0 \tag{1}$$

the velocity of a fluid particle at any point of the boundary relative to the surface must be tangential to the boundary. Thus the normal component of the velocity of the particle relative to the boundary is zero.



Let u be the velocity of the point P of the surface, q the fluid velocity, and n the normal unit vector drawn at the point P of the surface so that

$$(\mathbf{q}-\mathbf{u}) . \mathbf{n}=0 \text{ or } (\mathbf{q}-\mathbf{u}) . \nabla F=0 \tag{2}$$

Since the surface is in motion, its position at time $t+\delta t$ is given by

$$F(\mathbf{r}+\delta\mathbf{r}, t+\delta t)=0 \tag{3}$$

Adding and subtracting $F(\mathbf{r}, t+\delta t)$ to (3) and using (1) we get

$$[F(\mathbf{r}+\delta\mathbf{r}, t+\delta t)-F(\mathbf{r}, t+\delta t)]+[F(\mathbf{r}, t+\delta t)-F(\mathbf{r}, t)]=0$$

or $\quad \delta\mathbf{r} . \nabla F(\mathbf{r}, t+\delta t)+(\partial F/\partial t)\delta t=0 \quad$ (by Taylor's theorem)

Proceeding to the limits as $\delta\mathbf{r}\to 0$, $\delta t\to 0$, this yields

$$\mathbf{u} . \nabla F(\mathbf{r}, t)+(\partial F/\partial t)=0 \qquad (\because \ \mathbf{u}=d\mathbf{r}/dt) \tag{4}$$

From (2) and (4) follows the result

$$(\mathbf{q} . \nabla F)+(\partial F/\partial t)=0, \text{ or } dF/dt=0. \tag{5}$$

The component form is

$$u(\partial F/\partial x)+v(\partial F/\partial y)+w(\partial F/\partial z)+(\partial F/\partial t)=0. \tag{5'}$$

Cor. When the liquid is in contact with a *rigid* surface $F(\mathbf{r})=0$, in order that the contact is preserved, the fluid and the surface must have the same velocity normal to the surface. Hence the condition is

$$\mathbf{q} . \nabla F=0.$$

NOTES: 1. The boundary condition, viz.

$$(\partial F/\partial t)+u(\partial F/\partial x)+v(\partial F/\partial y)+w(\partial F/\partial z)=0$$

is Lagrange's linear equation and as such its solution is given by

$$\frac{dt}{1}=\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w} \text{ or } \frac{dx}{dt}=u, \frac{dy}{dt}=v, \frac{dz}{dt}=w.$$

But these equations define the path lines $\mathbf{r}=f(\mathbf{r}_0, t)$ and hence, once a particle is in the surface, it never leaves it.

2. The velocity of the moving surface $F(\mathbf{r}, t)=0$, normal to itself is

$$-(\partial F/\partial t)/|\nabla F|.$$

This follows at once for

$$\mathbf{q} . \nabla F=-\partial F/\partial t ; \ \mathbf{q} . (\nabla F/|\nabla F|)=-(\partial F/\partial t)/|\nabla F|,$$

and this is nothing but the velocity of the surface normal to it, equal to U_n.

xp. *Find the differential equation of all the boundary surfaces of a fluid.*

Show that $$\frac{x^2}{a^2}\tan^2 t+\frac{y^2}{b^2}\cot^2 t=1$$

is a possible form for the bounding surface of a liquid, and find an expression for the normal velocity.

[*Ag 1964, 59, 55, 46 ; Del 61, 50 ; Jab 61 ; Mar 62 ; Osm 59 ; Pb 56, 52, 50 ; Raj 60*]

Sol. The surface $F(x, y, z)=x^2\tan^2 t/a^2+y^2\cot^2 t/b^2-1=0$ can be a *possible form* for the bounding surface of a liquid, if it satisfies the boundary condition

$$(\partial F/\partial t)+u(\partial F/\partial x)+v(\partial F/\partial y)+w(\partial F/\partial z)=0. \tag{1}$$

Now $\partial F/\partial x=2x\tan^2 t/a^2,\ \partial F/\partial y=2y\cot^2 t/b^2,\ \partial F/\partial z=0$

Thus $$|\nabla F|=2(b^4x^2\tan^4 t+a^4y^2\cot^4 t)^{\frac{1}{2}}/a^2b^2, \tag{2}$$

Also $\partial F/\partial t=2a^{-2}x^2\tan t\sec^2 t-2b^{-2}y^2\cot t\operatorname{cosec}^2 t$

Substitutions in (1) lead to

$$a^{-2}x(u\tan t+x\sec^2 t)\tan t+b^{-2}y(v\cot t-y\operatorname{cosec}^2 t)\cot t=0 \tag{3}$$

If we accept

$$u=-x\sec^2 t\cot t\ ;\ v=y\operatorname{cosec}^2 t\tan t\ ;\ w=w(x, y, t) \tag{4}$$

(3) is obviously satisfied. Also satisfied is the equation of continuity $\partial u/\partial x+\partial v/\partial y+\partial w/\partial z=0$, by velocity components given in (4). Hence the given surface is a possible form for the bounding surface of a liquid.

$$\text{normal velocity}=-\frac{(\partial F/\partial t)}{|\nabla F|}=\frac{a^2y^2\cot t\operatorname{cosec}^2 t-b^2x^2\tan t\sec^2 t}{\sqrt{(b^4x^2\tan^4 t+a^4y^2\cot^4 t)}}.$$

Exp. (2) *Show that the variable ellipsoid*

$$\frac{x^2}{a^2k^2t^4}+kt^2\left[\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2\right]=1$$

is a possible form for the boundary surface of a liquid motion at any time t.

[*Ald 1964 ; Alig 62, 58 ; Bcm 62 ; Cal 54 ; Del 54 ; Gti 55 ; Gor 60 ; Lkn 62, 60 ; Pna 64 ; Pb 64, 58 ; Raj 65 ; Osm 62 ; Sag 60*]

Determine the restriction on f_1, f_2, f_3 if

$$\frac{x^2}{a^2}f_1(t)+\frac{y^2}{b^2}f_2(t)+\frac{z^2}{c^2}f_3(t)=1$$

is a possible boundary surface of a liquid. (*Del 1954, 50*)

Sol. The surface

$$F(x, y, z, t)=\frac{x^2}{a^2k^2t^4}+kt^2\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)-1=0$$

can be a possible boundary surface of a liquid, if it satisfies the condition

$$\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}=0. \tag{1}$$

Here $$\frac{\partial F}{\partial t}=-\frac{4x^2}{a^2k^2t^5}+2kt\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)$$

$$\frac{\partial F}{\partial x}=\frac{2x}{a^2k^2t^4}\ ;\ \frac{\partial F}{\partial y}=\frac{2kyt^2}{b^2}\ ;\ \frac{\partial F}{\partial z}=\frac{2kzt^2}{c^2}$$

whence the condition (1) reduces to

$$\frac{2x}{a^2k^2t^4}\left(u-\frac{2x}{t}\right)+\frac{2kyt}{b^2}(vt+y)+\frac{2kzt}{c^2}(wt+z)=0$$

If we accept : $u=2x/t$; $v=-y/t$; $w=-z/t$

the above equation is identically satisfied.

Also $(\partial u/\partial x)=(2/t)$; $(\partial v/\partial y)=(-1/t)$; $(\partial w/\partial z)=(-1/t)$.

The equation of continuity

$$\partial u/\partial x+\partial v/\partial y+\partial w/\partial z=0$$

is clearly satisfied with the above values of u, v, w, so that the velocity components are possible ; and the given surface is a possible boundary.

To determine the restriction on f_1, f_2 and f_3, we proceed as under ; using dashes for differentiation with regard to t.

$$\frac{\partial F}{\partial t}=\frac{x^2}{a^2}f_1'(t)+\frac{y^2}{b^2}f_2'()+\frac{z^2}{c^2}f_3'(t)$$

$$\frac{\partial F}{\partial x}=\frac{2x}{a^2}f_1(t) \; ; \; \frac{\partial F}{\partial y}=\frac{2y}{b^2}f_2(t) \; ; \; \frac{\partial F}{\partial z}=\frac{2z}{c^2}f_3(t).$$

Substituting in (1), we get

$$\frac{x^2}{a^2}f_1'(t)+\frac{y^2}{b^2}f_2'(t)+\frac{z^2}{c^2}f_3'(t)+\frac{2xu}{a^2}f_1(t)+\frac{2yv}{b^2}f_2(t)+\frac{2zw}{}f_3(t)=0.$$

If we accept

$$u=-\frac{xf_1'}{2f_1} \; ; \; v=-\frac{yf_2'}{2f_2} \; ; \; w=-\frac{zf_3'}{2f_3} \; ;$$

then the required restriction will be obtained if these velocity components satisfy the equation of continuity. Thus

$$-\frac{f_1'}{2f_1}-\frac{f_2'}{2f_2}-\frac{f'_3}{2f_3}=0 \quad \text{or} \quad \frac{f_1'}{f_1}+\frac{f_2'}{f_2}+\frac{f_3'}{f_3}=0$$

i.e. $$\frac{d}{dt}\log(f_1 \cdot f_2 \cdot f_3)=0 \quad \text{or} \quad \log(f_1 \cdot f_2 \cdot f_3)=\text{const.}$$

Thus, $f_1 \cdot f_2 \cdot f_3=\text{constant}$; the required restriction.

Exp. 3. *Show that all necessary conditions can be satisfied by a velocity potential of the form*

$$\phi=\alpha x^2+\beta y^2+\gamma z^2,$$

and a bounding surface of the form

$$F\equiv ax^4+by^4+cz^4-\chi(t)=0$$

where $\chi(t)$ is a given function of the time, and α, β, γ ; a, b, c, suitable functions of the time. [Ag 1951 ; Ald 63 ; Del 59, 48 ; Mad 59 ; Osm 63]

Sol. The following conditions are to be satisfied :

(i) $\nabla^2\phi=0$, for incompressible fluid flow,

(ii) $\partial F/\partial t+u(\partial F/\partial x)+v(\partial F/\partial y)+w(\partial F/\partial z)=0$, for the material surface,

(iii) The mass of the fluid contained in the material surface is constant.

Since $\phi=\alpha x^2+\beta y^2+\gamma z^2$, we get

$$\mathbf{q}=-\text{grad}\;\phi=(-2\alpha x,-2\beta y,-2\gamma z)=(u, v, w). \tag{1}$$

$$\therefore \quad \nabla^2\phi=0, \Rightarrow \alpha+\beta+\gamma=0 \tag{2}$$

Since $$F=ax^4+by^4+cz^4-\chi(t)=0 \tag{3}$$

where $\frac{\partial F}{\partial t}=\dot{a}x^4+\dot{b}y^4+\dot{c}z^4-\dot{\chi}$, [illegible]

$$\frac{\partial F}{\partial x}=4ax^3,\quad \frac{\partial F}{\partial y}=4by^3,\quad \frac{\partial F}{\partial z}=4cz^3.$$

Putting these values in (1) and using (2) we get

$$(\dot{a}-8\alpha a)\,x^4+(\dot{b}-8\beta b)\,y^4+(\dot{c}-8\gamma c)\,z^4-\dot{\chi}=0 \tag{4}$$

For the locus of (x, y, z) which has always to lie on the material surface (3) and simultaneously to satisfy (4), we must have

$$\frac{\dot{a}-8\alpha a}{a}=\frac{\dot{b}-8\beta b}{b}=\frac{\dot{c}-8\gamma c}{c}=\frac{\dot{\chi}}{\chi}.$$

These relations yield

$$\alpha=\tfrac{1}{8}\left[\frac{\dot{a}}{a}-\frac{\dot{\chi}}{\chi}\right],\ \beta=\tfrac{1}{8}\left[\frac{\dot{b}}{b}-\frac{\dot{\chi}}{\chi}\right],\ \gamma=\tfrac{1}{8}\left[\frac{\dot{c}}{c}-\frac{\dot{\chi}}{\chi}\right]. \tag{5}$$

From (2) and (5) follows the result

$$\frac{\dot{a}}{a}+\frac{\dot{b}}{b}+\frac{\dot{c}}{c}-\frac{3\dot{\chi}}{\chi}=0,\ \Rightarrow\ \frac{da}{a}+\frac{db}{b}+\frac{dc}{c}=\frac{3d\chi}{\chi}.$$

If we integrate this last expression, the result is

$$abc\,\chi^{-3}=\text{constant}=K \tag{6}$$

For condition (iii), we have for the mass M

$$M=\rho V=\rho\iiint dx\,dy\,dz \qquad \text{over the +ve octant of } ax^4+by^4+cz^4=\chi(t)$$

$$=\frac{8}{64}\,\frac{\chi^{\frac{3}{4}}}{(abc)^{\frac{1}{4}}}\iiint \lambda^{-\frac{3}{4}}\mu^{-\frac{3}{4}}\nu^{-\frac{3}{4}}\,d\lambda\,d\mu\,d\nu \text{ over } \lambda+\mu+\nu=1 \text{ (putting } ax^4=\lambda\chi,\text{ etc.)}$$

$$=\tfrac{1}{8}\,\frac{\chi^{\frac{3}{4}}}{(abc)^{\frac{1}{4}}}\int_0^1 l^{-\frac{1}{4}}\,dl\int_0^1 m^{-\frac{1}{2}}(1-m)^{-\frac{3}{4}}\,dm\int_0^1 n^{-\frac{3}{4}}(1-n)^{-\frac{3}{4}}\,dn$$

where $\lambda+\mu+\nu=l,\ \mu+\nu=lm,\ \nu=lmn.$

Thus $M=A\,\chi^{\frac{3}{4}}/8(abc)^{\frac{1}{4}}$ where A is the value of the integral involved,
= constant, by virtue of (6).

Hence all the conditions are satisfied.

NOTE : Conditions (i) and (iii) are, in fact, identical ; but serve verifications.

Ex. Explain the significance of the operator

$$\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}.$$

Find the condition that the surface $F(x, y, z, t)=0$ may be a [illegible] of a fluid in motion.

Prove that a surface of the form $ax^4+by^4+cz^4-\chi(t)=0$ is a boundary surface of a homogeneous liquid at time t, the [illegible] of the liquid motion being

$$\phi=(\beta-\gamma)\,x^2+(\gamma-\alpha)\,y^2+(\alpha-\beta)z^2$$

where $\chi, \alpha, \beta, \gamma$ are given functions of time and a, b, c are suit[illegible] time.

Ex. 1. Show that the variable ellipsoid

$$\frac{x^2}{a^2k^2t^{2n}}+kt^n\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=1$$

is a possible form of the boundary surface of a liquid at time t.

[Born 1956, 53 ; Jab 60]

Ex. 2(a). Show that $(x^2/a^2)\phi(t)+(y^2/b^2)[1/\phi(t)]=1$ is a possible form of the bounding surface of a liquid.

Ex. 2(b). Uniform incompressible inviscid liquid is contained within a flexible boundary that contains the surface to be an ellipsoid with principal axes in fixed directions. Verify that the velocity potential for the irrotational motion of the liquid is

$$-\tfrac{1}{2}\left(\frac{x^2}{a}\frac{da}{dt}+\frac{y^2}{b}\frac{db}{dt}+\frac{z^2}{c}\frac{dc}{dt}\right).$$

[Del 1966]

Ex. 3. The parabolic profile $y=k\sqrt{x}$ moves in the negative x-direction with a velocity U, through a fluid which was initially stationary. If u, v are the instantaneous velocity components of a fluid particle on the boundary, show that

$$2vy=k^2(u-\mathrm{U}).$$

Ex. 4. A sphere of radius r moves with the steady velocity components (u', v', w') through an initially stationary fluid. If t be measured from the instant the sphere was at the origin, show that the equation of its surface is

$$F=(x-u't)^2+(y-v't)^2+(z-w't)^2-r^2=0$$

and establish the boundary condition equation

$$(u-u')(x-u't)+(v-v')(y-v't)+(w-w')(z-w't)=0.$$

Ex. 5. Liquid is moving irrotationally in three dimensions is bounded by the ellipsoid $(x/a)^2+(y/b)^2+(z/c)^2=1$, where a, b, c are functions of the time, such that the volume of the ellipsoid remains constant. Prove that if the ellipsoid is rotating with angular velocities ω_1, ω_2, ω_3 about its principal axes, and u, v, w are the component velocities of the liquid parallel to the principal axes, the equation of continuity and the boundary conditions are satisfied if

$$u=(\dot{a}x/a)+y\omega_3[(a^2-b^2)/(a^2+b^2)]+z\omega_2[(c^2-a^2)/(c^2+a^2)],$$

with similar expressions for v and w.

1.90. Helmholtz's vorticity equation. *If the body forces are conservative and ρ is a function of p only, then*

$$\frac{d}{dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right)=\left(\frac{\boldsymbol{\omega}}{\rho}\cdot\nabla\right)\mathbf{q}$$

where $\boldsymbol{\omega}=\mathbf{curl\ q}$ *is the vorticity vector.*

PROOF : The Lagrange's acceleration relation is

$$\mathbf{a}=d\mathbf{q}/dt=\partial\mathbf{q}/\partial t+\nabla(\tfrac{1}{2}\mathbf{q}^2)+\boldsymbol{\omega}\times\mathbf{q}. \quad (1)$$

Taking **curl** of the above formula we obtain

$$\begin{aligned}\mathbf{curl\ a}&=\partial\boldsymbol{\omega}/\partial t+\mathbf{curl}\ (\boldsymbol{\omega}\times\mathbf{q}) \qquad [\because\ \mathbf{curl\ grad}\equiv 0]\\ &=\partial\boldsymbol{\omega}/\partial t+[\boldsymbol{\omega}\ \mathrm{div}\ \mathbf{q}-\mathbf{q}\ \mathrm{div}\ \boldsymbol{\omega}+(\mathbf{q}.\nabla)\boldsymbol{\omega}-(\boldsymbol{\omega}.\nabla)\mathbf{q}]\\ &=\partial\boldsymbol{\omega}/\partial t+(\mathbf{q}.\nabla)\boldsymbol{\omega}-(\boldsymbol{\omega}.\nabla)\mathbf{q}+\boldsymbol{\omega}\ \mathrm{div}\ \mathbf{q} \quad [\because\ \mathrm{div}\ \boldsymbol{\omega}\equiv 0]\\ &=d\boldsymbol{\omega}/dt-(\boldsymbol{\omega}.\nabla)\mathbf{q}+\boldsymbol{\omega}\ \mathrm{div}\ \mathbf{q} \quad (2)\end{aligned}$$

Now differentiating (ω/ρ) we get

$$\frac{d}{dt}\left(\frac{\omega}{\rho}\right)=\frac{1}{\rho}\frac{d\omega}{dt}-\frac{\omega}{\rho^2}\frac{d\rho}{dt}$$

$$=\frac{1}{\rho}\frac{d\omega}{dt}-\frac{\omega}{\rho^2}(-\rho\,\text{div}\,\mathbf{q}),\ \text{by continuity equation}$$

$$=\frac{1}{\rho}\left[\frac{d\omega}{dt}+\omega\,\text{div}\,\mathbf{q}\right]$$

$$=\frac{1}{\rho}[(\omega.\nabla)\mathbf{q}+\textbf{curl a}]\quad\text{by using (2)}$$

$$=\left(\frac{\omega}{\rho}.\nabla\right)\mathbf{q}+\frac{1}{\rho}\ \textbf{curl a}. \tag{3}$$

If the body forces are conservative and ρ is a function of p only, then the acceleration vector is irrotational* so that **curl a**$\equiv$**0**. Hence the relation :

$$\frac{d}{dt}\left(\frac{\omega}{\rho}\right)=\left(\frac{\omega}{\rho}.\nabla\right)\mathbf{q}. \tag{4}$$

Cor. 1. CARTESIAN EQUIVALENTS : If $\mathbf{q}=(u, v, w)$; $\omega=(\xi, \eta, \zeta)$, then (4) is equivalent to

$$\left.\begin{aligned}\frac{d}{dt}\left(\frac{\xi}{\rho}\right)&=\frac{\xi}{\rho}\frac{\partial u}{\partial x}+\frac{\eta}{\rho}\frac{\partial u}{\partial y}+\frac{\zeta}{\rho}\frac{\partial u}{\partial z}\\ \frac{d}{dt}\left(\frac{\eta}{\rho}\right)&=\frac{\xi}{\rho}\frac{\partial v}{\partial x}+\frac{\eta}{\rho}\frac{\partial v}{\partial y}+\frac{\zeta}{\rho}\frac{\partial v}{\partial z}\\ \frac{d}{dt}\left(\frac{\zeta}{\rho}\right)&=\frac{\xi}{\rho}\frac{\partial w}{\partial x}+\frac{\eta}{\rho}\frac{\partial w}{\partial y}+\frac{\zeta}{\rho}\frac{\partial w}{\partial z}\end{aligned}\right\}\text{(Nanson's equations).}$$

Cor. 2. *In the two-dimensional liquid motion the vorticity of any particle remains constant.* For $\omega=\textbf{curl q}=[(\partial v/\partial x)-(\partial u/\partial y)]\mathbf{k}=\omega\mathbf{k}$ (say) and hence $\omega.\nabla\equiv0$. This implies that the Helmholtz's vorticity equation is simply

$$\frac{d}{dt}\left(\frac{\omega}{\rho}\right)=0\quad\text{or}\quad\omega=\text{constant}.$$

1·91. Permanence of rotational and irrotational fluid motions. We start with the Helmholtz vorticity equation, viz.

$$\frac{d}{dt}\left(\frac{\omega}{\rho}\right)=\left(\frac{\omega}{\rho}.\nabla\right)\mathbf{q} \tag{1}$$

Taking the scalar product of both sides of (1) with (ω/ρ) we get

$$\frac{\omega}{\rho}.\ \frac{d}{dt}\left(\frac{\omega}{\rho}\right)=\frac{1}{2}\frac{d}{dt}\left(\frac{\omega^2}{\rho^2}\right)=\left(\frac{\omega}{\rho}\right.$$

*In fact, $d\mathbf{q}/dt=\mathbf{F}-(1/\rho)\nabla p$ is the Euler's equation . 74) which can be set as $d\mathbf{q}/dt=-\nabla(\chi+P)$, under the

Dividing both sides of the last equations by ω^2/ρ^2 and setting the unit vector $(\omega/\rho)/|\omega/\rho|$ equal to e we get

$$\frac{d}{}\left(\frac{\omega^2}{\rho^2}\right)\Big/\left(\frac{\omega^2}{\rho^2}\right)=\frac{d}{dt}\left(\log\frac{\omega^2}{\rho^2}\right)=2\left[\frac{(\omega/\rho)}{|\omega/\rho|}.\nabla\right]\mathbf{q}.\frac{(\omega/\rho)}{|\omega/\rho|},$$

i.e.
$$\frac{d}{dt}\left(\log\frac{\omega^2}{\rho^2}\right)=2(\mathbf{e}.\nabla)\mathbf{q}.\mathbf{e} \qquad (2)$$

Let us assume that q possesses continuous first order partial derivatives, then there exists some constant $k>0$, such that

$$|2(\mathbf{e}.\nabla)(\mathbf{q}.\mathbf{e})|\leqslant k.$$

This implies
$$-k\leqslant 2(\mathbf{e}.\nabla)(\mathbf{q}.\mathbf{e})\leqslant k$$

or
$$-k\leqslant D(\log \omega^2/e^2)\leqslant k \qquad \text{[by (2)]} \qquad (3)$$

where $D=d/dt$. Now, if at time t_0, the vorticity is ω_0, and the density is ρ_0, then integration of (3) provides

$$-\int_{t_0}^{t} k\,dt\leqslant \log\frac{\omega^2}{\rho^2}-\log\frac{\omega_0^2}{\rho_0^2}\leqslant\int_{t_0}^{t} k\,dt.$$

The two results which follow are

$$\omega^2/\rho^2\leqslant(\omega_0^2/\rho_0^2)\,e^{k(t-t_0)} \qquad (4)$$

and
$$\omega^2/\rho^2\geqslant(\omega_0^2/\rho_0^2)\,e^{-k(t-t_0)} \qquad (5)$$

From (4) we infer that if $\omega_0=0$ at any instant, then $\omega=0$ for all times. Thus, if the motion is initially irrotational, motion with vorticity cannot be generated in a barotropic fluid under conservative body forces : conditions very essential under which (1) is valid.

Clearly, vortex motion in a barotropic fluid initially at rest cannot be generated.

From (5) we infer that if $\omega_0\neq 0$, then $\omega\neq 0$; i.e. the vortex motion cannot be destroyed.

It may be noted that ω can never reverse its direction, for this would imply that ω could vanish which is contrary to permanency of vorticity.

1·92. Velocity field induced by a vortex tube. Let q be the velocity vector with components (u, v, w) parallel to the coordinates axes. For incompressible fluid, whether the flow is steady or not, the equation of continuity is given by div $\mathbf{q}=0$. Since div curl$\equiv 0$, there is a vector field A such that

$$\mathbf{q}=\text{curl } \mathbf{A}=\nabla\times\mathbf{A}. \qquad (1)$$

Further, the vector A is not unique, because for any scalar function Ψ, curl grad $\Psi\equiv 0$, i.e.

$$\text{curl}(\mathbf{A}+\text{grad } \Psi)=\text{curl } \mathbf{A}.$$

Hence, to determine A uniquely, we impose a restriction on the vector A, viz. we set div A=0. Now

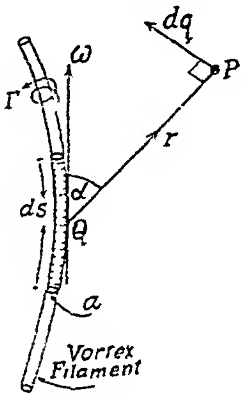
$$\boldsymbol{\omega}=\text{curl } \mathbf{q}=\text{curl curl } \mathbf{A}=\text{grad div } \mathbf{A}-\nabla^2\mathbf{A}$$

Since div A=0, this result reduces to

$$\nabla^2\mathbf{A}=-\boldsymbol{\omega}. \tag{2}$$

This is the vector Poisson's equation. If the velocity and vorticity fields are supposed to occupy all space and sufficient conditions are satisfied at infinity, then the solutions of (2), i.e. the value of A at a point P is given by

$$\mathbf{A}=\frac{1}{4\pi}\int\frac{\boldsymbol{\omega}}{r}\,dv \tag{3}$$



where r is the distance of P *from* the volume element dv at Q (say) : which means that the coordinates of Q are the integration variables.

If a is the small cross-section of the vortex tube, and ds an element of length along it, then $dv=ads$. The constancy of the vortex strength (Γ) then supplies

$$\boldsymbol{\omega}\,dv=a\boldsymbol{\omega}\,d\mathbf{s}=a\omega d\mathbf{s}=\Gamma d\mathbf{s}$$

[where $|\boldsymbol{\omega}|=\omega$]

whence (3) may be rewritten as

$$\mathbf{A}=\frac{\Gamma}{4\pi}\int_L\frac{d\mathbf{s}}{r} \tag{4}$$

taken along the vortex tube. From (1) we observe that

$$\mathbf{q}=\int d\mathbf{q}=\text{curl } \mathbf{A} \tag{5}$$

where

$$d\mathbf{q}=d(\text{curl } \mathbf{A})=\text{curl } d\mathbf{A}=\text{curl}\left(\frac{\Gamma}{4\pi}\frac{d\mathbf{s}}{r}\right)=\frac{\Gamma}{4\pi}\text{ curl }\left(\frac{1}{r}\,d\mathbf{s}\right)$$

$$=\frac{\Gamma}{4\pi}\left[\frac{1}{r}\text{ curl } d\mathbf{s}+\nabla\left(\frac{1}{r}\right)\times d\mathbf{s}\right]=\frac{\Gamma}{4\pi r^3}d\mathbf{s}\times\mathbf{r} \tag{6}$$

since $d\mathbf{s}$ is a constant, so far as this operation is concerned (curl $d\mathbf{s}$ $=0$). The velocity relation thus appears, from (5)

$$\mathbf{q}=\frac{1}{4\pi}\int_L\frac{\Gamma\,d\mathbf{s}\times\mathbf{r}}{r^3}=\frac{1}{4\pi}\int_V\frac{\boldsymbol{\omega}\times\mathbf{r}}{r^3}\,dv. \tag{7}$$

Now setting $\mathbf{q}=(u, v, w)$: $\mathbf{r}=(x-x', y-y', z-z')$, $\boldsymbol{\omega}=(\xi', \eta', \zeta')$, we get from (7),

$$u=\frac{1}{4\pi}\int_V\frac{[\eta'(z-z')-\zeta'(y-y')]\,dx'\,dy'\,dz'}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}} \tag{8}$$

with similar relations for v and w, obtained by cyclic permutations of the letters involved.

Cor. From (6) we observe that the velocity relation appears in the incremental form

$$d\mathbf{q}=\frac{\Gamma}{4\pi}\frac{d\mathbf{s}\times\mathbf{r}}{r^3}=\frac{\Gamma\, ds\sin\alpha}{4\pi r^2}\,\mathbf{e},$$

where α is the angle between the vortex-filament direction and the radius vector to the point P under consideration (vide Fig. p. 71). The velocity increment due to the element of the filament is at right angles to $d\mathbf{s}$ and $\mathbf{r}$ and corresponds with a right-handed screw (for right handed axes) about the direction of ds. In fact, its sense is in accord with the rotation of Γ. The induced velocity due to the complete vortex filament is

$$\mathbf{q}=\frac{\Gamma}{4\pi}\int_L\frac{d\mathbf{s}\times\mathbf{r}}{r^3}=\frac{\Gamma}{4\pi}\int_L\frac{\sin\alpha}{r^2}\,ds.$$

This may be compared with the *Law of Biot and Savart* in Electromagnetism' where the magnet field strength **H** induced by a current i in a wire is given by

$$\mathbf{H}=i\int\frac{d\mathbf{r}\times\mathbf{r}}{r^3}.$$

Note. The derivation of *the Biot-Savart law*, in all its diversity, is explained in § 7.32, p. 392 of *Mathematical Theory of Electromagnetism* by Bansi Lal : the author.

Ex (i). If the vorticity be given at all points within an incompressible fluid extending to infinity where it is at rest, prove that the velocity **V** is given by

$$\mathbf{V}=\text{curl }\mathbf{A},\text{ where }\mathbf{A}=\frac{1}{4\pi}\int\frac{\boldsymbol{\omega}\,dv}{r}$$

where $\boldsymbol{\omega}$ is the vorticity at the volume element dv, r is the distance between this element and the point at which **A** is evaluated, and the integral extends throughout the liquid. Can the vorticity $\boldsymbol{\omega}$ be assigned arbitrarily? [*Cal 1956* ; *Del 55*]

Ex. (ii). If the vorticity is given at all points within an incompressible fluid, prove that a possible value of the velocity **V** is given by

$$\mathbf{V}=\text{curl }\mathbf{A}$$

where if $(2\xi, 2\eta, 2\zeta)$ are the components of the vorticity, the components of **A** are

$$\frac{1}{2\pi}\iiint\xi\,\frac{dx\,dy\,dz}{r}\;;\;\frac{1}{2\pi}\iiint\eta\,\frac{dx\,dy\,dz}{r}\;;\;\frac{1}{2\pi}\iiint\zeta\,\frac{dx\,dy\,dz}{r}\;;$$

and the integrals extend throughout the liquid. [*Del 1958*]

Ex. (iii). If **q** is the velocity at a point P in an unbounded liquid which is at rest at infinity, show that

$$\mathbf{q}=\text{curl }\mathbf{A}\text{ when }\mathbf{A}=\frac{1}{2\pi}\int\frac{\boldsymbol{\omega}'}{r}\,dv'$$

where the integration is taken over the whole of space and dv' is an element of volume at a point Q where the vorticity is $\boldsymbol{\omega}'$ and $PQ=r$.

Hence derive the formula

$$\delta q=(\Gamma/4\pi)\sin\alpha\;\delta s'/r^2,$$

for the magnitude of the velocity induced by an element $\delta s'$ of a vortex filament of strength Γ at a point distant r from the element. [*Del* 1963, 61]

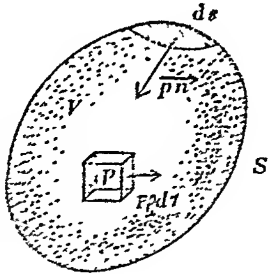
2 : *Equations of Motion of Inviscid Hydrodynamics*

2.00. Introductory remarks. In this chapter, we consider in an elementary way the dynamics of uniform and incompressible inviscid fluids, and the hydrokinematic-techniques already dealt with in chapter 1, are thoroughly made use of. For any unconventional matter, the reference (for a proper back-ground in mathematical techniques appropriate for elegant discussions) may be made to the 'Subsidiary Results' of zero chapter. This will be an absolute necessity for some topics, i.e. Lagrange's equation of motion, permanence of vorticity, etc.

Throughout the text, problems are scattered lavishly, many are profusely illustrated and solutions are presented in detail. Some theorems of general interest are also included ere the chapter is closed down.

2.10. Euler's equation of motion. *We shall now obtain Euler's equation of motion for a perfect fluid.*

Consider any arbitrary closed surface S drawn in the region occupied by the non-viscous fluid and moving with it, so that it contains the same fluid particles at every time.



Let ρ be the density of the fluid particle P within S and dv be the volume element surrounding P. Then the mass ρdv of this element always remains constant. If q is the velocity of P, then the momentum of the volume V is

$$M=\int_V q\,\rho\,dv.$$

$\therefore$ the time rate of change of momentum is

$$\frac{dM}{dt}=\int_V \frac{dq}{dt}\,\rho\,dv+\int_V q\,\frac{d}{dt}(\rho\,dv)$$

$$=\int_V \frac{dq}{dt}\rho\,dv \qquad (1)$$

since the mass ($\rho\,dv$) remains constant

If F is the external force (i.e. body force, such as gravity) per unit mass acting on the fluid ; then the total body force acting on the

fluid within the surface S at time t is

$$\int_V \mathbf{F} \rho \, dv. \tag{2}$$

If p is the pressure at a point of the surface element dS having the *outward* drawn unit normal $\mathbf{n}$, then the total surface force

$$\int_S p(-\mathbf{n}) \, dS$$

i.e.

$$-\int_V \nabla p \, dv. \qquad \text{(by Gauss Theorem)} \tag{3}$$

In (3) we have taken negative sign for the surface force acts inwards, and unit normal $\mathbf{n}$ is drawn outwards.

Thus the total force acting on V

$$= \int_V \mathbf{F} \rho \, dv - \int_V \nabla p \, dv$$

$$= \int_V (\rho F - \nabla p) \, dv. \tag{4}$$

By *Newton's Second Law of Motion*, viz. the rate of change of linear momentum is equal to the total force acting on this mass of fluid, we get by equating (1) and (4)

$$\int_V \frac{d\mathbf{q}}{dt} \rho \, dv = \int_V (\rho \mathbf{F} - \nabla p) \, dv$$

or

$$\int_V \left(\rho \frac{d\mathbf{q}}{dt} - \rho \mathbf{F} + \nabla p \right) dv = 0$$

As the surface S is arbitrary, it follows that

$$\rho \frac{d\mathbf{q}}{dt} - \rho \mathbf{F} + \nabla p = 0$$

or

$$\frac{d\mathbf{q}}{dt} = \mathbf{F} - \frac{1}{\rho} \nabla p \tag{5}$$

which is the famous Euler's equation of motion.

Since

$$\frac{d\mathbf{q}}{dt} = \frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla) \mathbf{q} \qquad [\S 1.31 \, (5), \text{p. } 23]$$

we can re-write the equation of motion or pressure equation as

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla) \mathbf{q} = \mathbf{F} - \frac{1}{\rho} \nabla p. \tag{6}$$

NOTE. Using Lagrange acceleration relation, viz.

$$\frac{d\mathbf{q}}{dt} = (\partial \mathbf{q}/\partial t) + \nabla (\tfrac{1}{2} q^2) + \boldsymbol{\omega} \times \mathbf{q} \qquad [\S 1.31 \, (6), \text{p.} 24]$$

Euler's equation (5) can be set as

$$(\partial \mathbf{q}/\partial t)+\nabla(\tfrac{1}{2}q^2)+\boldsymbol{\omega}\times\mathbf{q}=\mathbf{F}-(1/\rho)\nabla p \tag{6'}$$

This is called *Lamb's hydrodynamical equation.* The chief advantage of (6′) is that it is *invariant under a change of coordinate system.*

Cor. 1. *Cartesian Equivalents* : we set

$$\mathbf{q}=(u, v, w),\ \mathbf{F}=(X, Y, Z),\ \nabla p=\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right).$$

Substituting in (5) and equating the coefficients of **i**, **j**, **k** we get

$$\left.\begin{aligned} du/dt&=X-(1/\rho)\,(\partial p/\partial x)\\ dv/dt&=Y-(1/\rho)\,(\partial p/\partial y)\\ dw/dt&=Z-(1/\rho)\,(\partial p/\partial z)\end{aligned}\right\}. \tag{7}$$

We may observe that $du/dt=\ddot{x}$, $dv/dt=y$, $dw/dt=z$. Should we use (6), then since

$$(\mathbf{q}.\ \nabla)\ \mathbf{q}=\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{z}\right)\mathbf{q}$$

the coefficients of **i**, **j**, **k** when equated on either side of (6) give

$$\frac{du}{dt}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=X-\frac{1}{\rho}\frac{\partial p}{\partial x},$$

with two more relations for v and w.

Cor. 2. *Equations of motion in cylindrical coordinates* : With velocity components (u, v, w) in the (r, θ, z) directions, we have [vide § 1.31, p. 24]

$$\frac{d\mathbf{q}}{dt}=\left(\frac{du}{dt}-\frac{v^2}{r},\ \frac{dv}{dt}+\frac{uv}{r},\ \frac{dw}{dt}\right)$$

$$\mathbf{F}=(F_r, F_\theta, F_z),\ \nabla p=\left(\frac{\partial p}{\partial r}, \frac{1}{r}\frac{\partial p}{\partial \theta}, \frac{\partial p}{\partial z}\right)$$

Substituting in (5) and equating the coefficients of **i**, **j**, **k**, we get

$$\left.\begin{aligned}\frac{du}{dt}-\frac{v^2}{r}&=F_r-\frac{1}{\rho}\frac{\partial p}{\partial r}\\ \frac{dv}{dt}+\frac{uv}{r}&=F_\theta-\frac{1}{\rho}\frac{\partial p}{r\partial\theta}\\ \frac{dw}{dt}&=F_z-\frac{1}{\rho}\frac{\partial p}{\partial z}\end{aligned}\right\} \tag{8}$$

Cor. 3. *Equations of motion in space polar coordinates*: With velocity components (u, v, w) in the (r, θ, φ) directions, we have [vide §1.31, p. 24]

$$\therefore \frac{d\mathbf{q}}{dt}=\left(\frac{du}{dt}-\frac{v^2+w^2}{r}, \frac{dv}{dt}-\frac{w^2 \cot\theta}{r}+\frac{uv}{r}, \frac{dw}{dt}+\frac{vw \cot\theta}{r}\right)$$

$$\mathbf{F}=(F_r, F_\theta, F_\varphi), \nabla p=\left(\frac{\partial p}{\partial r}, \frac{1}{r}\frac{\partial p}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial p}{\partial \varphi}\right).$$

Substituting in (5) and equating the coefficients of **i**, **j**, **k**, we get

$$\left.\begin{aligned} \frac{du}{dt}-\frac{v^2+w^2}{r}&=F_r-\frac{1}{\rho}\frac{\partial p}{\partial r}\\ \frac{dv}{dt}-\frac{w^2\cot\theta}{r}+\frac{uv}{r}&=F_\theta-\frac{1}{\rho}\frac{\partial p}{r\partial\theta}\\ \frac{dw}{dt}+\frac{vw\cot\theta}{r}&=F_\varphi-\frac{1}{\rho}\frac{1}{r\sin\theta}\frac{\partial p}{\partial\varphi} \end{aligned}\right\} \qquad (9)$$

Cor. 4. *Equations of motion referred to rotating axes.* If the axes rotate with *uniform* angular velocity $\omega=(\omega_1, \omega_2, \omega_3)$, then the expression for acceleration is, [by § 0.81, p. 15]

$$\frac{d\mathbf{q}}{dt}=\left[\frac{\partial\mathbf{q}}{\partial t}\right]+(\mathbf{q}'.\nabla)\mathbf{q}=\frac{\partial\mathbf{q}}{\partial t}+\omega\times\mathbf{q}+(\mathbf{q}'.\nabla)\,\mathbf{q}.$$

Hence, the Euler's equation of motion then is

$$\frac{\partial\mathbf{q}}{\partial t}+\omega\times\mathbf{q}+(\mathbf{q}'.\nabla)\mathbf{q}=\mathbf{F}-\frac{1}{\rho}\nabla p.$$

Here $\mathbf{q}$ and $\mathbf{q}'$ are the absolute and relative velocities.

Cor. 5. *Acceleration potential.* When the body forces are conservative so that $\mathbf{F}=-\nabla\chi$ and the fluid is barotropic, i.e. density is a function of pressure, so that $\rho^{-1}\nabla p=\nabla\int(dp/\rho)$, then Euler's equation of motion may be expressed as

$$d\mathbf{q}/dt=-\nabla\chi-\nabla\int(dp/\rho)$$

or

$$\mathbf{a}=-\nabla\left(\chi+\int\frac{dp}{\rho}\right)=-\text{grad}\,\Phi, \text{ (say)}.$$

This result shows that the acceleration vector **a** possesses acceleration potential $\chi+\int(dp/\rho)=\Phi$.

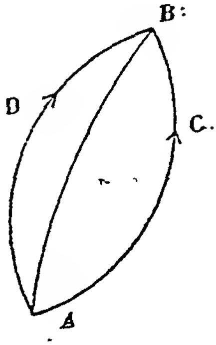
Ex. Find the equations of motion of an inviscid fluid in cylindrical polar coordinates by:

(i) Using the expressions for grad, div, and curl from curvilinear coordinates

(ii) Considering the motion of an element of volume in these coordinates.

NOTE. Conservative field of force. *In a conservative field of force,*

the work done by the force **F** *of the field in taking a unit mass from A to B is independent of the path.*



Thus

$$\int_{ACB} \mathbf{F} \cdot d\mathbf{r} = \int_{ADB} \mathbf{F} \cdot d\mathbf{r} = -\chi \text{ (say)}$$

where χ is a scalar point function whose value depends upon the initial and final position of A and B.

As in dynamics, it can be easily shown that

$$\mathbf{F} = -\nabla \chi.$$

χ is known as force potential and measures the potential energy of the field.

Exp. 1. *Prove that if*

$$\lambda = \frac{\partial u}{\partial t} - v\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + w\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

and μ, ν are two similar expressions, then $\lambda dx + \mu dy + \nu dz$ is a perfect differential, if the forces are conservative and the density is constant. [*Ban 1961*]

Sol. If λ, μ, ν are supposed to be components of some vector **N**, then the given equation is the first-component equation of the single vector equation

$$\mathbf{N} = \partial \mathbf{q}/\partial t - \mathbf{q} \times \text{curl } \mathbf{q}. \tag{1}$$

Now, the Lagrange's acceleration relation is

$$d\mathbf{q}/dt = (\partial \mathbf{q}/\partial t) + \nabla(\tfrac{1}{2}q^2) - \mathbf{q} \times \boldsymbol{\omega} \tag{2}$$

Hence, from (1) and (2), we get

$$\mathbf{N} = (d\mathbf{q}/dt) - \tfrac{1}{2}\nabla(q^2). \tag{3}$$

From Euler's equation of motion

$$\frac{d\mathbf{q}}{dt} = \mathbf{F} - \frac{1}{\rho}\nabla p = -\nabla\left(\chi + \frac{p}{\rho}\right), \Rightarrow \frac{d\mathbf{q}}{dt} \cdot d\mathbf{r} = -d\left(\chi + \frac{p}{\rho}\right).$$

Thus, (3) reduces to

$$\mathbf{N} \cdot d\mathbf{r} = -d\left(\chi + \frac{1}{\rho} p + \tfrac{1}{2}q^2\right).$$

And this is equivalent to the fact that $\lambda dx + \mu dy + \nu dz$ is a perfect differential.

Exp. 2. *A quantity of liquid of density ρ occupies a length 2a of a long straight tube of uniform small cross-section, and is under the action of a force kx per unit mass towards a fixed point O in the tube, where x is the distance from O. Show that, when the nearer free surface is at a distance z from O, the pressure at a distance x exceeds atmospheric pressure Π by*

$$k\rho(x-z)(a - \tfrac{1}{2}x + \tfrac{1}{2}z)$$ [*Del 1937 ; Pb 63*]

Sol. The continuity-condition for incompressible fluids (div $\mathbf{q} = 0$) in the present case of one-dimensional flow is simply $\partial(\rho u)/\partial x = 0$. Thus, u is a function of time t only. Further, z is a function of t only, so that $u = \dot{z}$ and the particle acceleration at time t will be $\ddot{z}$. Thus, Euler's equation of motion is

$$\partial u/\partial t = \ddot{z} = -kx - \rho^{-1}(\partial p/\partial x). \tag{1}$$

Integrating with regard to x, keeping t constant, we get

$$\left(\frac{\partial u}{\partial t}\right) x = A - \tfrac{1}{2}kx^2 - (p/\rho).$$

Since $p=\Pi$ when $x=z$ and when $x=2a+z$, the above yields

$$\ddot{z} = -k(z+a) \tag{2}$$

which reveals that the liquid performs simple harmonic motion of period $2\pi/\sqrt{k}$. From (1) and (2) we now get

$$\partial p/\partial x = \rho k(z+a-x)$$

Integrating with regard to x, keeping t constant, we get

$$p = -\tfrac{1}{2}k\rho(z+a-x)^2 + B$$

where B is independent of x (it may depend on t). To determine it we observe that $p=\Pi$ when $x=z$ (and also when $z=x+2a$), then

$$B = \Pi + \tfrac{1}{2}k\rho a^2$$

Hence $$p = \Pi + k\rho(x-z)\,(a - \tfrac{1}{2}x + \tfrac{1}{2}z).$$

Exp. 3. *Air, obeying Boyle's Law, is in motion in a uniform tube of small section; prove that if ρ be the density and v the velocity at a distance x from a fixed point at the time t,*

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x^2}[\rho\,(v^2+k)]$$

[*Del 1949 ; Gti 58, 53 ; Pna 63 (Old)*]

Sol. At time t, let p be the pressure and v the velocity at a distance x from the end of the tube. The equations of motion and continuity are

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \tag{2}$$

Since, by *Boyle's Law*, $p = k\rho$, the equation of motion can be written as

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{k}{\rho}\frac{\partial \rho}{\partial r} \tag{3}$$

Differentiating (2) with respect to t, we get

$$\frac{\partial^2 \rho}{\partial t^2} = -\frac{\partial^2(\rho v)}{\partial x \partial t}$$

$$= -\frac{\partial}{\partial x}\left[\rho\frac{\partial v}{\partial t} + v\frac{\partial \rho}{\partial t}\right]$$

$$= -\frac{\partial}{\partial x}\left[-\rho\left(\frac{1}{\rho}\frac{\partial p}{\partial x} + v\frac{\partial v}{\partial x}\right) - v\frac{\partial}{\partial x}(\rho v)\right], \text{ by (1) and (2)}$$

or $$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial}{\partial x}\left[\frac{\partial k\rho}{\partial x} + \rho v\frac{\partial v}{\partial x} + v\frac{\partial}{\partial x}(\rho v)\right] \qquad \textit{Boyle's Law.]}$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial k\rho}{\partial x} + \frac{\partial}{\partial x}(\rho v^2)\right] = \frac{\partial^2}{\partial x^2}[k\rho + \rho v^2] = \qquad (k+v^2)].$$

Ex. Air, obeying Boyle's law, is in motion in a uniform tube of small section, prove that if ρ be the density and v the velocity at a distance r from a fixed point; at time t,

$$\frac{\partial^2 \rho}{\partial t^2} = 4v\frac{\partial v}{\partial r}\cdot\frac{\partial \rho}{\partial r} + 2\rho\left(\frac{\partial v}{\partial r}\right)^2 + 2\rho v\frac{\partial^2 v}{\partial r^2} + k\frac{\partial^2 \rho}{\partial r^2} + v^2\frac{\partial^2 \rho}{\partial r^2}. \qquad [\textit{Pb 1950}]$$

Exp. 4 *Obtain the equation of continuity, and expressions for the total components of acceleration of a fluid particle in cylindrical coordinates. Derive the equations of motion expressed in these coordinates.*

If liquid of density ρ *rotates like a rigid body with constant angular velocity* ω *about the z-axis which is vertical, deduce from the above equations of motion that the pressure is given by*

$$p/\rho=\tfrac{1}{2}\omega^2 r^2-gz+const.,$$

where r is the distance from the axis. Show that the surfaces of equal pressure are paraboloids with the same latus rectum. [*Pna 1959*]

Sol. Since the liquid revolves with constant angular velocity $\boldsymbol{\omega}=\omega\mathbf{k}$ about z-axis, its velocity distribution $\mathbf{q}=\boldsymbol{\omega}\times\mathbf{R}$ is that of a rigid body. Hence

$$\mathbf{a}=\boldsymbol{\omega}\times\mathbf{q}=\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{R})=(\boldsymbol{\omega}\cdot\mathbf{R})\,\boldsymbol{\omega}-\omega^2\mathbf{R}=\omega^2(z\mathbf{k}-x\mathbf{i}-y\mathbf{j}-z\mathbf{k})$$

since $\mathbf{R}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. Thus,

$$\mathbf{a}=-\omega^2(x\mathbf{i}+y\mathbf{j})=-\omega^2(x\nabla x+y\nabla y)=-\tfrac{1}{2}\omega^2\nabla(x^2+y^2)=-\tfrac{1}{2}\omega^2\nabla r^2,$$

because, in cylindrical coordinates, $x=r\cos\theta, y=r\sin\theta$ so that $x^2+y^2=r^2$. Hence, equation of motion yields $[\mathbf{F}=-\nabla(gz)=-\mathbf{k}g]$

$$\mathbf{a}=-\tfrac{1}{2}\omega^2\nabla r^2=-\nabla[gz+(p/\rho)]$$

or
$$\nabla[gz+(p/\rho)-\tfrac{1}{2}\omega^2 r^2]=0$$

Integration provides : $\quad p/\rho=\tfrac{1}{2}\omega^2r^2-gz+\text{const}.$

For a surface of equal pressure (i.e. free surface), p=const. wh nce the free surface is a paraboloid of revolution with oz as axis, each having the same latus rectum $(2g/\omega^2)$.

Ex. 1. In a large pond each particle of water traverses a horizontal circle whose centre is on a fixed vertical axis. The speed of a particle at distance r from the axis is ωr for $r<a$, but $\omega a^2/r$ for $r>a$. Find the form of the surface of water, given that ω and a are constant.

Ex. 2. Assuming that the earth is a fluid sphere of radius a, of constant density ρ, and without rotation, show that the pressure at a distance r from the centre is $\tfrac{1}{2}g\rho a(1-r^2/a^2)$.

Ex. 3. A mass of liquid is revolving about z-axis with the angular velocity $f(r)\,\mathbf{k}$ where r is the perpendicular distance from the axis. If $\mathbf{e}$ is a unit vector perpendicular to the axis, prove that the velocity of the liquid is

$$\mathbf{q}=rf(r)\,\mathbf{k}\times\mathbf{e},\text{ and curl }\mathbf{q}=[rf'(r)+2f(r)]\mathbf{k}.$$

If the motion is irrotational, show that

$$\boldsymbol{\omega}=a\mathbf{k}/r^2,\ \mathbf{q}=(a/r)\mathbf{k}\times\mathbf{e}\,;\ \phi=b-a\,\theta \qquad (a, b\text{ const.}),$$

where ϕ is the multivalued velocity potential and θ is the polar angle.

For a liquid of constant density, under constant gravity, show that

$$gz+(p/\rho)+\tfrac{1}{2}(a^2/r^2)=\text{constant}.$$

2·11. Cauchy's pressure equation : integrals of the equation of motion. To obtain the solution of Euler's equation of motion, which is non-linear, we will have to entertain simplifying assumptions. Firstly we assume that the external forces form a conservative system so that $\mathbf{F}=-\nabla\chi$. Secondly we assume that the fluid is barotropic so that

$$\frac{1}{\rho}\nabla p=\nabla\int\frac{dp}{\rho}=\nabla P\text{ (say)} \qquad \text{(Vide Note. (iv)p. 2).}$$

Since $\quad \dfrac{d\mathbf{q}}{dt}=\dfrac{\partial\mathbf{q}}{\partial t}+\boldsymbol{\omega}\times\mathbf{q}+\nabla(\tfrac{1}{2}q^2)\quad$ [Lagrange acceleration relation]

the equations of motion can be set as

$$\frac{\partial \mathbf{q}}{\partial t}+\boldsymbol{\omega}\times\mathbf{q}+\nabla(\tfrac{1}{2}\mathbf{q}^2)=-\nabla\chi-\nabla P$$

or
$$\frac{\partial \mathbf{q}}{\partial t}=\mathbf{q}\times\boldsymbol{\omega}-\nabla(\chi+P+\tfrac{1}{2}\mathbf{q}^2). \tag{1}$$

This last expression is the final result. Since this result, as it stands, is not very useful, several special cases will be considered.

Special cases :

(i) When the motion is *irrotational*, $\boldsymbol{\omega}=\text{curl }\mathbf{q}=0$, and $\mathbf{q}=-\nabla\phi$. and (1) reduces to

$$\nabla\left(\frac{\partial\phi}{\partial t}\right)=\nabla H \qquad [\text{where } H=\chi+P+\tfrac{1}{2}\mathbf{q}^2] \tag{2}$$

since the operators ∇ and $\partial/\partial t$ are interchangeable. The solution of (2), viz. grad $(H-\partial\phi/\partial t)=0$ is

$$\chi+\int\frac{dp}{\rho}+\tfrac{1}{2}\mathbf{q}^2-\frac{\partial\phi}{\partial t}=C(t), \text{ constant.} \tag{3}$$

The constant $C(t)$ shall be a function of time t only.

(ii) **Bernoulli's theorem.** When the motion is *steady* as well as *irrotational*, $\partial\mathbf{q}/\partial t=0$ and $\boldsymbol{\omega}=0$; (1) reduces to

$$\text{grad }(\chi+P+\tfrac{1}{2}\mathbf{q}^2)=0.$$

The solution of this equation is

$$\chi+P+\tfrac{1}{2}\mathbf{q}^2=C.$$

Here C is an absolute constant ; i.e. is independent of time also. It may be remarked that Bernoulli's theorem *is still true even if the velocity potential ϕ does not exist.*

If the fluid is incompressible and homogeneous, $\rho=$const. then

$$P=\int\frac{dp}{\rho}=\frac{p}{\rho}$$

$\omega=\alpha\mathbf{q}$, where α is any function such that $(\mathbf{q}\cdot\nabla)\alpha=0$. In such a case, $\mathbf{q}\times\omega=0$ even if $\omega\neq0$, i e. even if the motion is not irrotational. Hence for a steady flow we get by virtue of (1)

$$\nabla(H)=0, \Rightarrow \chi+p+\tfrac{1}{2}q^2=\text{constant}$$

where the constant has the *same* value throughout the fluid.

(v) **Adiabatic compressible flow :** For adiabatic changes, (i.e. expansion or contraction without loss or gain of heat), the variable density is related to variable pressure p by relation

$$p=k\rho^{\gamma},$$

where k is constant for any particular gas ; γ is the ratio of the specific heat of a gas at constant pressure to specific heat at constant volume. Thus

$$\int\frac{dp}{\rho}=k\gamma\int\rho^{\gamma-2}\,d\rho=\frac{k\gamma}{\gamma-1}\,\rho^{\gamma-1}=\frac{\gamma}{\gamma-1}\,\frac{p}{\rho}.$$

And Bernoulli's equation gives

$$\frac{\gamma}{\gamma-1}\,\frac{p}{\rho}+\tfrac{1}{2}q^2+\chi=\text{constant.}$$

Ex 1. Form the general equations of motion of a fluid under any forces. In the case of a homogeneous liquid moving irrotationally under the action of conservative forces, establish the equation

$$\frac{p}{\rho}+\tfrac{1}{2}\,q^2-\frac{\partial\phi}{\partial t}+\chi=C$$

where the symbols have their usual meanings. [*Del 1951, 34*]

Ex. 2. State and prove Euler's hydrodynamical equations. Establish Bernoulli's theorem that in the case of steady motion of a homogeneous inelastic fluid

$$(p/\rho)+\tfrac{1}{2}\,q^2+V=K,$$

where K is a constant along a stream-line, but varies from stream line to stream line.

Prove further, that if the external forces, whose potential is V, are conservative, and the motion be steady, the portion of the stream-line along which the velocity q is constant is geodesic on the surface

$$\int\frac{dp}{\rho}+V=\text{constant.}$$ [*Pna 1958*]

2.12. Problems with solutions

(1) *A sphere is at rest in an infinite mass of homogeneous liquid of density* ρ, *the pressure at infinity being* Π. *Show that, if the radius R of the sphere varies in any manner, the pressure at the surface of the sphere at any time is*

$$\Pi+\tfrac{1}{2}\rho\left[\frac{d^2(R^2)}{dt^2}+\left(\frac{dR}{dt}\right)^2\right].$$

[*Boni 1964 (Old), 58; Gti 61, 55 ; Kuru 65 ; Pna 60 ; Pb 50(S), 49*]

Sol. In the incompressible liquid outside the sphere, the fluid velocity q will be radial, and thus q will be a function of r, the radial distance from the centre of

the sphere (the origin), and time t only. The continuity equation div q=0, in spherical polar coordinates becomes

$$\frac{1}{r^2}\frac{d}{dr}(r^2 q)=0, \Rightarrow r^2 q=\text{const.}=f(t)=R^2\dot{R} \tag{1}$$

where f is a function of time t only. We notice that $q \to 0$ as $r \to \infty$, as required. Clearly, curl q=0, so that the motion is irrotational, the velocity potential being $\phi=f/r$. The pressure equation for irrotational non-steady fluid motion, in the absence of body forces is

$$\frac{p}{\rho}-\frac{\partial \phi}{\partial t}+\tfrac{1}{2}q^2=C(t) \tag{2}$$

where $C(t)$ is an arbitrary function of time t. As $r \to \infty$, $p \to \Pi$, $q=f/r^2 \to 0$, $\phi \to 0$, so that $C(t)=\Pi/\rho$, for all t. Thus, using (1), we get from (2)

$$\frac{p}{\rho}=\frac{\Pi}{\rho}+\frac{\partial}{\partial t}\left(\frac{f}{r}\right)-\tfrac{1}{2}\left(\frac{R^2\dot{R}}{r^2}\right)^2. \tag{3}$$

Now $$\frac{\partial f}{\partial t}=\frac{d}{dt}(R^2\dot{R})=\ddot{R}R^2+2R\dot{R}^2$$

Hence, at the surface of the sphere, $r=R$ and (3) yields

$$\frac{p}{\rho}=\frac{\Pi}{\rho}+\frac{1}{R}(2R\dot{R}^2+\ddot{R}R^2)-\tfrac{1}{2}\dot{R}^2$$

or $$p=\Pi+\tfrac{1}{2}\rho\{d^2(R^2)/dt^2+(dR/dt)^2\};$$

because $\ddot{(R^2)}=(2R\dot{R})^{\cdot}=2(\dot{R}^2+R\ddot{R})$

(2) *An infinite mass of homogeneous incompressible fluid is at rest subject to a uniform pressure Π, and contains a spherical cavity of radius a, filled with gas at a pressure $m\Pi$; prove that if the inertia of the gas be neglected, and Boyle's law be supposed to hold throughout the ensuing motion, the radius of the sphere will oscillate between the values a and na; where n is determined by the equation*

$$1+3m\log n-n^3=0.$$

If m be nearly equal to 1, the time of an oscillation will be $2\pi\sqrt{(a^2\rho/3\Pi)}$, ρ being the density of the fluid.

[*Ag 1959; Alig 61; Bom 52; Del 55; 48 (Special); Gti 62; Lkn 62; Osm 60; Pra 65.*]

Sol. In the incompressible fluid outside the spherical cavity, the fluid velocity q will be radial and shall be a function of r, the radial distance from the centre of the cavity (the origin), and time t only. The continuity equation div q=0, in spherical polar coordinates becomes

$$\frac{1}{r^2}\frac{d}{dr}(r^2 q)=0, \Rightarrow r^2 q=\text{const.}=f(t)=R^2\dot{R}\ (\text{say}) \tag{1}$$

where f is a function of time t only. We observe that $q \to 0$ as $r \to \infty$, as required. Clearly, curl q=0, so that the motion is irrotational and hence velocity potential exists and is given by $\phi=f/r$. The pressure equation for irrotational non-steady fluid motion, in the absence of body forces, is

$$\frac{p}{\rho}-\frac{\partial \phi}{\partial t}+\tfrac{1}{2}q^2=C(t) \tag{2}$$

where $C(t)$ is an arbitrary function of time t. As $r \to \infty$, $p \to \Pi$, $q = f/r^2 \to 0$, $\phi \to 0$, so that $C(t) = \Pi/\rho$, for all t. Putting for ϕ and C in (2) we get

$$\frac{p}{\rho} - \frac{1}{r}\frac{\partial f}{\partial t} + \frac{1}{2}q^2 = \frac{\Pi}{\rho} \tag{3}$$

Now when the cavity expands to radius r, Boyle's law provides $pv = \text{const.}$ (v=volume) so that

$$\tfrac{4}{3}\pi a^3 m\,\Pi = \tfrac{4}{3}\pi r^3 p, \;\Rightarrow\; p = m\Pi a^3/r^3 \tag{4}$$

$$\partial f/\partial t = d(r^2 q)/dt = 2rq^2 + r^2 q\, dq/dr \tag{5}$$

From (3), (4) and (5) we obtain

$$m\Pi a^3/\rho r^3 - (2q^2 + rq\;dq/dr) + \tfrac{1}{2}q^2 = \Pi/\rho$$

or
$$3q^2r^2 + 2r^3 q\; dq/dr = (2\Pi/\rho)\,(m\,a^3/r - r^2)$$

or
$$d(r^3q^2)/dr = (2\Pi/\rho)\,(m\,a^3/r - r^2).$$

Integration with respect to r yields

$$r^3q^2 = 2\Pi\rho^{-1}(m\,a^3 \log r - \tfrac{1}{3}r^3) + A$$

where A is constant of integration; determined by the fact that at $r=a$, $q=0$ and hence $A = -2\Pi\rho^{-1}(m\,a^3 \log a - \tfrac{1}{3}a^3)$. Thus,

$$r^3q^2 = 2\Pi\,\rho^{-1}\,a^3\{3m \log(r/a) - (r/a)^3 + 1\}/3. \tag{6}$$

Now, q shall be zero again where $r = na$; provided n is given by

$$1 + 3m \log n - n^3 = 0.$$

Special Case: If $m \doteq 1$, let $r = a + x$ where x is small. Then, $\dot{x} = \dot{r} = q$. We now get from (6)

$$(\dot{x})^2(a+x)^3 \doteq \tfrac{2}{3}\Pi\rho^{-1}a^3\{3\log(1+x/a) - (1+x/a)^3 + 1\}$$

$$(\dot{x})^2(1+3y+3y^2+\ldots) \doteq \lambda\{3(y - \tfrac{1}{2}y^2 + \ldots) - (1+3y+3y^2+\ldots) + 1\}$$

where $\lambda = \tfrac{2}{3}\Pi\rho^{-1}$, $y = x/a$ and y^3 is neglected. Thus

$$(\dot{x})^2(1+3y+3y^2) \doteq \lambda(-9y^2/2)$$

or
$$(\dot{x})^2 \doteq \lambda(-9y^2/2)\,(1+3y+3y^2)^{-1} \doteq -9\lambda y^2/2$$

i.e.
$$(\dot{x})^2 = -3\Pi\rho^{-1}(x^2/a^2).$$

Differentiating with respect to t gives

$$\ddot{x} = -(3\Pi/\rho a^2)x.$$

This is a simple harmonic motion of periodic time $2\pi\sqrt{a^2\rho/3\Pi}$.

(3). *An infinite mass of liquid, containing a spherical bubble, is initially in equilibrium under constant pressure* Π. *The radius of the bubble is then disturbed from its equilibrium value* R_0, *and during the subsequent oscillatory motion, the pressure* p *and volume* v *of the bubble are related by the equation* $pv^\gamma = \text{constant}$, *where* γ *is a constant greater than unity. Show that, if the amplitude of the oscillations is sufficiently large, the maximum and minimum values of the radius,* a *and* b, *are related approximately by the equation*

$$a/b = (\gamma - 1)^{-\frac{1}{2}}\,(R_0/b)^\gamma.$$

Sol. Let the centre of the bubble be taken at the origin. Then in the incompressible liquid *outside* the bubble, the fluid velocity q will be radial and hence will be a function of radial distance r and time t only. The continuity condition div $\mathbf{q}=0$, in spherical polar coordinates then provides

$$\frac{1}{r^2}\frac{d}{dr}(r^2 q)=0, \Rightarrow r^2 q=\text{const.}=f(t)=R^2\dot{R} \text{ (say)} \tag{1}$$

where $f(t)$ is a function of t only. Clearly, when $r\to\infty$, $q\to 0$ as required. We note that curl $\mathbf{q}=0$, so that motion is irrotational and hence the velocity potential ϕ exists and is clearly $\phi=+f(t)/r$. The pressure equation, in the absence of body forces and with constant density, viz.

$$\frac{p}{\rho}+\tfrac{1}{2}q^2-\frac{\partial\phi}{\partial t}+\chi=C(t).$$

now gives

$$\frac{p}{\rho}=C(t)-\tfrac{1}{2}\frac{f^2}{r^4}+\frac{1}{r}\frac{\partial f}{\partial t}$$

where $C(t)$ is an arbitrary function of time. Now when $r\to\infty$, $p\to\Pi$ as provided so that $C(t)=\text{constant}=\Pi/\rho$. Thus, the preceeding equation reduces to

$$\frac{p}{\rho}=\frac{\Pi}{\rho}-\tfrac{1}{2}\frac{f^2}{r^4}+\frac{1}{r}\frac{\partial f}{\partial t}. \tag{2}$$

At the surface of the bubble, $r=R(t)$ say; the pressure is *continuous*. Thus, at $r=R$, $pv^\gamma=$ constant, gives

$$\Pi\left(\tfrac{4}{3}\pi R_0^3\right)^\gamma=p\left(\tfrac{4}{3}\pi R^3\right)^\gamma, \text{ i.e. } p=(R_0/R)^{3\gamma}\,\Pi. \tag{3}$$

Further, the normal velocity of the surface of the bubble at $r=R$ is also equal to the normal velocity of the liquid there; so that $f(t)=R^2\dot{R}$, [by (1)]. Notice that R is a function of t only. From (1), (2) and (3) we get

$$\frac{d}{dt}(R^2\dot{R})-\tfrac{1}{2}R\dot{R}^2=\frac{R\Pi}{\rho}\left\{\left(\frac{R_0}{R}\right)^{3\gamma}-1\right\}.$$

or

$$R^2\ddot{R}+\tfrac{3}{2}R\dot{R}^2=\frac{R\Pi}{\rho}\left\{\left(\frac{R_0}{R}\right)^{3\gamma}-1\right\}.$$

To make this differential equation exact, multiply both sides by $2R\dot{R}$; then since $2R^3\dot{R}\ddot{R}+3R^2\dot{R}^3=d(R^3\dot{R}^2)/dt$, the above yields

$$\frac{d}{dt}(R^3\dot{R}^2)=\frac{2R^2\Pi}{\rho}\left\{\left(\frac{R_0}{R}\right)^{3\gamma}-1\right\}\dot{R}$$

or

$$\frac{d}{dR}(R^3\dot{R}^2)=\frac{2R^2\Pi}{\rho}\left\{\left(\frac{R_0}{R}\right)^{3\gamma}-1\right\}. \tag{4}$$

We now integrate (4) between $R=a$ and b; and notice that $\dot{R}=0$ at both limits. Thus, the result obtained is

$$\frac{R_0^{3\gamma}\,a^{-3(\gamma-1)}}{3(\gamma-1)}+\frac{a^3}{3}=\frac{R_0^{3\gamma}\,b^{-3(\gamma-1)}}{3(\gamma-1)}+\frac{b^3}{3}.$$

This is the exact relation between R_0, a and b. To obtain the required result we remember that the amplitude of the oscillations is sufficiently large so that $b\ll R_0$ and $R_0\ll a$. This gives

$$\frac{a^3}{3}\doteq\frac{R_0^{3\gamma}\,b^{-3(\gamma-1)}}{3(\gamma-1)}, \Rightarrow \frac{a}{b}=(\gamma-1)^{-\frac{1}{3}}\left(\frac{R_0}{b}\right)^\gamma.$$

(4) *Liquid is contained between two parallel planes; the free surface is a circular cylinder of radius a whose axis is perpendicular to the planes. All the liquid within a concentric circular cylinder of radius b is suddenly annihilated. Prove that if* Π *be the pressure at the outer surface, the initial pressure at any point of the liquid distance r from the centre is*

$$\Pi(\log r-\log b)/(\log a-\log b).$$

(*Ag 1965*; *Jab 60*; *Kr 60*; *Raj 62*; *Osm 63*)

Sol. In the incompressible liquid outside the cylinder $|z|=b$, the fluid velocity q will be radial, and q will be a function of r, the radial distance from the centre of that cylinder (the origin) $r<a$, and time t only. The continuity equation div $\mathbf{q}=0$, in cylindrical coordinates becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rq\right)=0, \Rightarrow rq=\text{const.}=f(t)=R\dot{R} \qquad \text{(say)} \tag{1}$$

where f is a function of time t only. We note that $q\to 0$ as $r\to\infty$, as required. Clearly, curl $\mathbf{q}=0$, so that motion is irrotational and hence velocity potential ϕ exists. In fact, $\phi=-f(t)\log r$. The pressure equation for irrotational non-steady fluid motion, in the absence of body forces, is

$$\frac{p}{\rho}-\frac{\partial\phi}{\partial t}+\tfrac{1}{2}q^2=C(t) \tag{2}$$

where $C(t)$ is an arbitrary function of time t. Initially, $t=0$, $q=0$, $p=P$, so that (2) yields, on using $\phi=-f(t)\log r$

$$\frac{P}{\rho}+\dot{f}(0)\log r=C(0) \tag{3}$$

Now $p=\Pi$ when $r=a$; $p=0$ when $r=b$, so that

$$\left.\begin{aligned}(\log a-\log b)\,\dot{f}(0)&=-\Pi/\rho\\ \text{and}\qquad (\log r-\log b)\,\dot{f}(0)&=-P/\rho\end{aligned}\right\} \tag{4}$$

Dividing we get $P=\Pi(\log r-\log b)/(\log a-\log b)$.

(5). *A centre of force attracting inversely as the square of the distance is at the centre of a spherical cavity within an infinite mass of incompressible fluid, the pressure on which at any infinite distance is* ϖ, *and is such that the work done by this pressure on a unit of area through a unit of length is one-half the work done by the attractive force on a unit of volume of the fluid from infinity to the initial boundary of the cavity. Prove that the time of filling up the cavity will be*

$$\pi a\sqrt{(\rho/\varpi)}[2-(\tfrac{3}{2})^{\frac{3}{2}}]$$

a being the initial radius of the cavity, and ρ *the density of the fluid.* (*Del 1966*)

Sol. In the incompressible fluid outside the spherical cavity, the fluid velocity q will be radial, and hence a function of r: the radial distance from the centre of the cavity (the origin), and time t only. The continuity equation div $\mathbf{q}=0$, in spherical polar coordinates becomes

$$\frac{1}{r^2}\frac{d}{dr}(r^2q)=0, \Rightarrow r^2q=\text{const.}=f(t)=R^2\dot{R} \qquad \text{(say)} \tag{1}$$

where f is a function of time t only. We note that $q\to 0$, as $r\to\infty$ as required and that motion is irrotational because curl $\mathbf{q}=0$; consequently velocity potential exists and is given by $\phi=f/r=R^2\dot{R}/r$. The pressure equation for irrotational non-steady fluid motion under the conservative forces $\mathbf{F}=-\nabla\chi$ is

$$\frac{p}{\rho}-\frac{\partial\phi}{\partial t}+\chi+\tfrac{1}{2}q^2=C(t) \tag{2}$$

where $C(t)$ is an arbitrary function of time. Now $F=-\mu/r^2$, $\Rightarrow \chi=-\mu/r$. As $r\to\infty$, $p\to\varpi$, $q=f/r^2\to 0$, $\phi\to 0$, $\chi\to 0$, so that $C(t)=\varpi/\rho$, for all t. Putting for ϕ,χ,C we get from (2)

$$\frac{p}{\rho}-\frac{1}{r}\frac{\partial}{\partial t}(R^2\dot{R})-\frac{\mu}{r}+\tfrac{1}{2}q^2=\frac{\varpi}{\rho}. \tag{3}$$

Now, the *defining-relation* between ϖ and μ gives

$$\varpi\times 1=\tfrac{1}{2}\rho\int_\infty^a(-\mu/r^2)\,dr=\frac{\mu\rho}{2a};\Rightarrow \mu=\frac{2a\varpi}{\rho}. \tag{4}$$

Since $\partial(R^2\dot{R})/\partial t=R^2\ddot{R}+2R\dot{R}^2$; and when $r=R$, $p=0$, $\dot{R}=q=dR/dt$, the relation (3) yields

$$\tfrac{1}{2}\dot{R}^2-\frac{1}{R}\left[R^2\left(\frac{d\dot{R}}{dR}\,\frac{dR}{dt}\right)+2R\dot{R}^2\right]=\frac{\varpi}{\rho}+\frac{2a\varpi}{R\rho}$$

or
$$3\dot{R}^2+2R\dot{R}(d\dot{R}/dR)+(2\varpi/\rho)+(4a\,\varpi/\rho R)=0.$$

Multiply by R^2dR; the result is

$$d(R^3\dot{R}^2)+2(\varpi/\rho)R^2dR+(4a\varpi/\rho)RdR=0.$$

Integrating and employing the conditions that when $R=a$, $\dot{R}=0$, we get

$$R^3\dot{R}^2+2(\varpi/3\rho)\,R^3+2(a\,\varpi/\rho)R^2=2(\varpi/3\rho)a^3+2(a^3\varpi/\rho).$$

Thus,
$$\dot{R}^2=2(\varpi/\rho)\,[\tfrac{1}{3}(a^3-R^3)+a(a^2-R^2)]/R^3.$$

Since R decreases as t increases, $\dot{R}$ must be negative: due to the process of filling the cavity. We thus have on *inverting* the limits

$$\sqrt{\frac{2\varpi}{3\rho}}\,t=\int_0^a\frac{R^{3/2}dR}{(a-R)^{\frac{1}{2}}\,(2a+R)}=\int_0^{\frac{1}{2}\pi}\frac{2a\sin^4\theta\,d\theta}{2+\sin^2\theta}\qquad(R=a\sin^2\theta$$

$$=2a\int_0^{\pi/2}\left(\sin^2\theta+\frac{4}{2+\sin^2\theta}-2\right)d\theta$$

$$=2a\left[\frac{\pi}{4}-\pi+4\int_0^{\pi/2}\frac{d\theta}{2+\sin^2\theta}\right]$$

$$=2a\left[-\frac{3\pi}{4}+4\int_0^{\pi/2}\frac{\sec^2\theta\,d\theta}{2+3\tan^2\theta}\right]$$

$$=2a\left[-\frac{3\pi}{4}+\frac{4}{3}\sqrt{\frac{3}{2}}\cdot\frac{\pi}{2}\right]$$

$$\therefore\qquad t=\pi a\sqrt{\frac{\rho}{\varpi}}\left\{2-\left(\frac{3}{2}\right)^{3/2}\right\};\qquad\text{as required.}$$

(6) *A homogeneous incompressible fluid enclosed in a boundary which can change both in shape and area, but not in volume enclosed, is acted on by a force whose compontents are*

$$y+z+[k/(x+y+z)],\ z+x+[k/(x+y+z)],\ x+y+[k/(x+y+z)],$$

respectively; at time $t=0$, fluid is at rest, and pressure is $k\rho\log[(x+y+z)/h]$; *afterwards the pressure at the boundary is always*

$$k\rho\log[(x+y+z)/h]-\rho t^2\,(x^2+y^2+z^2+xy+yz+zx)-\rho F(t).$$

Prove that the velocity components will always be

$$t(y+z)\ ;\ t(z+x)\ ;\ t(x+y),$$

and that the curves dercribed by the particle, whose coordinates when $t=0$ were (x_0, y_0, z_0) has for its equation

$$\left(\frac{x-y}{x_0-y_0}\right)^2=\left(\frac{y-z}{y_0-z_0}\right)^2=\left(\frac{x_0+y_0+z_0}{x+y+z}\right). \qquad (Pb\ 1954)$$

Sol. FIRST PART: we find the function χ such that $\mathbf{F}=-\nabla\chi$. Since

$$-\partial\chi/\partial x=X=y+z+k/(\Sigma x), \text{ etc.} \qquad [\Sigma x=x+y+z]$$

and
$$d\chi=(\partial\chi/\partial x)\,dx+(\partial\chi/\partial y)\,dy+(\partial\chi/\partial z)\,dz$$

$\therefore$
$$-d\chi=\Sigma[y+z+k/(x+y+z)]\,dx$$

Integrating we get

$$-\chi=xy+yz+zx+k\log(x+y+z)+A=\Sigma xy+k\log(\Sigma x) \qquad (1)$$

where initial conditions being so chosen that $A=0$.

Let the velocity potential be ϕ so that $\mathbf{q}=-\nabla\phi$. Then the pressure equation for irrotational non-steady fluid motion under conservative system of body forces is

$$(p/\rho)-(\partial\phi/\partial t)+\tfrac{1}{2}q^2+\chi=C(t). \qquad (2)$$

where $C(t)$ is an arbitrary function of time only. Now

$$p=k\rho\log[(\Sigma x)/h]-\rho t^2(\Sigma x^2+\Sigma xy)-\rho F(t), \text{ at time } t,$$

and
$$p_0=k\rho\log[\Sigma x)/h] \text{ at } t=0 \text{ as given.}$$

This implies that $F(0)=0$. Putting for p and χ in (2) we get

$$k\log[(\Sigma x)/h]-t^2(\Sigma x^2+\Sigma xy)-F(t)-(\partial\phi/\partial t)+\tfrac{1}{2}q^2-\Sigma xy-k\log(\Sigma x)=C(t)$$

or
$$t^2(\Sigma x^2+\Sigma xy)+F(t)+(\partial\phi/\partial t)-\tfrac{1}{2}q^2+C(t)+\Sigma xy+k\log h=0$$

This equation is true for *all t* and hence

$$t^2(\Sigma x^2+\Sigma xy)+F(t)+\partial\phi/\partial t-\tfrac{1}{2}q^2=0 \qquad \text{(i)}$$

$$\Sigma xy+k\log h=0 \qquad \text{(ii)}$$

where we have assumed that $\partial\phi/\partial t$ is purely function of t. Should it contain terms independent of t, those would be added up in (ii). We now have

$$t^2(\Sigma x^2+\Sigma xy)-\tfrac{1}{2}q^2=-F(t)-\partial\phi/\partial t=\text{a } \textit{purely} \text{ function of } t$$

Since q may contain spatial coordinates, (3) implies

$$q^2\equiv 2t^2(\Sigma x^2+\Sigma xy)$$

or
$$u^2+v^2+w^2\equiv t^2[(y+z)^2+(z+x)^2+(x+y)^2]$$

Thus,
$$u=t(y+z),\ v=t(z+x),\ w=t(x+y).$$

SECOND PART: Curves described by the particle are given by

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}=dt$$

or
$$\frac{dx}{y+z}=\frac{dy}{z+x}=\frac{dz}{x+y}=t\,dt$$

or
$$\frac{dx-dy}{y-x}=\frac{dy-dz}{z-y}=\frac{dz-dx}{x-z}=\frac{dx+dy+dz}{2(x+y+z)}=t\,dt$$

Integrating these simultaneous equations and multiplying each member by -2 we get

$$\log (x-y)^2=\log (y-z)^2=\log (z-x)^2=-\log (x+y+z)=-t^2+\log A.$$

Thus $$(x-y)^2=Ae^{-t^2}\ ;\ A(x+y+z)=e^{t^2}, \text{ etc.}$$

when $t=0$, $x=x_0$, $y=y_0$, $z=z_0$, so that

$$(x_0-y_0)^2=A=(y_0-z_0)^2=(z_0-x_0)^2=1/(x_0+y_0+z_0).$$

Thus $$\left(\frac{x-y}{x_0-y_0}\right)^2=\left(\frac{y-z}{y_0-z_0}\right)^2=\frac{x_0+y_0+z_0}{x+y+z} \qquad (\text{each}=e^{-t^2}.).$$

(7). *Infinite inviscid liquid of constant density is attracted towards a fixed point O by a force $f(r)$ per unit mass, r being the distance from O. Initially the liquid is at rest, and there is a cavity bounded by a sphere $r=a$. If there is no pressure at infinity or in the covity, prove that the radius R of the cavity at time t is such that*

$$\frac{d}{dt}\,[R^3\dot{R}^2]+2R^2\dot{R}\int_R^\infty f(r)dr=0. \qquad (\dot{R}=dR/dt)$$

If $f(r)=\mu r^{-3/2}$, μ constant, show that the cavity will be filled up after an interval of time $(2/5\mu)^{\frac{1}{2}}\, a^{5/4}$.

Sol. In the incompressible liquid outside the cavity, the fluid velocity q will be radial, and q will be a function of r, the radial distance from the centre of the cavity (the origin). The continuity equation div q=0, in space polar coordinates becomes

$$\frac{1}{r^2}\,\frac{d}{dr}\,(r^2q)=0, \Rightarrow r^2q=\text{const.}=f(t)=R^2\dot{R} \qquad (\text{Say}) \qquad (1)$$

where $f(t)$ is a function of t only. We notice that $q\to 0$ as $r\to\infty$, as required. Clearly, curl q=0, so that the motion is irrotational. In fact, the velocity potential is $\phi=f/r$. The pressure equation for irrotational non-steady fluid motion is

$$\frac{p}{\rho}-\frac{\partial\phi}{\partial t}+\chi+\tfrac{1}{2}q^2=C\,(t) \qquad (2)$$

where $C(t)$ is an arbitrary function of time and $\mathbf{F}=-\nabla\chi$, i.e. $\chi=-\int_r f(r)\,dr$. As

$$r\to\infty,\ p\to 0,\ q=f/r^2\to 0,\ \phi\to 0,\ \chi\to 0$$

so that $C(t)=0$, for all t. At the surface of the cavity at time t, $r=R$, $p=0$ we have from (2) and (1),

$$0-\frac{\partial}{\partial t}\left(\frac{R^2\dot{R}}{R}\right)+\int_R^\infty f(r)dr+\tfrac{1}{2}\left(\frac{R^2\dot{R}}{R^2}\right)^2=0.$$

Since R is a function of t *only*, this can be written as

$$\frac{d}{dt}\,(R\dot{R})+\tfrac{1}{2}\dot{R}^2+\int_R^\infty f(r)dr=0.$$

Multiply both sides by $2R^2\dot{R}$; the first two terms being $2R^3\dot{R}\,\ddot{R}+3R^2\dot{R}^3 \Rightarrow d\,(R^3\dot{R}^2)/dt$; hence the result

$$\frac{d}{dt}(R^3\dot{R}^2)+2R^2\dot{R}\int_R^\infty f(r)dr=0 \qquad (3)$$

Now, when $f(r)=\mu r^{-3/2}$, $\int_R^\infty f(r)dr=2\mu R^{-\frac{1}{2}}$ then (3) reduces to

$$\frac{d}{dR}(R^3\dot{R}^2)\frac{dR}{dt}+4\mu R^{3/2}\frac{dR}{dt}=0$$

This clearly integrates to

$$R^3\dot{R}^2+\frac{8}{5}\mu R^{5/2}=A=\frac{8}{5}\mu a^{5/2},$$

for $\dot{R}=0$ when $R=a$. We rewrite this as

$$\dot{R}^2=\frac{8\mu}{5R^3}(a^{5/2}-R^{5/2})$$

or $$\left(\frac{8\mu}{5}\right)^{\frac{1}{2}}t=\int_0^a\frac{R^{3/2}\,dR}{\sqrt{(a^{5/2}-R^{5/2})}}=-\frac{4}{5}\left|(a^{5/2}-R^{5/2})^{\frac{1}{2}}\right|_0^a$$

i.e. $$t=(2/5\mu)^{\frac{1}{2}}a^{5/4}.$$

NOTE: The working can be shortened by using Energy Principle to obtain a first integral directly. The problem has again been worked out by this method in §2.32.

(8) *A homogeneous liquid is contained between two concentirc spherical surfaces, the radius of the inner being a and that of the outer indefinitely great. The fluid is attracted to the centre of these surfaces by a force $\phi(r)$, and a constant pressure Π is exerted at the outer surface. Suppose $\int\phi(r)\,dr=\chi(r)$, and that $\chi(r)$ vanishes when r is infinite. Show that if the inner surface is suddenly removed, the pressure at the distance r is suddenly diminished by*

$$\Pi\frac{a}{r}-\frac{a\rho}{r}\chi(a).$$

Find $\phi(r)$ so that the pressure immediatly after the inner surface is removed may be the same as it would be if no attractive force existed. Also with this value of $\phi(r)$, find the velocity of the inner bourdary of the fluid at any period of the motion.

Sol. In the incompressible liquid outside the inner spherical surface, the fluid velocity q will be radial, hence a function of r: the radial distance from the centre of the inner sphere (the origin), and time t only. The continuity equation div q=0, in spherical polar coordinates becomes

$$\frac{1}{r^2}\frac{d}{dr}(qr^2)=0,\Rightarrow r^2q=\text{const.}=f(t) \tag{1}$$

where f is a function of t only. We note that $q\to0$ as $r\to\infty$, as required. Clearly, curl q=0, so that the motion is irrotational and hence velocity potential V exists. In fact, $V=f(t)/r$. Further, the relation $\int\phi(r)dr=\chi(r)\Rightarrow$ that $\chi(r)$ is the force potential, i.e. $\phi(r)$ is a conservative system of forces. The pressure equation for irrotational non-steady fluid motion under the conservative body forces is

$$\frac{p}{\rho}-\frac{\partial V}{\partial t}+\chi+\tfrac{1}{2}q^2=C(t) \tag{2}$$

where $C(t)$ is an arbitrary function of time. Now as $r\to\infty$, $q\to 0$; $\chi(\infty)\to 0$ (as given); $\partial V/\partial t\to 0$, (as $V\to 0$) ; $p\to\Pi/\rho$; so that $C(t)=\Pi/\rho$, for all t. Putting for $V=f/r$ and $C=\Pi/\rho$ we get

$$\frac{p}{\rho}-\frac{1}{r}\ \frac{\partial f}{\partial t}+\chi+\tfrac{1}{2}q^2=\frac{\Pi}{\rho}. \tag{3}$$

Initially, at $t=0$, $q=0$, so that pressure at a distance r is given by

$$(p-\Pi)/\rho=\dot{f}(0)/r - \chi(r).$$

When $r=a$, $p=0$ at the initial stage $t=0$, so that

$$-\Pi/\rho=\dot{f}(0)/a-\chi(a)$$

Eliminating $\dot{f}(0)$ yields

$$p-\Pi=-\rho\chi(r)+a\{\rho\chi(a)-\Pi\}/r$$

Now, hydrostatic pressure p_0 (i.e. when liquid is at rest) is given by

$$dp_0=-\rho\phi(r)dr, \Rightarrow p_0=A-\rho\chi(r)=\Pi-\rho\chi(r)$$

since as $r\to\infty$, $p\to\Pi$. Thus, the *decrease* in pressure is

$$p_0-p=a[\Pi-\rho\chi(a)]$$

If no external forces exist, terms containing χ shall be missing ; and the pressure at a distance r shall be, from (4)

$$p=\Pi(1-a/r). \tag{5}$$

If now, this pressure equals that given by (4) [as provided in the problem] we must have, on equating p's,

$$\chi(r)=a\chi(a)/r \tag{6}$$

Differentiating with respect to r yields

$$\chi'(r)=\phi(r)=-a\chi(a)/r^2 : \text{ the required result.}$$

With $\chi(r)$ as given by (6) and setting $a\chi(a)=A$, $p=0$, we get from (3)

$$-[\dot{f}(t)/r]+A/r+\tfrac{1}{2}q^2=\Pi/\rho \tag{7}$$

Now $\dot{f}(t)=d(r^2q)/dt=\dot{r}\,d(r^2q)/dr=2rq^2+r^2q\,dq/dr$. Hence (7) gives

$$3q^2+r\frac{d}{dr}(q^2)=\frac{2A}{r}-\frac{2\Pi}{\rho}$$

Multiplying by r^2, the left side becomes a perfect differential of r^3q^2 ; hence integrating we get

$$r^3q^2=Ar^2-\tfrac{2}{3}(\Pi r^3/\rho)+\lambda \qquad (\lambda=\text{const.})$$

To find λ : when $r=a$, $q=0$, so that $\lambda=(2\Pi a^3/3\rho)-Aa^2$. Hence the velocity of the inner boundary of the fluid is

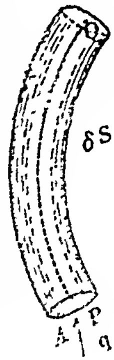
$$r^3q^2=A(r^2-a^2)+\left(\frac{2\Pi}{3\rho}\right)(a^3-r^3)$$

i.e.
$$q^2=a\chi(a)(r^2-a^2)/r^3-(2\Pi/3\rho)(1-a^3/r^3).$$

2·13. Bernoulli's theorem. For the *steady* motion of an inviscid barotropic fluid under conservative body forces, the pressure at a point is given by

$$\int \frac{dp}{\rho} + \tfrac{1}{2}q^2 + \chi = C.$$

PROOF : Consider a short and very slender cylindrical element of fluid whose axis is parallel to the flow at the element. Let the normal cross-sectional area of the cylinder be A and its length be δs, where s is the arc of the stream-line on which the element lies, measured from a fixed point on it. The pressures on the curved surface cancel out and hence contribute nothing to the resultant force in the direction of motion. The thrust on the rear end A of the element is pA in the direction of motion, while on the other end, the pressure thrust is $-[p+(\partial p/\partial s)\delta s]A$ in the direction of motion. Thus, the resultant forward thrust is $-(\partial p/\partial s) A\delta s$. Next, let the component of the body force in the direction of motion be F; the total body force on this element of mass $\rho A\delta s$ being $F\rho A\delta s$. Then if q be the velocity, the equation of motion, by Newton's second law *total propelling force = mass × acceleration of the element*, is



$$\rho A\delta s(dq/dt) = F\rho A\delta s - (\delta p/\delta s)A\delta s$$

Using $$\mathbf{F} = -\nabla\chi, \Rightarrow F = -\partial\chi/\partial s$$

we get $$dq/dt = F - (1/\rho)(\partial p/\partial s). \tag{1}$$

Since $dq/dt = \partial q/\partial t + q(\partial q/\partial s)$ and $\partial q/\partial t = 0$ for steady flow, (1) provids

$$q(\partial q/\partial s) = -\partial\chi/\partial s - (1/\rho)\partial p/\partial s$$

or $$\frac{\partial}{\partial s}\left(\tfrac{1}{2}q^2 + \int \frac{dp}{\rho} + \chi\right) = 0.$$

The rate of change of the quantity inside the bracket along the stream-line is zero, so that

$$\tfrac{1}{2}q^2 + \int \frac{dp}{\rho} + \chi = C \tag{2}$$

where C is a constant for the particular stream line (or vortex line) chosen, but varies from one stream line to the other.

NOTES. (1) If the motion is irrotational, velocity potential exists. In this particular case, C is an absolute constant.

(2) If ρ is constant, there results the simplest case :

$$\tfrac{1}{2}q^2 + (p/\rho) + \chi = C$$

(3) If the body force is due to gravity, $\chi = gh$ where h is the position

(height) above some fixed horizontal datum plane. The result (3) may then be written as

$$\tfrac{1}{2}q^2+p/\rho+gz=\text{const.}$$

or $$(q^2/2g)+(p/w)+h=\text{const.}$$

or in the language of hydraulics

velocity head+pressure head+position head=total head,

where the total head is constant along any stream line.

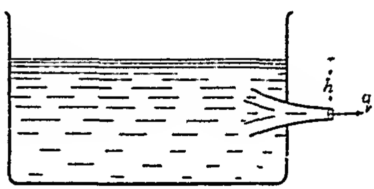
Ex. *For the steady motion of a fluid when a velocity potential does not exist, establish Bernoulli's theorem in the form*

$$\int\frac{dp}{\rho}+\frac{1}{2}q^2+\chi=C,$$

where C is constant along a stream-line. [*Pb 1958*]

Exp. 1. *Prove Bernoulli's theorem, that in a steady motion,* $\int\frac{dp}{\rho}+\frac{1}{2}q^2+\chi$; is constant along a stream line. Deduce the *theorem of Torricelli.* [*Mad 1953*]

Consider the efflux of liquid from a small orifice in one of the walls of a vessel kept filled to a constant level (giving steady motion). Let h be the depth of the *vena contracta* (the contraction), q the speed of efflux there at, and Π the atmospheric pressure. Then, by Bernoulli's theorem



$$(\Pi/\rho)+gh=(\Pi/\rho)+\tfrac{1}{2}q^2 \qquad (1)$$

because velocity is practically zero at the free surface of the water in the vessel, and the pressure is Π, both there and on the walls of the escaping jet. Hence (1) yields

$$q^2=2gh \qquad \textit{(Torricelli's theorem).}$$

Ex. 1. Hour glass. Show that the curve generating the shape of a vessel for use as hour-glass is given by $y=ax^4$.

Ex. 2. Gas flows radially from a point symmetrically in all directions, the pressure and density being connected by the law $p=k\rho$. If M is the rate of emission of mass, supposed constant, prove that

$$4\pi V r^2=M \exp[(V^2-V_1^2)/2k]$$

where V is the speed at distance r and V_1, the speed where $\rho=1$.
[*Bom 1950* ; *Kr 61* ; *Pna 65*]

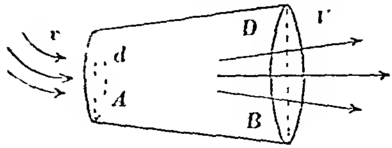
[*Hint* : Use Torricelli's theorem.]

Exp. 2. *Steam is rushing from a boiler through a conical pipe, the diameters of the ends of which are D and d; if V and v be the corresponding velocities of the stream, and if the motion be supposed to be that of divergence from the vertex of the cone, prove that*

$$\frac{v}{V}=\frac{D^2}{d^2}e^{(v^2-V^2)/2k}$$

where k is the pressure divided by the density and supposed constant.
[*Alig 1960* ; *Pb 53*]

Sol. This is a problem on Bernoulli's theorem, for the steam steadies down soon after and external forces like gravity are neglected.



As $$\int \frac{dp}{\rho}+\tfrac{1}{2}q^2=\text{const.} \qquad (1)$$

and $$p=k\rho \qquad (2)$$

$$\therefore \qquad k\int \frac{d\rho}{\rho}+\tfrac{1}{2}q^2=C.$$

Integrating, we get

$$k \log+\tfrac{1}{2}q^2=C.$$

If at the sections A and B, $\rho=\rho_1$ and ρ_2, we get

$$k \log(\rho_2/\rho_1)=\tfrac{1}{2}(v^2-V^2)$$

or $$\rho_2/\rho_1=e^{(v^2-V^2)/2k} \qquad (1)$$

From the equation of continuity, viz.

Flux across the section A=Flux across the section B,

we get $$\pi(\tfrac{1}{2}d)^2\,\rho_1 v=\pi(\tfrac{1}{2}D)^2\,\rho_2\,V$$

or $$\frac{\rho_2}{\rho_1}=\left(\frac{d}{D}\right)^2\frac{v}{V}. \qquad (2)$$

From (1) and (2) we get

$$\frac{v}{V}=\frac{D^2}{d^2}\,e^{(v^2-V^2)/2k}$$

which is the required result.

Exp. 3. *A stream in a horizontal pipe, after passing a contraction in the pipe at which its sectional area is A, is delivered at atmospheric pressure at a place where the sectional area is B. Show that if a side tube is connected with the pipe at the former place, water will be sucked up through it into the pipe from a reservoir at a depth*

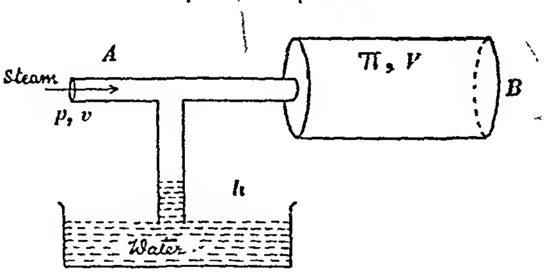
$$\frac{S^2}{2g}\left\{\frac{1}{A^2}-\frac{1}{B^2}\right\}$$

below the pipe ; S being the delivery per second.

[Alig 1960, 56 ; Del 47 ; Gti 54 ; Jad 58 ; Mad 59 ; Osm 63]

Sol. If v be the velocity in the tube of section A and p the pressure thereat ; V and Π being the corresponding quantities at the section B, then Bernoulli's theorem, viz. $(p/\rho)+\frac{1}{2}q^2=\text{const.}$ gives

$$\frac{p}{\rho}+\tfrac{1}{2}v^2=\frac{\Pi}{\rho}+\tfrac{1}{2}V^2$$



By the equation of continuity flux across the sections A and B are equal,

$$Av = BV = S : \text{delivery per second.}$$

$\therefore$ $$v = S/A, \text{ and } V = S/B$$

whence (1) gives

$$\tfrac{1}{2}S^2\left(\frac{1}{A^2}-\frac{1}{B^2}\right)=\frac{1}{\rho}\,(\Pi-p) \tag{2}$$

Now if h be the height through which water is sucked up, then

$$g\rho h = \text{difference of pressure} = \Pi - p.$$

Substituting in (2), we have

$$gh=\frac{S^2}{2}\left\{\frac{1}{A^2}-\frac{1}{B^2}\right\} \quad\text{or}\quad h=\frac{S^2}{2g}\left\{\frac{1}{A^2}-\frac{1}{B^2}\right\}.$$

Ex. Show that, if a river has a bend, the flow velocity is larger at the shore A(say) on the inner side of the bend and that the level is lower than at the shore B(say) on the outer side.

[*Hint* : Take A for the origin of cylindrical coordinates and use Bernoulli-Euler integral, viz. $(p/\rho)+\frac{1}{2}q^2+\chi=\text{const.}$]

2·20. Equation of impulsive motion. We shall now find the *relation between impulsive pressure and change of velocity.*

Let ϖ denote the impulsive pressure and $\mathbf{I}$ the extraneous impulse per unit mass of fluid. Let $\mathbf{q}_1$ and $\mathbf{q}_2$ be the velocities just before and just after the impulsive action.

Newton's second law for impulsive motion applied to the fluid within closed surface S states :

Extraneous impulse = Change of momentum.

Then, if $\mathbf{n}$ is inward unit normal we must have

$$\int_S \varpi\,\mathbf{n}\,dS+\int_V \mathbf{I}\,\rho\,dv=\int_V \rho\,(\mathbf{q}_2-\mathbf{q}_1)\,dv.$$

But $$\int_S \varpi\,\mathbf{n}\,dS=-\int_V \text{grad}\,\varpi\,dv \qquad \text{(by Gauss theorem)}$$

$\therefore$ $$\int_V\left[\mathbf{I}\,\rho-\nabla\varpi-\rho(\mathbf{q}_2-\mathbf{q}_1)\right]dv=0.$$

Since the surface is arbitary, we must have

$$\mathbf{I}-(1/\rho)\nabla\varpi=(\mathbf{q}_2-\mathbf{q}_1). \tag{1}$$

Cor. 1. *Interpretation of potential as impulsive pressure.* Let us suppose that ϕ is the velocity potential of a motion generated from rest by impulsive pressure ϖ and that external impulses are non-operative, then

$$\mathbf{I}=0\;;\; \mathbf{q}_1=0\;;\; \mathbf{q}_2=-\nabla\phi\;;$$

With these values, the above equation (1) reduces to

If ρ be constant, integration provides the result

$$\varpi = \rho\phi + \text{constant}. \tag{2}$$

The constant may be omitted, as an extra pressure, constant throughout the fluid. produces no effect on the motion.

Cor. 2. In the case of a *liquid*, ρ is constant; and if the external impulses are superficial to the liquid, $\mathbf{I}=0$. Then, taking the divergence of both members of (1) and using div $\mathbf{q}_1=0$, div $\mathbf{q}_2=0$, we get

$$\nabla^2\varpi = 0 \qquad \text{(Laplace's equation)} \tag{3}$$

Cor. 3. For the *liquid* motion started from *rest* by impulsive pressure *alone* we obtain from (1), $\mathbf{q} = -\text{grad}\,(\varpi/\rho)$; hence the motion is necessarily irrotational and velocity potential ϕ exists and is given by $\phi = \varpi/\rho$.

Cor. 4. In the absence of external impulses, (1) provides

$$\mathbf{q}_2 - \mathbf{q}_1 = -\nabla(\varpi/\rho). \tag{4}$$

Now, if the fluid motion before the action of the instantaneous forces is irrotational, i.e. $\mathbf{q}_1 = -\nabla\phi_1$, then obviously, $\mathbf{q}_2 = -\nabla[(\varpi/\rho)+\phi_1]$ so that the fluid motion remains irrotational after these forces have ceased to operate. Setting $\mathbf{q}_2 = -\nabla\phi_2$ we immediately obtain

$$\phi_2 = \phi_1 + (\varpi/\rho) + C. \tag{5}$$

If $\mathbf{q}_2 (=\nabla\phi_2)$ is costant, (4) provides,

$$\mathbf{q}_1 = \nabla(\varpi/\rho) \Rightarrow \text{curl}\,\mathbf{q} = 0. \tag{6}$$

Thus, the given irrotational motion can be established completely throughout the fluid after the action of impulsive pressure $\varpi = (\rho\phi_1 + C)$ and that it is impossible to create or destroy by rotational motion any combination of instantaneous pressure forces.

Ex. 1. Find the equations of motion of a perfect fluid under extraneous impulses and impulsive pressure. Deduce that any actual irrotational motion of a liquid can be produced instantaneously from rest by a set of impulses properly applied. [*Del 1955*]

[The single vector equation for impulsive motion, viz.

$$\mathbf{q}_2 - \mathbf{q}_1 = \mathbf{I} - \frac{1}{\rho}\nabla\varpi \tag{1}$$

is equivalent to three Cartesian equations. For the *liquid* motion started from *rest* under a set of impulses which form a conservative system ($\mathbf{I} = -\nabla J$, say), (1) may be rewritten as ($\mathbf{q}_2 = \mathbf{q}$)

$$\mathbf{q} = -\nabla J - (1/\rho)\nabla\varpi = -\nabla(J + \varpi/\rho)$$

Thus, curl $\mathbf{q}=0$ and hence actual irrotational motion is produced; the velocity potential being $J + \varpi/\rho$].

Ex. 2. If $\tilde{\omega}$ is the impusive pressure; ϕ, ϕ' the velocity potentials immediately before and after an impulse acts, V the potential of the impulses, prove that

$$\tilde{\omega}+\rho V+\rho(\phi-\phi')=\text{const.} \qquad [Del\ 1934]$$

Exp. 1. *Obtain the equation of continuity of a perfect fluid in the form* $\rho\dfrac{\partial(x, y, z)}{\partial(a, b, c)}=\rho_0$, *where the symbols have their usual meanings. Obtain also the equations of motion of fluid in Lagrange's form.*

A mass of fluid of density ρ is bounded by two concentric spherical free surfaces of radii r_1 and r_2 and the fluid being at rest, impulsive pressures $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are applied to these surfaces. Show that the surfaces begin to move with velocities

$$\frac{\tilde{\omega}_1-\tilde{\omega}_2}{\rho(r_2-r_1)}\frac{r_2}{r_1} \text{ and } \frac{\tilde{\omega}_1-\tilde{\omega}_2}{\rho(r_2-r_1)}\frac{r_1}{r_2}. \qquad [Del\ 1952]$$

Sol. In the case of a fluid with constant density ρ and no external impulses, the impulsive pressure $\tilde{\omega}$ satisfies Laplace's equation $\nabla^2\tilde{\omega}=0$, which because of spherical symmetry reduces to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\tilde{\omega}}{\partial r}\right)=0.$$

Integrating twice we get

$$\tilde{\omega}=A/r+B \qquad (1)$$

where A and B are constants of integration, to be determined shortly.

Since the fluid motion is started from *rest*, it is necessarily irrotational and the velocity potential ϕ exists and is given by $\phi=\tilde{\omega}/\rho$. Hence

$$\mathbf{q}=-\nabla(\tilde{\omega}/\rho) \Rightarrow q=-\frac{1}{\rho}\frac{\partial}{\partial r}\left(\frac{A}{r}+B\right)=\frac{A}{\rho r^2}.$$

Thus the two velocities with which the surfaces begin to move are $A/\rho r_1^2$ and $A/\rho r_2^2$. And to determine the unknown constant A, we use the conditions that at $r=r_1$, $\tilde{\omega}=\tilde{\omega}_1$; and at $r=r_2$, $\tilde{\omega}=\tilde{\omega}_2$. We then get from (1)

$$\tilde{\omega}_1=A/r_1+B\,;\ \tilde{\omega}_2=A/r_2+B$$

so that $A=(\tilde{\omega}_1-\tilde{\omega}_2)\,r_1r_2/(r_2-r_1)$. Thus

$$q_1=\frac{(\tilde{\omega}_1-\tilde{\omega}_2)}{\rho(r_2-r_1)}\frac{r_2}{r_1}, \quad q_2=\frac{(\tilde{\omega}_1-\tilde{\omega}_2)}{\rho(r_2-r_1)}\frac{r_1}{r_2}.$$

Exp. 2. *Prove that if the boundaries of a liquid at rest are suddenly set in motion the resulting motion of the liquid is irrotational.*

A cylinder of any form of section is filled with fluid and is suddenly given an angular velocity ω about any point. Show that if the kinetic energy acquired by the fluid is $\frac{1}{2}I\omega^2$, the angular momentum acquired is $I\omega$.

Sol. A motion generated from rest by impulsive pressure $\tilde{\omega}$ by boundaries only is necessarily *irrotational* (vide Cor 3, p. 95). Let ϕ be the velocity potential so that $\mathbf{q}=-\text{grad}\,\phi$. If $\mathbf{h}$ is the angular momentum, (i.e. moment of momentum) about the point O, then

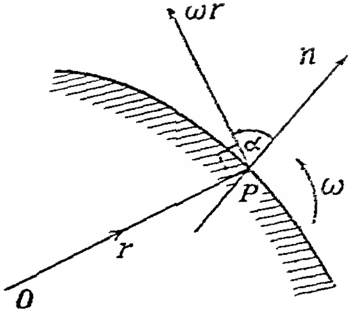
$$\mathbf{h}=\int_V \mathbf{r}\times(\rho\mathbf{q}\,dv)=-\rho\int_V \mathbf{r}\times\text{grad}\,\phi\,dv \qquad (1)$$

the integral being taken through the interior of the cylinder. Since it is only on the boundary that the velocity ωr is known, we need transform (1) to a surface integral. Now

$$\mathbf{r} \times \text{grad}\,\phi = \text{grad}\,(\tfrac{1}{2}r^2) \times \text{grad}\,\phi = -\text{curl}\,(\phi\ \text{grad}\,\tfrac{1}{2}r^2) \tag{2}$$

because *curl grad* $(\frac{1}{2}r^2) \equiv 0$.

Hence from (1) and (2) we get



$$\mathbf{h} = \rho \int_V \text{curl}\,(\phi\ \text{grad}\,\tfrac{1}{2}r^2)\,dv$$

$$= \rho \int_S \mathbf{n} \times (\phi\,\text{grad}\,\tfrac{1}{2}r^2)\,dS$$

by Gauss's theorem

$$= \int_S (\mathbf{n} \times \phi \mathbf{r})\,dS \qquad [\because \text{grad}\,(\tfrac{1}{2}r^2) = \mathbf{r}] \quad (3)$$

where $\mathbf{n}$ is the unit normal to the surface drawn *out* of the liquid. If α is the angle between the directions of $\mathbf{n}$ and the velocity $\mathbf{w} \times \mathbf{r}$ at the point P of the boundary, (3) may be written as

$$h = \rho \int_S \phi r \sin(90 + \alpha)\,dS = -\rho \int_S \phi r \cos \alpha\,dS$$

Since the kinetic energy of the irrotational motion is

$$T = \tfrac{1}{2}\rho \int_S \phi \frac{\partial \phi}{\partial n}\,dS$$

and on the boundary, the *inward normal velocity* $\partial\phi/\partial n = -r\omega \cos \alpha$.

Hence $\qquad T = -\frac{1}{2}\omega\rho \int_S \phi r \cos \alpha\,dS = \frac{1}{2}\omega h$ by (4)

Thus if $T = \frac{1}{2}I\omega^2$, we must have $h = I\omega$.

Exp. 3. A sphere of radius a is surrounded by infinite liquid of density ρ, the pressure at infinity being Π. The sphere is suddenly annihilated. Show that the pressure at distance r from the centre immediately falls to

$$\Pi(1 - a/r).$$

Show further that if the liquid is brought to rest by impinging on a concentric sphere of radius $\frac{a}{2}$, the impulsive pressure sustained by the surface of this sphere is

$$\sqrt{7\Pi\rho a^2/6}$$

[*Ag 1964, 58 ; Alg 63; Bcm 63 (old), 61, 53 ; Del 65, 39 ; Jab 59; Jad 59 ; Mar 62 ; Pbi 66 ; Raj 63 ; Ut 62*]

Sol. In the incompressible liquid outside the sphere, the fluid velocity is radial and hence a function of r, the radial distance from the centre of the sphere (the origin) and t only. The continuity equation div $\mathbf{q} = 0$, in polar spherical coordinates become

$$\frac{1}{r^2}\frac{d}{dr}(r^2 q) = 0, \Rightarrow r^2 q = \text{const.} = f(t) = R^2 \frac{dR}{dt} \tag{1}$$

where $f(t)$ is a function of time t only. We observe that $q \to 0$ as $r \to \infty$, as required. Clearly, *curl* $\mathbf{q} = 0$, so that the motion is irrotational and hence the velocity potential ϕ exists and is given by $\phi = f/r$. The pressure equation for irrotational non-steady fluid motion, in the absence of body forces, is

$$\frac{p}{\rho} - \frac{\partial \phi}{\partial t} + \frac{1}{2} q^2 = C(t) \tag{2}$$

where $C(t)$ is an arbitrary function of time t. As $r \to \infty$, $p = \Pi$, $q = f/r^2 \to 0$, $\phi \to 0$ so that $C(t) = \Pi/\rho$, for all t. Putting for ϕ and C in (2) we get

$$\frac{p}{\rho} = \frac{\Pi}{\rho} + \frac{1}{r}\frac{\partial f(t)}{\partial t} - \frac{1}{2}\frac{f^2}{r^4}. \tag{3}$$

At time $t=0$, the pressure p_0 (when $q=0$) is given by

$$\frac{p_0}{\rho}=\frac{\Pi}{\rho}+\frac{1}{r}\left(\frac{\partial f}{\partial t}\right)_0. \tag{4}$$

Also when $r=a$ at time $t=0$, $p=0$, so that

$$0=\frac{\Pi}{\rho}+\frac{1}{a}\left(\frac{\partial f}{\partial t}\right)_0. \tag{4'}$$

Eliminating $(\partial f/\partial t)_0$ between (4) and (4′), we get

$$p_0=\Pi(1-a/r).$$

SECOND PART. Since $p=0$ at the surface of hollow sphere of radius r, we get from (3) on using (1),

$$\frac{\Pi}{\rho}+\frac{1}{r}\frac{\partial}{\partial t}(r^2 q)-\tfrac{1}{2}q^2=0.$$

Now $d(r^2q)/dt=2rq^2+rqdq/dr$, so that above may be expressed as

$$\frac{d}{dr}\left(r^3q^2\right)=-\frac{2\Pi}{\rho}r^2$$

Integrating and applying the conditions that at $r=a$, $q=0$, we get

$$r^3q^2=(2\Pi/3\rho)(a^3-r^3).$$

To get the velocity at the surface of the sphere $r=\frac{1}{2}a$, we put $r=a/2$, and get

$$q^2=14\Pi/3\rho. \tag{5}$$

From equations of impulsive motion, viz.

$$\mathbf{q}_2-\mathbf{q}_1=\mathbf{I}-\rho^{-1}\nabla\tilde{\omega}$$

putting $\mathbf{q}_2=0$ (liquid brought to rest); $\mathbf{I}=0$ (no extraneous impulses), we get $\mathbf{q}_1=\nabla(\tilde{\omega}/\rho)$ which, by virtue of (5), gives

$$\frac{d\tilde{\omega}}{dr}=\rho q=\rho\sqrt{\frac{14\Pi}{3\rho}}$$

$$\therefore \qquad \tilde{\omega}=\sqrt{\frac{14\Pi\rho}{3}}\int_0^{a/2}dr=\sqrt{\frac{7\rho\Pi a^2}{6}}.$$

Exp. 4. A portion of homogeneous liquid is confined between two concentric spheres of radii A and a ; and is attracted towards their centre by a force varying inversely as the square of the distance. The inner spherical surface is suddenly annihilated, and when the radii of the inner and outer surfaces of the fluid are r and R, the liquid impinges on a solid ball concentric with their surfaces ; prove that the impulsive pressure at any point of the ball for different values of r and R varies as

$$\sqrt{(a^2-r^2-A^2+R^2\left(\frac{1}{r}-\frac{1}{R}\right)}.$$

[*Del 1954 ; Mad 58 ; Osm 59 ; Raj 65*]

Sol. In the incompressible liquid outside the inner spherical surface, the fluid velocity q will be radial and hence a function of η : the radial distance from the centre of the spherical surface (the origin), and at time t. The continuity equation div $\mathbf{q}=0$, in spherical polar coordinates becomes

$$\frac{1}{\eta^2}\frac{d}{dr}(\eta^2 q)=0, \Rightarrow \eta^2 q=\text{const.}=f(t)=x^2\frac{dx}{dt}\text{ (say)} \tag{1}$$

where f is a function of t only. We note that *curl* $\mathbf{q}=0$, so that the motion is irrotational and hence velocity potential exists and is given by $\phi=f/\eta=x^2(dx/dt)/\eta$. The pressure equation for irrotational non-steady flow under the conservative body force $-\mu/\eta^2$, $(-\nabla\chi=\mathbf{F}$ and hence force potential $\chi=-\mu/\eta)$, is

$$\frac{p}{\rho}-\frac{\partial\phi}{\partial t}+\chi+\tfrac{1}{2}q^2=C(t)$$

where $C(t)$ is an arbitrary function of time. Putting for ϕ and χ we get

$$\frac{p}{\rho}-\frac{1}{\eta}\frac{\partial f}{\partial t}-\frac{\mu}{\eta}+\tfrac{1}{2}q^2=C(t). \tag{2}$$

Now, when $\eta=r$, $q=\dot{r}$, $p=0$; and also when $\eta=R$, $q=\dot{R}$, $p=0$ again.

Hence, the above equation, when $C(t)$ is eliminated, gives

$$\left(\frac{1}{r}-\frac{1}{R}\right)\frac{\partial f}{\partial t}+\mu\left(\frac{1}{r}-\frac{1}{R}\right)+\tfrac{1}{2}(\dot{R}^2-\dot{r}^2)=0. \tag{3}$$

Now $\partial f/\partial t=(df/dr)\dot{r}=d(r^2\dot{r})/dr$. Also, since mass between the two concentric spheres is conserved, we have

$$(4/3)\pi R^3\rho-(4/3)\pi r^3\rho=(4/3)\pi A^3\rho-(4/3)\pi a^3/\rho.$$

This gives : $R^3-r^3=A^3-a^3=c^3$ so that $R^3=r^3+c^3$. From this also follows $R^2\dot{R}=f=r^2\dot{r}$ which is otherwise evident. Thus, (3) may be written as

$$\left(\frac{1}{r}-\frac{1}{R}\right)\dot{r}\frac{d}{dr}(r^2\dot{r})-\tfrac{1}{2}\dot{r}^2\left(1-\frac{r^4}{R^4}\right)=-\mu\left(\frac{1}{r}-\frac{1}{R}\right)$$

Multiplying by r^2 throughout, then observing that $r^2\dot{r}=f$, the above may be rewritten as

$$\left(\frac{1}{r}-\frac{1}{R}\right)\frac{d}{dr}(\tfrac{1}{2}f^2)-\tfrac{1}{2}.\frac{f^2}{r^2}\left(1-\frac{r^4}{R^4}\right)=-\mu r^2\left(\frac{1}{r}-\frac{1}{R}\right)$$

or

$$\frac{d}{dr}\left(\frac{1}{r}-\frac{1}{R}\right)(\tfrac{1}{2}f^2)=-\mu r^2\left(\frac{1}{r}-\frac{1}{R}\right)$$

$$=-\mu\left(r-\frac{r^2}{(r^3+c^3)^{1/3}}\right)\left[\text{as } \frac{dR}{dr}=\frac{r^2}{R^2}\right]$$

Putting $f=r^2\dot{r}$, we get

$$\left(\frac{1}{r}-\frac{1}{R}\right)r^4\dot{r}^2=[r^2-(r^3+c^3)^{2/3}]+\lambda$$

Initially, at the instant $t=0$, $r=a$, $R=A$, $(dr/dt)=0$, so that $\lambda=\mu(a^2-A^2)$.

Thus,
$$r^4\left(\frac{dr}{dt}\right)^2\left(\frac{1}{r}-\frac{1}{R}\right)=\mu(a^2-r^2-A^2+R^2). \tag{4}$$

From the usual relation, $\mathbf{q}_2-\mathbf{q}_1=\mathbf{I}-(1/\rho)\nabla\varpi$, we find that when the liquid impinges on a solid ball, $\mathbf{q}_2=0$; also $\mathbf{I}=0$, so that $\mathbf{q}=(1/\rho)\nabla\varpi$ or $d\varpi/d\eta=\rho\dot{\eta}$,

Thus
$$\frac{d\varpi}{d\eta}=\rho\frac{d\eta}{dt}=\frac{\rho\eta^2(d\eta/dt)}{\eta^2}=\frac{\rho r^2(dr/dt)}{\eta^2}.$$

The impulsive pressure is accordingly

$$\varpi=\rho r^2\frac{dr}{dt}\int_r^R\frac{d\eta}{\eta^2}=\rho r^2\frac{dr}{dt}\left(\frac{1}{r}-\frac{1}{R}\right) \tag{5}$$

Putting for $r^2\dot{r}$ from (4) into (5) yields

$$\varpi=\rho\sqrt{\mu(a^2-r^2-A^2+R^2)\left(\frac{1}{r}-\frac{1}{R}\right)}$$

Hence ϖ varies as given.

Ex. 1. Prove that if ϖ be the impulsive pressure, ϕ, ϕ' the velocity potentials immediately before and after an impulse acts, V the potential of the impulses,

$$\varpi+\rho V+\rho(\phi'-\phi)=\text{const.}$$

Ex. 2$_1$. Show that in the absence of extraneous impulses, the impulsive pressure at any point of a liquid satisfies Laplace's equation.

An explosion takes place at a point O at some distance below the surface of deep water. If O' is the image of O in the free surface, show that the velocity potential of the initial motion at any point P varies as

$$(1/OP)-(1/O'P).$$

Determine the initial velocity of the free surface at any point.

[*Lkn 1960 ; Mad 60*]

2$_2$. If a bomb shell explode at a great depth beneath the surface of the sea, prove that the impulsive pressure at any point varies inversely as the distance from the centre of the shell. [*Ban 1953 ; Pb 61 ; I.A.S. 51*]

Ex. 3. Prove that the acceleration field of an inviscid fluid in motion is

$$\mathbf{F}-(1/\rho)\nabla p$$

where $\mathbf{F}$ is the specific body force, ρ is the density and p is the pressure.

Show that if a body immersed in an incompressible inviscid fluid at rest is *jerked* into motion, the resulting flow is initially irrotational. [Lkn 1956]

Ex. 4. A mass of liquid surrounds a solid sphere of radius a and its outer surface which is a concentric sphere of radius b, is subject to a given constant pressure p_0, no other forces being in action on the liquid. The solid sphere suddenly shrinks into a concentric sphere. Find the subsequent motion and the impulsive action on the sphere.

[*Ag 1961, 53 ; Ald 64 ; Cal 56 ; Del 34 ; I.A.S. 54*]

Ex. 5. Show that the impulsive pressure at a point in an incompressible fluid of density ρ is $\rho(\phi_2-\phi_1)$ where ϕ_1 and ϕ_2 are the velocity potentials of the motions just before and just after the impulses have acted.

A solid sphere of mass M and density ρ_1 is at rest in an infinite liquid of density ρ. Show that an impulse $Mu\,(2\rho_1+\rho)/2\rho_1$, is required to set the sphere in motion with velocity u. [Del 1940]

2.30. *Intrinsic or elastic strain energy.* It is the energy stored in the fluid by virtue of compression and is analogous to the one stored in a stretched elastic string. *Intrinsic energy 'E' per unit mass measures the work done by unit mass of the fluid against external pressure, as it passes, under the supposed relation between p and ρ, from its actual state to some standard state in which the pressure and density are p_0 and ρ_0.* For the incompressible fluid, $E=0$.

Since the work done in changing the shape of any volume V to V_0 is $\int_V^{V_0} p\,dv$ and $\rho=mv$, we may set

$$\text{work done}=\int_V^{V_0} p\,dv=\int_\rho^{\rho_0} p\,d(m/\rho).$$

Hence $$E=\int_\rho^{\rho_0} p\,d(1/\rho) \qquad \text{since } m=1.$$

The total intrinsic energy of a fluid body is often called *internal energy* and is obviously given by $\int_V \rho E\,dv$.

Ex. Show that the rate per unit of time at which work is done by the internal pressure between the parts of a compressible fluid is

$$\int p(\nabla\cdot\mathbf{q})\,dv$$

where p is the pressure, and $\mathbf{q}$ the velocity at any point, and the integration extends through the volume of the fluid.

2.31. *The Energy equation.* We shall now show that the *rate of change of total energy (kinetic, potential and intrinsic) of any portion of a compressible inviscid fluid as it moves about is equal to the rate at which work is being done by the pressure on the boundary. The potential due to the extraneous forces is supposed to be independent of time.*

We multiply scalarly both sides of the Euler's equation of motion under conservative system of forces, viz.

$$\frac{d\mathbf{q}}{dt}=-\nabla\Omega-\frac{1}{\rho}\nabla p$$

with $\rho\mathbf{q}$ so as to obtain

$$\rho\frac{d}{dt}(\tfrac{1}{2}q^2)+\rho\mathbf{q}.\nabla\chi=-\mathbf{q}\cdot\nabla p$$

Since $$\frac{d\chi}{dt}=\frac{\partial\chi}{\partial t}+(\mathbf{q}.\nabla)\chi=(\mathbf{q}.\nabla)\chi \text{ as } \frac{\partial\chi}{\partial t}=0$$

by hypothesis, the above may be rewritten as

$$\rho\frac{d}{dt}\left(\tfrac{1}{2}q^2+\chi\right)=-\mathbf{q}.\nabla p=-\nabla.(p\mathbf{q})+p\nabla\cdot\mathbf{q} \tag{1}$$

Integrating both sides of (1) we get

$$\frac{d}{dt}\left\{\int_V(\tfrac{1}{2}q^2+\chi)\rho dv\right\}=-\int_V\nabla.(p\mathbf{q})dv+\int_V p(\nabla.\mathbf{q})dv \tag{2}$$

the left hand side being valid as $d(\rho dv)/dt=0$ by continuity condition. By virtue of the divergence theorem and the equation of continuity the right side of (2) may be simplified to yield

$$\frac{d}{dt}\left\{\int_V(\tfrac{1}{2}q^2+\chi)\rho dv\right\}=+\int_S\mathbf{n}.p\mathbf{q}dS-\int_V\frac{p}{\rho}\frac{d\rho}{dt}dv \tag{3}$$

where $\mathbf{n}$ is unit *inward* normal. Now, by definitions

$$T=\int_V\tfrac{1}{2}\rho q^2dv: \quad V=\int_V \rho\chi dv\,;\, I=\int_V\rho E dv \tag{4}$$

are the kinetic, potential and intrinsic (internal) energies respectively, then (3) may be written as

$$\frac{d}{dt}(T+V)=\int_S(\mathbf{n}\,.\,\mathbf{q})pdS-\frac{dI}{dt} \tag{5}$$

because

$$\frac{dI}{dt}=\int_V\frac{dE}{dt}\rho dv \qquad [\text{by (4) and } \frac{d}{dt}(\rho v)=0]$$

$$=\int_V\frac{dE}{d\rho}\,\frac{d\rho}{dt}\rho dv$$

$$=\int\frac{p}{\rho}\,\frac{d\rho}{dt}dv\ \left[\text{as } E=\int_\rho^{\rho_0}pd\left(\frac{1}{\rho}\right)=\int_{\rho_0}^{\rho}\frac{p}{\rho^2}\,d\rho\Rightarrow\frac{dE}{d\rho}=\frac{p}{\rho^2}\right].$$

Also, the work done by the fluid pressure on an element dS being $pdS\mathbf{n}\,d\mathbf{r}$ and the rate at which this is being done is $pdS\mathbf{n}.\mathbf{q}$, ($\mathbf{q}=d\mathbf{r}/dt$), it follows that for the space of volume V, the rate at which work is being done by the fluid pressure is $\int_S(\mathbf{n}.\mathbf{q})pdS=R$ (say). Thus (5) may be put as

$$\frac{d}{dt}(T+V+I)=R. \tag{6}$$

The statement embodied in (6) is what we were interested in and is often quoted as "the *Volume integral form of Bernoulli's equation.*"

Ex. 1. Establish the energy equation

$$\frac{d}{dt}(T+V+I)=\int p(\mathbf{q}.\mathbf{n})dS$$

for a compressible inviscid fluid. *(Ban 1965)*

Ex. 2. Prove that the rate of change of total energy (kinetic, potential and intrinsic) of any portion of a compressible inviscid fluid as it moves about is equal to the rate at which work is being done by the pressure on the boundary. [The potential due to the extraneous forces is supposed to be independent of time]. [*Del 1958, 55*]

Exp. 1. *Show that for irrotational motion with velocity potential* ϕ, *the kinetic energy of a liquid enclosed in a stream tube with a small cross-section between the normal sections* $\phi=a$, $\phi=b$ *can be expressed as* $\frac{1}{2}M(b-a)$, *where* M *is the mass of fluid flowing per second through the tube.*

Sol. Let S_1 and S_2 be the areas of the sections of the stream tube. Since kinetic energy T of the liquid enclosed by the surface S is given by $T=-\frac{1}{2}\rho\int\phi(\partial\phi/\partial n)dS$, we get

$$T=-\tfrac{1}{2}\rho[a(\partial\phi/\partial n)_1S_1+b(\partial\phi/\partial n)_2S_2] \tag{1}$$

where the contribution to T due to lateral surface of the tube is neglected, as along that part, $(\partial\phi/\partial n)=0$. Now $(\partial\phi/\partial n)_1=q_1$, $(\partial\phi/\partial n)_2=-q_2$, whence (1) reduces to

$$T=\tfrac{1}{2}\rho(q_2S_2b-q_1S_1a)$$

However, $\rho q_1S_1=\rho q_2S_2=M$ (mass) : by Continuity equation, therefore,

$$T=\tfrac{1}{2}M(b-a).$$

Exp. 2. *A space is bounded by an ideal fixed surface S drawn in a homogeneous incompressible fluid satisfying the conditions for the continued existence of a velocity potential* ϕ *under conservative forces. Prove that the rate per unit time at which energy flows across S into the space bounded by S is*

$$-\rho\int\int\frac{\partial\phi}{\partial t}\cdot\frac{\partial\phi}{\partial n}dS$$

where ρ *is the density and* δn *an element of the normal to dS drawn into the space considered.* [*Del 1959 ; Gti 61, 58*]

Sol. By an *ideal* surface we mean a surface which is free from any hydrodynamical singularity, i.e. there are no sources, sinks or vortices in the region enclosed by it.

Since the fluid is incompressible, its intrinsic energy (i.e. energy stored in the fluid by compression) is zero. If χ is the force-potential (i.e. $\mathbf{F}=-\nabla\chi$) then its potential energy shall be $\rho\int\chi dv$. We shall take it that χ is independent of t, so that $\partial\chi/\partial t=0$. Now the total energy $\mathcal{E}$ is

$$\mathcal{E}\equiv K.E.+P.E.+I.E.=\tfrac{1}{2}\int\rho q^2dv+\rho\int\chi dv+0$$

$$\therefore \quad \frac{\partial\mathcal{E}}{\partial t}=\int_V\mathbf{q}\cdot\dot{\mathbf{q}}dv+0 \quad [\partial\chi/\partial t=0]$$

$$=\rho\int_V(\nabla\phi)\cdot(\nabla\frac{\partial\phi}{\partial t})dv, \quad [\mathbf{q}=-\nabla\phi \text{ ; irrotational motion}]$$

$$=-\rho\int_V\frac{\partial\phi}{\partial t}\cdot\nabla^2\phi dv-\rho\int_S\frac{\partial\phi}{\partial t}\cdot\frac{\partial\phi}{\partial n}dS \quad \text{by Green's theorem}$$

$$=-\rho\int_S\frac{\partial\phi}{\partial t}\cdot\frac{\partial\phi}{\partial n}dS \qquad [\text{Since } \nabla^2\phi=0].$$

2.32. *Applications of dynamical principles.* The ordinary laws of dynamics may often be used with advantage to shorten the solution of a particular problem. Among the basic principles to be adopted is one that relates to energy and states :

The rate of increase of energy in a system is the rate at which work is done on the system : or increase in energy = work done.

The following examples illustrate the above remarks.

Exp. 1. An infinite mass of fluid is acted on by a force ($\mu/r^{3/2}$) *per unit mass directed to the origin. If initially the fluid is at rest and there is a cavity in the form of a sphere $r=a$ in it, show that the cavity will be filled up after an interval of time*

$$(2/5\mu)^{1/2}a^{5/4}$$

[*Ag 1955, 51 ; Alig 64, 58 ; Ald 63 ; Ban 64 ; Bom 62, 55 ; Del 62, 38 ; Gti 64 ; Gor 59 ; Kar 59 ; Lkn 63 ; Mad 59 ; Pb 58, 52 ; Raj 60.*]

Sol. In the incompressible liquid outside the cavity, the fluid velocity q will be radial and hence function of r: the radial distance from the centre of the cavity (the origin). The continuity equation div $\mathbf{q}=0$, in space polar co-ordinates becomes

$$\frac{1}{r^2}\frac{d}{dr}(r^2q)=0, \Rightarrow r^2q=\text{const.}=f(t)=R^2\dot{R} \text{ (say)} \tag{1}$$

where $f(t)$ is a function of t only. We notice that $q\to 0$ as $r\to\infty$, as required. Clearly, *curl* $\mathbf{q}=0$, so that motion is irrotational and hence velocity potential exists and is given by $\phi=f/r$. The kinetic energy, at the instant when the radius of the cavity is R, is

$$T=\tfrac{1}{2}\rho\int\phi\frac{\partial\phi}{\partial n}dS=-\tfrac{1}{2}\rho\left|\phi\frac{\partial\phi}{\partial r}\cdot 4\pi r^2\right|_{r=R}$$

because, ϕ and $\partial\phi/\partial r$ are constant over $r=R$. Thus,

$$T=2\pi\rho R^3\dot{R} \qquad \text{[using (1)].} \tag{2}$$

The rate of flow must be the same across every concentric spherical surface; across the one of radius r is, $-4\pi r^2 dr/dt$ (*negative*, as flow is *inwards*).

Thus, the total rate of doing work is

$$\rho\int_R^\infty 4\pi r^2\frac{dr}{dt}\frac{\mu}{r^{3/2}}dr=\rho\int_R^\infty 4\pi R^2\dot{R}\frac{\mu}{r^{3/2}}dr \quad (\because r^2\dot{r}=R^2\dot{R})$$

$$=4\rho\pi R^2\dot{R}(2\mu/R^{1/2})=8\mu\rho\dot{R}R^{3/2}\pi \tag{3}$$

Initial value of kinetic energy is zero; intrinsic energy is zero due to incompressibility and flow is determined only by virtue of motion and hence potential energy is also zero. Thus, the rate of increase of kinetic energy is equal to the rate of doing work. This gives

$$\frac{d}{dt}(2\pi\rho R^3\dot{R}^2)=-8\pi\mu\rho\dot{R}R^{2/2} \text{ or } \frac{d}{dr}(R^3\dot{R})+4\mu R^{3/2}=0.$$

Integrating we get

$$R^3\dot{R}^2=A-(8/5)\mu R^{5/2}=(8\mu/5)[a^{5/2}-R^{5/2}]$$

because $\dot{R}=0$ when $R=a$. The above further gives

$$\frac{dR}{dt}=-\left(\frac{8\mu}{5}\right)^{1/2}\left(\frac{a^{5/2}-R^{5/2}}{R^3}\right)^{1/2}$$

whence $t=\left(\dfrac{5}{8\mu}\right)^{1/2}\displaystyle\int_0^a\frac{R^{3/2}dR}{\sqrt{a^{5/2}-R^{5/2}}}=\frac{4}{5}\left(\frac{5}{8\mu}\right)^{1/2}a^{5/4}\int_0^{\frac{1}{2}\pi}\sin\theta\,d\theta$

Thus $t=(2/5\mu)^{1/2}a^{5/4}$.

NOTES : (1) The kinetic energy could also be determined as :

$$T=\tfrac{1}{2}\rho\int_R^\infty 4\pi r^2\dot{r}^2\,dr \qquad (r^2\dot{r}=R^2\dot{R})$$

$$=2\pi\rho R^4\dot{R}^2\int_R^\infty\frac{dr}{r^2}=2\pi\rho R^3\dot{R}^2.$$

(2) The solution may also be obtained by the general method for irrotational fluid motion. *See* Problem 7, p 88.

Exp. 2. An infinite fluid in which is a spherical hollow of radius a is initially at rest under the action of no forces. If a constant pressure Π is applied at infinity, show that the time of filling up the cavity is

$$\pi^2 a\left(\frac{\rho}{\Pi}\right)^{1/2}(2)^{5/6}[\Gamma(\tfrac{1}{3})]^{-3}.$$

[*Ag 1960, 56, 46 ; Ban 56, 54 ; Alig 57 ; Del 48 ; Gti 63 ; Gor 60 ; Jad 60 ; Lkn 59 ; Osm 61 ; Pb 64 ; I.A.S. 1952.*]

Sol. In the incompressible fluid outside the hollow sphere, the fluid velocity q will be radial and hence function of r : the radial distance from the centre of the sphere (the origin). The continuity equation div $\mathbf{q}=0$, in space polar coordinates, becomes

$$\frac{1}{r^2}\frac{d}{dr}(r^2 q)=0, \quad \Rightarrow \quad r^2 q=\text{const.}=f(t)=R^2\dot{R} \text{ (say)} \tag{1}$$

where $f(t)$ is a function of t only. We observe that $q \to 0$ as $r \to \infty$, as required. Clearly *curl* $\mathbf{q}=0$, so that motion is irrotational and hence velocity potential exists and is given by $\phi=f/r$. If T be the kinetic energy, then

$$T=\tfrac{1}{2}\rho\int \phi\,\frac{\partial\phi}{\partial n}\,dS=-\tfrac{1}{2}\rho\left[\phi\,\frac{\partial\phi}{\partial r}\,4\pi r^2\right]_{r=R}$$

because, ϕ and $\partial\phi/\partial r$ are constant over $r=R$. Thus,

$$T=2\pi\rho R^3\dot{R}^2 \qquad \text{[using (1)]} \tag{2}$$

The liquid motion starts from rest and so this energy is the work done by the pressure in expanding the cavity from initial radius a to R.

The work done by pressure p in expanding from radius R to $R+dR$ is $4\pi R^2 p\,dR$, and here $p=-\Pi$, giving

$$\text{total work}=-\int_a^R 4\pi R^2\Pi\,dR=\frac{4\pi}{3}\Pi(a^3-R^3) \tag{3}$$

Equating (2) and (3) gives

$$\dot{R}^2=\frac{2\Pi}{3\rho}\,\frac{a^3-R^3}{R^3}.$$

Thus, $$\frac{dR}{dt}=-\left(\frac{2\Pi}{3\rho}\right)^{1/2}\left(\frac{a^3-R^3}{R^3}\right)^{1/2} \text{ or } t=\left(\frac{3\rho}{2\Pi}\right)^{1/2}\int_0^a \frac{R^{3/2}\,dR}{(a^3-R^3)^{1/2}}.$$

Putting $r^3=a^3\sin^2\theta$, the above reduces to

$$t=\div\sqrt{\frac{3\rho}{2\Pi}}\cdot\frac{2a}{3}\int_0^{\frac{1}{2}\pi}\sin^{2/3}\theta\,d\theta$$

$$=\frac{2a}{3}\sqrt{\frac{3\rho}{2\Pi}}\cdot\frac{\Gamma(\frac{5}{6}).\Gamma(\frac{1}{2})}{2\Gamma(\frac{8}{6})}$$

$$=a\sqrt{\frac{3\rho}{2\Pi}}\cdot\frac{\Gamma(\frac{5}{6}).\sqrt{\pi}}{\Gamma(\frac{1}{3})} \tag{4}$$

Now from Integral Calculus

$$\Gamma(p).\Gamma(p-\tfrac{1}{2})=\frac{\sqrt{\pi}.\Gamma(2p)}{2^{2p-1}} \qquad \textit{(Duplication formula)} \tag{i}$$

and $$\Gamma(p).\Gamma(1-p)=\frac{\pi}{\sin p\pi} \tag{ii}$$

Putting, $p=\frac{1}{3}$ in (i) and (ii), we get

$$\Gamma(\tfrac{1}{3}).\Gamma(\tfrac{5}{6})=\frac{\sqrt{\pi}\,\Gamma(\frac{2}{3})}{2^{\frac{2}{3}-1}} \quad \text{and} \quad \Gamma(\tfrac{1}{3}).\Gamma(\tfrac{2}{3})=\frac{2\pi}{\sqrt{3}}$$

$$\therefore \qquad \Gamma(\tfrac{5}{6})=\frac{2\sqrt{\pi}}{2^{\frac{2}{3}}}\cdot\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}=\frac{2\sqrt{\pi}}{2^{\frac{2}{3}}}\cdot\frac{2\pi}{\sqrt{3}}\cdot\frac{1}{[\Gamma(\frac{1}{3})]^2}.$$

Substituting, in (4) we get

$$t=a\sqrt{\frac{3\rho}{2\Pi}}\cdot\frac{\sqrt{\pi}}{[\Gamma(\frac{1}{3})]^3}\cdot\frac{2\pi}{\sqrt{3}}\cdot\frac{2\sqrt{\pi}}{2^{\frac{2}{3}}}\cdot$$

$$=\pi^2 a\left(\frac{\rho}{\Pi}\right)^{\frac{1}{2}}\cdot(2)^{\frac{5}{6}}\cdot\Gamma(\tfrac{1}{3})]^{-3}$$

Ex. 1. An infinite mass of liquid acted upon by no forces is at rest, and a spherical portion of radius c is suddenly annihilated; the pressure ϖ at an infinite distance being supposed to remain constant, prove that the pressure at the distance r from the centre of the sphere is instancaneously diminished in the ratio $r-c : r$, and that the cavity will be filled up in time

$$\left(\frac{\pi\rho\, c^2}{6\varpi}\right)^{\frac{1}{2}}\frac{\Gamma(5/6)}{\Gamma(4/3)}. \qquad \text{[Del 1932]}$$

Ex. 2. Determine the pressure at the fluid particles adjacent to the surface of a spherical bomb, exploding in an incompressible fluid.

Exp. 3. An infinite mass of inviscid liquid of constant density ρ is initially at rest, and has a spherical cavity of radius a. The liquid is made to move outwards by a pressure applied uniformly over the surface of the cavity; there is no pressure at infinity and no body forces act. If the radius of the cavity at time t is R, and the pressure applied is μ/R^3, μ constant, show that

$$\rho R^3(dR/dt)^2=2\mu\log(R/a)$$

and that the pressure at distance r from the centre of the sphere is

$$(\mu/R^2r)[1+\{1-R^3/r^3\}\log(R/a)].$$

Sol. In the incompressible liquid outside the cavity, the fluid velocity q will be radial and hence function of r: the radial distance from the centre of the cavity (the origin). The continuity equation div $\mathbf{q}=0$, in space polar coordinates becomes

$$\frac{1}{r^2}\frac{d}{dr}(r^2q)=0, \;\Rightarrow\; r^2q=\text{const.}=f(t)=R^2\dot{R}\;\text{(say)} \tag{1}$$

where $f(t)$ is a function of t only. We notice that $q\to 0$ as $r\to\infty$, as required. Obviously, *curl* $\mathbf{q}=0$, so that motion is irrotational and hence velocity potential exists and is given by $\phi=f/r=R^2\dot{R}/r$ by (1). The kinetic energy T at the instant when the radius of the cavity is R is

$$T=\tfrac{1}{2}\rho\int\phi\,\frac{\partial\phi}{\partial n}\,dS=-\tfrac{1}{2}\,\rho\,\phi\left|\frac{\partial\phi}{\partial r}.\,4\pi r^2\right|_{r=R}$$

because ϕ and $\partial\phi/\partial r$ are constant over $r=R$. Thus

$$T=2\pi\rho R^3\dot{R}^2. \tag{2}$$

The liquid motion starts from rest (the initial kinetic energy thus being zero) and so this energy is the work done by the pressure in expanding the cavity from initial radius a to radius R.

The work done by pressure p in expanding the cavity from radius R to radius $R+dR$ is $4\pi R^2 p dR$, and here $p=\mu/R^3$; thus the total work done W is given by

$$W=\int_a^R \frac{\mu}{R^3}.4\pi R^2 dR=4\pi\mu \log (R/a).$$

Equating the kinetic energy and the work done,

$$T=W, \Rightarrow \rho R^3 \dot{R}^2=2\mu \log (R/a). \tag{3}$$

Now, the pressure equation for irrotational non-steady fluid motion under no external forces is

$$\frac{p}{\rho}-\frac{\partial \phi}{\partial t}+\frac{1}{2}q^2=C(t) \tag{4}$$

where $C(t)$ is an arbitrary function of time t only. As $r\to\infty$, $p\to 0$, $q=f/r^2\to 0$, $\phi\to 0$ so that $C(t)=0$, for all t. Now putting for ϕ, q and C, in (4) we get

$$\frac{p}{\rho}-\frac{R^2\ddot{R}+2R\dot{R}^2}{r}+\frac{1}{2}\ \frac{R^4\dot{R}^2}{r^4}=0$$

or
$$p=\frac{\rho}{r}\left(\frac{\mu}{\rho R^2}+\frac{1}{2}R\dot{R}^2\right)-\frac{\rho}{2}\ \frac{R^4\dot{R}^2}{r^4} \text{ by differentiating (3)}$$

$$=\frac{\mu}{rR^2}+\frac{\rho}{2}\ \frac{R}{r}\left[1-\frac{R^3}{r^3}\right]\dot{R}^2$$

$$=(\mu/rR^2)[1+\{1-R^3/r^3\}\log (R/a)] \text{ by (3).}$$

Exp. 4. A mass of fluid of density ρ and volume $(4/3)\pi c^3$ is in the form of a spherical shell. A constant pressure Π is exerted on the external surface of the shell. There is no pressure on the internal surface and no other forces act on the liquid. Initially the liquid is at rest and the internal radius of the shell is 2c. Prove that the velocity of the internal surface when its radius is c is

$$\sqrt{(14\Pi/3\rho)[2^{1/3}/(2^{1/3}-1)]}$$

(*Ag 1957; 45 ; Ald 61 ; Del 63 ; Osm 62*)

Sol In the incompressible fluid outside the spherical shell, the fluid velocity q will be radial and hence function of r : the radial distance from the centre of the shell (the origin). The continuity equation div $\mathbf{q}=0$, in space polar coordinate becomes

$$\frac{1}{r^2}\ \frac{d}{dr}(r^2 q)=0, \quad \Rightarrow \quad r^2 q=\text{const.}=f(t) \tag{1}$$

where $f(t)$ is a function of t only. We notice that $q\to 0$ as $r\to\infty$, as required. Clearly, *curl* $\mathbf{q}=0$, so that motion is irrotational and hence velocity potential ϕ exists. In fact, $\phi=f/r$.

Now, the initial kinetic energy is zero ; the final kinetic energy T being given by

$$T=\tfrac{1}{2}\int_r^R (4\pi r^2 dr\rho q^2)=2\pi\rho f^2 \int_r^R \frac{1}{r^2}\,dr \qquad [r^2 q=f=\text{const.}]$$

Thus,
$$T=2\pi\rho r^4 q^2(r^{-1}-R^{-1}). \tag{2}$$

Also, work done by the external pressure Π in decreasing the shell from radius r to radius $2c$ is W where

$$W=\int_{2c}^R 4\pi R^2\Pi(-dR)=\frac{4\pi}{3}(8c^3-r^3)\Pi. \tag{3}$$

From *conservation of mass* $\frac{4}{3}\pi\rho R^3-\frac{4}{3}\pi\rho r^3=\text{const.}=\frac{4}{3}\pi\rho c^3, \Rightarrow R^3=r^3+c^3$

Now equating the increase in energy to that work done by the pressure, the conclusion obtained is

$$2\pi\rho r^4 q^2\left(\frac{1}{r}-\frac{1}{(r^3+c^3)^{1/3}}\right)=\frac{4}{3}\pi\Pi(8c^3-r^3).$$

We need the value of q (velocity) when $r=c$. Thus, we get

$$\rho c^4 q^2 \left(\frac{1}{c}-\frac{1}{c\,2^{1/3}}\right)=\frac{2}{3}\,\Pi.(7c^3) \quad \text{or} \quad q^2=(14\Pi/3\rho)[2^{1/3}/(2^{1/3}-1)].$$

The positive square root of this result is our requirement.

Exp 5 Obtain the equation of motion of a fluid in Euler's form. How are these equations supplemented to make the general problem definite.

A volume $(4/3)\pi c^3$ of gravitating liquid, of density ρ is initially in the form of a spherical shell of infinitely great radius. If the liquid shell contracts under the influence of its own attraction, there being no external or internal pressure, show that when the radius of the inner spherical surface is x, its velocity will be given by

$$v^2=\frac{4\pi\Upsilon\rho z}{15x^3}\{2z^4+2z^3x+2z^2x^2-3zx^3-3x^4\}$$

where Υ is the constant of gravitation and $z^3=x^3+c^3$. (*Cal 1955 ; Del 56*)

Sol. We have already obtained the equations of motion from the Eulerian point of view in the form

$$d\mathbf{q}/dt=\mathbf{F}-(1/\rho)\,\nabla\,p. \qquad [\text{§ 2.10(5) p. 74}]$$

The general problem of hydrodynamics consists in finding the velocity $\mathbf{q}$, the pressure p and the density ρ as functions of $\mathbf{r}$ and t. Thus we need more relations to determine these quantities. These relations are provided by the equation of continuity, viz.

$$(\partial\rho/\partial t)+\nabla.(\rho\mathbf{q})=0 \quad \text{or} \quad (d\rho/dt)+\rho \operatorname{div} \mathbf{q}=0$$

and some other physical relations such as between density ρ and pressure p. Whence the solution of any hydrodynamical problem is arrived at with the help of boundary conditions.

In the incompressible gravitating liquid outside the spherical volume, the fluid velocity q will be radial and hence a function of r: the radial distance from the centre of the spherical volume (the origin), and time t only. The continuity equation div $\mathbf{q}=0$, in space polar coordinates becomes

$$\frac{1}{r^2}\,\frac{d}{dr}(r^2 q)=0, \quad\Rightarrow\quad r^2 q=\text{const.}=f(t)=r_1^2\dot{r}_1\ (\text{say}) \qquad (1)$$

where f is a function of t only. We note that $q\to 0$ as $r\to\infty$, as required. Clearly, *curl* $\mathbf{q}=0$, so that motion is irrotational and hence velocity potential exists and is given by $\phi=f/r^2$.

The relation $z^3=x^3+c^3$ implies the conservation of mass, viz.

$$(4/3)\pi\rho z^3-(4/3)\pi\rho x^3=(4/3)\pi\rho c^3$$

so that x and z are the internal and external radii of a spherical mass at any time t. It is understood that the initial kinetic energy is zero, the final kinetic energy T being

$$T=\tfrac{1}{2}\rho\int(4\pi\,r^2dr)\dot{r}^2=2\pi\rho(r^2\dot{r})^2\int_x^z \frac{1}{r^2}\,dr \qquad [r^2\dot{r}=\text{const.}]$$

Thus, $$T=2\pi\rho(r^2\dot{r})^2[x^{-1}-z^{-1}]. \qquad (2)$$

To find the work done W by the gravitating liquid, we use the familiar formula (vide p. 359 Loney's Statics)

$$W=(\tfrac{1}{2})\int_b^a \phi\,dm$$

where ϕ is the potential of the shell whose internal and external radii are a and b, given distant r from the centre by the relation (vide p. 335 Loney's Statics)

$$\phi=2\pi\Upsilon\rho[a^2-(1/3)r^2-(2/3)(b^3/r)].$$

Making substitutions, putting $a=z$, $b=x$ we get

$$W=(1\;2)\int_x^z 2\pi\gamma\rho[z^2-(1/3)r^2-(2/3)(x^3/r)]4\pi r^2\rho dr$$

$$=\frac{4\pi^2\gamma\rho^2}{3}\int_x^z (3r^2z^2-r^4-2rx^3)dr$$

$$=4\pi^2\gamma\rho^2\{5z^2(z^3-x^3)-(z^5-x^5)-5x^3(z^2-x^2)\}/15.$$

Equating the work done W and the change in energy $(T-0)$ we get

$$r^4\dot{r}^2(z-x)/xz=k\{5z^2(z^3-x^3)-(z^5-x^5)-5x^3(z^2-x^2)\}.$$

$$[2\pi\rho\gamma/15=k\,;\ r^2\dot{r}=x^2\dot{x}=\text{const.}].$$

Cancelling out the common factor $z-x$ from both sides, we get

$$\dot{x}^2=(k/x^3)\{5z^2(z^2+xz+x^2)-(z^4+z^3x+z^2x^2+x^3z+x^4)-5x^3(z+x)\}$$

$$=(2k/x^3)\{2z^4+2z^3x+2z^2x^2-3zx^3-3x^4\}.$$

Putting for $k=2\pi\rho\gamma/15$ and setting $\dot{x}=v$ we get the result

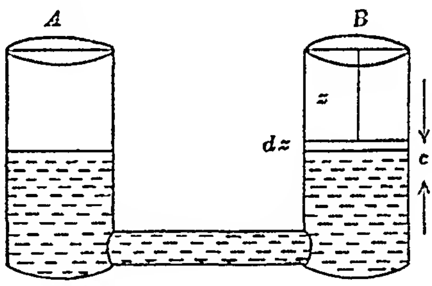
$$v^2=(4\pi\gamma\rho/15x^3)\{2z^4+2z^3x+2z^2x^2-3zx^3-3x^4\}.$$

Exp. 6. Two equal closed cylinders of heights c, with their bases in the same horizontal plane are filled, one with water and the other with air, of such a density as to support a column h of water, $h<c$. If a communication be opened between them at their bases, the height x to which the water rises, is given by the equation

$$cx-x^2+ch\log(c-x)/c=0.$$

[*Ag 1954 ; Ban 65 ; Del 46 ; Jab 62 ; Pb 59, 56* (S)]

Sol. Let A, B be two cylinders containing water and air respectively. Let the cross-section of each cylinder be S. Before communication is set up, the air and water are at rest and after communication when water has risen through x in cylinder B, water and air are again at rest. Thus, the *initial* and *final* kinetic energies are zero. Because of incompressibility, intrinsic energies also vanish. And because of the position of water, there is only potential energy.



Now, *initial* potential energy V due to water height in A is obtained by summing *mass* × *gravity* × *height*, i.e.

$$V_i=\int_0^c (\rho Sdy)gy=\tfrac{1}{2}g\rho S\,c^2. \tag{1}$$

After communication is opened, a height x of water rises in cylinder B and consequently a height $(c-x)$ of water remains in the cylinder A. And the final potential energy is thus obtained through (1) as

$$V_f=\tfrac{1}{2}g\rho S(c-x)^2+\tfrac{1}{2}g\rho Sx^2 \tag{2}$$

Thus, loss in potential energy, or work done *against* gravity

$$V_i-V_f=\tfrac{1}{2}g\rho S\{c^2-[(c-x)^2+x^2]\}=g\rho Sx(c-x) \tag{3}$$

Also, some work is done against the compression of air in the cylinder B; the external force being the pressure of the air inside B over the water column of height x. The length z of air is obviously $c-x$, i.e. $z=c-x$.

Let p be the pressure of the air of length z. Since the barometric height is h, the pressure must be given by $g\rho h$. We assume that temperature remains constant, so that Boyle's law applies, i.e. *pressure* × *volume* = *const.* This provides pressure at a *point*

$$(g\rho h)(cS)=p(c-x)S,\ \Rightarrow\ p=ghc\rho/(c-x) \tag{4}$$

The *down-ward* force pS works and work done W is

$$W=-\int_0^x \frac{\rho ghcSdx}{c-x}=\rho ghcS\log\left(\frac{c-x}{c}\right) \quad (5)$$

Equating 'gain of potential energy' to the work done we get

$$-g\rho Sx(c-x)=g\rho hcS\log[(c-x)/c]$$

or $$cx-x^2+ch\log[(c-x)/c]=0.$$

NOTE. In (4), we have used density ρ of water and not of mercury; for the simple reason that the density is such as to support a column h of *water*.

Aliter. A direct evaluation by the 'energy principle' is as follows: Applying this principal to the entire volume of the fluid, we obtain (K=kinetic energy ; P=potential energy)

$$\frac{d}{dt}(K+P)=-\int_S p\mathbf{v}dS=-pvS$$

where the integral is extended only over the surface of the water in the vessel B, since in the vessel A the pressure $p=0$, and along the rigid walls $\mathbf{v}.\mathbf{n}=0$. Now if x is the height to which water has risen in B, then $v=dx/dt$ and using for p from (4) we get, after integrating with regard to x

$$K+P=ghc\rho S\log(c-x)+A.$$

Now potential energy P of the fluid volume equals the weight of the entire volume multiplied by the height of the centre of gravity.

$$\therefore \quad P=[(c-x)S\rho g][(c-x)/2]+(xS\rho g)(x/2)=\tfrac{1}{2}Sg\rho(2x^2+c^2-2cx)$$

Substituting into the preceding equation, we obtain

$$T+\tfrac{1}{2}Sg\rho(2x^2+c^2-2cx)-Sg\rho ch\log(c-x)=A.$$

At the start of the motion, $x=0, T=0$ so that

$$A=\tfrac{1}{2}Sg\rho c^2-Sg\rho ch\log c.$$

Thus, $$T+Sg\rho(x^2-cx)-Sg\rho ch\log[(c-x)/c]=0.$$

When the water has risen to its highest level in vessel B, the velocity of every fluid particle reduces to zero at that instant and hence $T=0$ then. Consequently the maximum height attained by the fluid is the root of the equation

$$cx-x^2+ch\log[(c-x)/c]=0$$

2.40. *Lagrange's equations.* We shall now obtain the equation of motion of a fluid in Lagrangian form.

In Lagrangian system we fix our attention on a particular fluid particle and follow its course. Here the independent variables are, $\mathbf{r}_0$, the initial position vector of the particle to specify as to which particular particle has been chosen, and t, the time.

If the particle is at P at any time t, we have

$$\mathbf{r}=\mathbf{r}(\mathbf{r}_0, t)$$

so that the acceleration of the particle is $(\partial^2\mathbf{r}/\partial t^2)$ whence the equation of motion is

$$\frac{\partial^2\mathbf{r}}{\partial t^2}=\mathbf{F}-\frac{1}{\rho}\frac{\partial p}{\partial\mathbf{r}}$$

$$=\mathbf{F}-\frac{1}{\rho}\frac{\partial\,;\mathbf{r}_0}{\partial\mathbf{r}}\cdot\frac{\partial p}{\partial\mathbf{r}_0}\quad\text{for }\frac{\partial}{\partial\mathbf{r}}=\frac{\partial\,;\mathbf{r}_0}{\partial\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{r}_0}$$

(§ 0.60, p. 8)

or

$$\left(\frac{\partial^2\mathbf{r}}{\partial t^2}-\mathbf{F}\right)+\frac{\partial\,;\mathbf{r}_0}{\partial\mathbf{r}}\cdot\frac{\partial p}{\partial\mathbf{r}_0}\frac{1}{\rho}=0.$$

Now pre-multiplying $\dfrac{\partial\,;\mathbf{r}}{\partial\mathbf{r}_0}$ and remembering that

$$\frac{\partial\,;\mathbf{r}}{\partial\mathbf{r}_0}\cdot\frac{\partial\,;\mathbf{r}_0}{\partial\mathbf{r}_0}=1$$

(§ 0·60, p. 8)

we get

$$\frac{\partial\,;\mathbf{r}}{\partial\mathbf{r}_0}\cdot\left(\frac{\partial^2\mathbf{r}}{\partial t^2}-\mathbf{F}\right)+\frac{1}{\rho}\frac{\partial p}{\partial\mathbf{r}_0}=0,\tag{1}$$

which is the required Lagrangian form of motion. Here all differentiations are with respect to the *independent* variables $\mathbf{r}_0$ and t; the essentials of Lagrangian method.

This equation together with the equation of continuity

$$\rho\partial(x,y,z)/\partial(a,b,c)=\rho_0$$

constitutes *Lagrange's Hydrodynamical equations.*

Cor. Cartesian equivalents. To translate (1) into Cartesian co-ordinates, we set $\mathbf{F}=\nabla\chi$, $\ddot{\mathbf{r}}=-\partial^2\mathbf{r}/\partial t^2$, and rewrite (1) as

$$(\nabla_0\,;\mathbf{r})\ddot{\mathbf{r}}=-(\nabla_0\,;\mathbf{r})\,\nabla_0\chi-(1/\rho)\nabla_0\,p\tag{2}$$

For brevity, we write $\mathbf{i}\,;\mathbf{i}=\mathbf{ii}$, $\partial x/\partial a=x_a$, etc. $\ddot{\mathbf{r}}=\ddot{x}\mathbf{i}+\ddot{y}\mathbf{j}+\ddot{z}\mathbf{k}$,

$$\nabla_0:\mathbf{r}=x_a\mathbf{ii}+y_a\mathbf{ij}+z_a\mathbf{ik}+x_b\mathbf{ji}+y_b\mathbf{jj}+z_b\mathbf{jk}+x_c\mathbf{ki}+y_c\mathbf{kj}+z_c\mathbf{kk}$$

Thus, $(\nabla_0\,;\mathbf{r})\ddot{\mathbf{r}}=(x_a\ddot{x}+y_a\ddot{y}+z_a\ddot{z})\mathbf{i}+$ two similar expressions.

$$(\nabla_0\,;\mathbf{r})(\nabla_0\chi)=\frac{\partial;\mathbf{r}}{\partial\mathbf{r}_0}\cdot\frac{\partial\chi}{\partial\mathbf{r}}=\frac{\partial\chi}{\partial\mathbf{r}_0}.$$

Equating the coefficients of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ we finally get

$$\left.\begin{aligned}\ddot{x}x_a+\ddot{y}y_a+\ddot{z}z_a&=-\chi_a-(1/\rho)\,p_a\\ \ddot{x}x_b+\ddot{y}y_b+\ddot{z}z_b&=-\chi_b-(1/\rho)\,p_b\\ \ddot{x}x_c+\ddot{y}y_c+\ddot{z}z_c&=-\chi_c-(1/\rho)\,p_c\end{aligned}\right\}\tag{3}$$

where suffixes indicate partial derivatives e.g. $p_a=\partial p/\partial a$, etc.

Problems for solutions

1. Oscillations of water are observed in a bent uniform tube in a vertical plane. If O be the lowest point of the tube, AB the equilibrium level of the water; α, β the inclinations of the tube to the horizontal at A, B, show that the

period of oscillation is given by

$$2\pi\sqrt{\{(a+b)/g(\sin\alpha+\sin\beta)\}}$$

where a, b denote the lengths OA and OB.

2_1. A straight tube of small bore, ABC, is bent so as to make the angle ABC a right angle, $AB=BC$. The end C is closed; and the tube is placed with the end A upwards and AB vertical, and is filled with liquid. If the end C be opened, prove that the pressure at any point of the vertical tube is instantaneously diminished one-half; and find the instantaneous change of pressure at any point of the horizontal tube, the pressure of the atmosphere being neglected.

2_2. A straight pipe, of uniform cross-section, is bent to form a right angle and fixed with one arm vertically upwards. A valve is provided in the horizontal section distant l from the bend. The valve is closed and liquid of density ρ is poured into the vertical portion of the tube until it fills the pipe from the valve to a height l in the vertical portion. The valve is now opened, and after a time t has elapsed the level of the liquid in the vertical tube has fallen a distance x. Show that

$$x=l\ (1-\cos\omega t),\ \omega^2=g/2l.$$

Calculate the pressure at all points in the liquid at this instant.

2_3. A vertical tube AB of small constant cross-section divides at its lower end into two horizontal tubes BC and BD the cross-sections of which are also constant and each equal to one-half the cross-section of the vertical tube. At the joints of the tubes there are two valves which shut off the horizontal tubes. The valves are closed and the vertical tube is filled with fluid to a height $AB=a$. Investigate the motion after the valves have been opened simultaneously.

3. AB is a tube of small uniform bore forming a quadrantal arc of radius a and centre O; OA is horizontal and OB vertical. It is full of liquid of density ρ, the lower end B being closed. If B is suddenly opened, show that the pressure at a point, whose angular distance from A is θ, immediately drops to $\rho ga\ [\sin\theta-(2\theta/\pi)]$ above the atmospheric pressure and that, when the liquid remaining in the tube subtends an angle β at the centre

$$d^2\beta/dt^2=-(2g\sin^2\beta/2)/a\beta.$$

4. An elastic fluid, the weight of which is neglected, obeying *Boyle's law*, is in motion in a uniform straight tube; show that on the hypothesis of *parallel sections* the velocity at any time t at a distance r from a fixed point in the tube is defined by the equation

$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left(2v\frac{\partial v}{\partial t}+v^2\frac{\partial v}{\partial r}\right)=k\frac{\partial^2 v}{\partial r^2}.$$

[v denotes the velocity and $p=k\rho$.] (*Del 1960*)

5. A mass of homogeneous liquid is moving so that the velocity at any point is proportional to the time, and that the pressure is given by

$$(p/\rho)=\mu xyz-\tfrac{1}{2}t^2\ (y^2z^2+z^2x^2+x^2y^2).$$

Prove that this motion may have been generated from rest by finite natural forces independent of the time, and show that, if the direction of motion at every point coincide with the direction of the acting force, each particle of the liquid describes a curve which is the intersection of two hyperbolic cylinders.
[*Pna 1964 (Old), Sag 54*]

6. A solid sphere of radius a is surrounded by a mass of liquid whose volume is $\frac{4}{3}\pi c^3$ and its centre is a centre of attractive force varying directly as the square of the distance. If the solid sphere be suddenly annihilated, show that the velocity of the inner surface, when its radius is x, is given by

$$\dot{x}^2x^3\left[\left(x^3+c^3\right)^{\frac{1}{3}}-x\right]=\left(\frac{2\,\Pi}{3\rho}+\frac{2\mu c^3}{9}\right)\left(a^3-x^3\right)\left(c^3+x^3\right)^{\frac{1}{3}},$$

where ρ is the density, Π the external pressure and μ the absolute force.

7. An infinite mass of incompressible liquid contains a spherical bubble within which the pressure is zero. Initially the radius of the bubble is a and s increasing at a rate V. The pressure of the liquid at infinity is Π. Show that the bubble will expand to a maximum radius R given by

$$R^3 = a^3\,[1+(3\rho V^2/2\Pi)].$$

Find the pressure at a distance $r(>R)$ from the centre of the bubble when its radius is greatest.

8. A mass of uniform liquid is in the form of a thick spherical shell bounded by concentric spheres of radii a and b $(a<b)$. The cavity is filled with gas the pressure of which varies according to *Boyle's Law* and is initially equal to the atmospheric pressure Π, and the mass of which may be neglected. The outer surface of the shell is exposed to atmospheric pressure. Prove that if the system is symmetrically disturbed, so that each particle moves along the line joining it to the centre, the time of a small oscillation is

$$2\pi a\,\{\rho\,(b-a)/3\Pi b\}^{\frac{1}{2}}$$ [*Del 1961, 58*]

9. A given quantity of liquid moves, under no forces, in a smooth conical tube having a small vertical angle, and the distances of its nearer and farther extremities from the vertex at the time t are r and r'. Show that

$$2r\frac{d^2r}{dt^2}+\left(\frac{dr}{dt}\right)^2\left(3-\frac{r}{r'}-\frac{r^2}{r'^2}-\frac{r^3}{r'^3}\right)=0$$

the pressures at the two surfaces being equal.

Show also that the preceding equation results from supposing the vis viva of the mass of liquid to be constant; and that the velocity of the inner surface is given by the equation

$$V^2 = Ar'/r^3\,(r'-r)\,;\ r'^3-r^3=a^3,$$

where A and a are constants. [*Del 1959 ; Mad 56, 53 ; Pna 65 (Old)*]

10. An infinite perfect incompressible liquid of uniform density contains a spherical cavity which is allowed to collapse. The pressure Π at infinity is constant, and initially the liquid is at rest and the radius of the cavity is a. Prove that the radius R of the cavity at time t satisfies the equation

$$R^3(dR/dt)^3 = \tfrac{2}{3}(\Pi/\rho)(a^3-R^3).$$

Prove also that the pressure at a point in the liquid distant r from the centre of the cavity is

$$\left[1-\frac{R}{r}+\frac{1}{3}\,\frac{R}{r}\left(1-\frac{R^3}{r^3}\right)\left(\frac{a^3}{R^3}-1\right)\right].$$

11. A mass of gravitating fluid is at rest under its own attraction only; the free surface being a sphere of radius b and the inner surface a rigid concentric shell of radius a. Show that if this shell suddenly disappear, the initial pressure at any point of the fluid at distance r from the centre is

$$\frac{2}{3}\pi\,\rho^2(b-r)\,(r-a)\left(\frac{a+b}{r}+1\right).$$ [*Gor 1961 ; Pb 62, 48*]

12 A mass of perfect incompressible fluid of density ρ is bounded by concentric spherical surfaces. The outer surface is contained by a flexible envelope which exerts continuously a uniform pressure Π and contracts from radius R_1 to R_2 The hollow is filled with a gas obeying *Boyle's Law*, its radius contracts from c_1 to c_2 and the pressure of the gas is initially p_1. Initially the whole mass is at rest. Prove that, neglecting the mass of the gas, the velocity V of the inner surface when the configuration R_2, c_2 is reached is given by

$$\left(1-\frac{c_2}{R_2}\right)V^2 = 2\,\frac{c_1^3}{c_2^3}\left[\frac{1}{3}\left(1-\frac{c_2^3}{c_1^3}\right)\frac{\Pi}{\rho}-\frac{p_1}{\rho}\log\frac{c_1}{c_2}\right].$$

[*Bom 1950 ; Gti 56*]

13. A spherical hollow of radius a initially exists in an infinite fluid, subject to constant pressure Π at infinity. Show that

$$\frac{p}{\Pi}=\frac{3x^2r^4+(a^3-4x^3)\,r^3-(a^3-x^3)x^3}{3x^2r^4}$$

where p is the pressure at a distance r from the centre when the radius of the cavity is x. [*Del 1935 ; Gti 54 ; Jab 61 ; Pb 60 ; Sag 57 ; Ut 61*]

14. A spherical cavity of radius a containing gas at pressure p begins to expand in unbounded liquid otherwise at rest, the pressure at infinity being negligible. There are no external forces. If the gas expands adiabatically, prove that at time t the radius R of the cavity is given by

$$\frac{\rho}{p}\left(\frac{dR}{dt}\right)^2=\frac{2}{3(\gamma-1)}\left\{\left(\frac{a}{R}\right)^3-\left(\frac{a}{R}\right)^{3\gamma}\right\}$$

where ρ is the density of the liquid, and γ is the ratio of the specific heats of the gas.

15₁. An infinite liquid is initially everywhere at rest and contains a spherical cavity of radius c. Gas is suddenly introduced into the cavity at pressure p_0: during its subsequent expansion, according to the law $pv^n=$const., the pressure p $(<p_0)$ at infinity remains constant. Show that the maximum radius r subsequently attained by the cavity is given by the equation

$$1-x^{1-n}=A(1-x)$$

where $x=(r/c)^3$, and determine the constant A.

15₂. An infinite liquid contains a spherical cavity in which there is gas whose pressure and density are connected by the law $p=\lambda\rho^{4/3}$ where λ is constant. Initially the liquid is momentarily at rest with the cavity of radius a and the gas at pressure $n\Pi$ where Π is the pressure in the liquid at an infinite distance. Show that, neglecting the inertia of the gas, the radius of the cavity oscillates between the initial value a and a value ma where

$$m^3+m^2+m=3n.$$

16₁. Investigate an expression for the change in an indefinitely short time in the mass of fluid contained within a spherical surface of small radius. [*I.A.S. 1957*]

16₂. Fluid is contained within a sphere of small radius; prove that the momentum of the mass in the direction of the axis of x is greater than it would be if the whole were moving with the velocity at the centre by

$$(Ma^2/5\rho)\,[\rho_x u_x+\rho_y v_y+\rho_z u_z+\tfrac{1}{2}\,\rho\nabla^2 u]$$

where $\rho_x=d\rho/dx$, etc. [*Del 1944 ; I.A.S. 57*]

17. A spherical mass of liquid of radius b has a concentric spherical cavity of radius a, which contains gas at pressure p whose mass may be neglected; at every point of the external boundary of the liquid an impulsive pressure $\bar{\omega}$ per unit area is applied. Assuming that the gas obeys Boyle's Law, show that when the liquid first comes to rest, the radius of the internal spherical surface will be

$$a\ \exp.\,[-\bar{\omega}^2 b/2\,p\rho\,a^2\,(b-a)]$$

where ρ is the density of the liquid. [*Del 1957*]

18. A mass of fluid of density ρ and volume $(4/3)\pi\,c^3$ is in the form of a spherical shell. There is a constant pressure p on the external surface, and zero pressure on the internal surface. Initially the fluid is at rest, and the external radius is $2nc$. Show that when the external radius becomes nc, the velocity U of the external surface is given by

$$U^2=(14p/3\rho)\,\{(n^3-1)^{1/3}/[n-(n^3-1)^{1/3}]\}$$ [*Del 1950 ; Pna 64*]

19. A FINE tube whose section k is a function of its length s, in the form of a closed plane curve of area A, filled with ice, is moved in any manner. When the component angular velocity of the tube about a normal to its plane is Ω

the ice melts without change of volume. Prove that the velocity of the fluid relatively to the tube at a point where the section is K at any subsequent time when ω is the angular velocity, is

$$2A(\Omega-\omega)\div K\int\frac{ds}{k}$$

the integral being taken once round the tube.

20. A pipe of variable circular cross-section is given by $r=a(\cosh \alpha z)^{1/4}$ where (r, θ, z) are cylindrical polars, and the z-axis is vertical. Incompressible inviscid fluid is in steady irrotational motion, a volume Q passing every cross-section of pipe per unit time.

Assuming the vertical velocity to be a function of z only, and neglecting the horizontal velocity, show that the pressure will not be a monotonic function of z if $\alpha Q^2 > 4\pi^2 a^4 g$.

21. Fluid extending to infinity, surrounds a spherical boundary whose radius at time t is $a+b\sin nt$, the centre being fixed. If there are no external forces, show that the pressure at any point on the spherical boundary is

$$\Pi+1/4\rho\, b\, n^2(5b\cos 2nt-4a\sin nt+b),$$

where Π is the pressure at infinity. [*Bom 1964*; *Del 51*]

22₁. A sphere whose radius at time t is $b+a\cos nt$ is surrounded by liquid extending to infinity under no forces. Prove that the pressure at distance r from the centre is less than the pressure Π at an infinite distance by

$$\frac{\rho n^2 a}{r}(b+a\cos nt)\left\{a(1-3\sin^2 nt)+b\cos nt+\frac{1}{2}\frac{a}{r^3}\sin^2 nt\,(b+a\cos nt)^3\right\}.$$

Prove further that if the sphere is forced to vibrate radially keeping its spherical shape, the least pressure (assumed positive) at the surface of the sphere during the motion is $\Pi-n^2\rho a(a+b)$. [*Bom 1959*]

22₂. The radius of a sphere immersed in an infinite ocean of incompressible fluid of density ρ varies according to the equation $r=a+b\cos nt$, where a, b and n are constants. The fluid moves radially under no forces and the constant pressure at infinity is Π. If the velocity potential for the motion is given by $\phi=f(t)/r$, find $f(t)$ and use Bernoulli's theorem (general) for unsteady flow to obtain the pressure at any point on the sphere. Show that the maximum pressure attained is

$$p=\Pi+\rho n^2 b\{(3b/2)+(a^2/10b)\},\ \text{iff } a\leqslant 5b.$$

23. Incompressible ideal fluid of density ρ fills the space between the concentric cylinders $r=a$ and $r=b$ where $r^2=x^2+y^2$. The velocity of the fluid is

$$\mathbf{q}=[-y f(r), x f[r), o].$$

Find the pressure throghout the fluid and explain why for general $f(r)$ Bernoulli's theorem ($p/\rho+q^2/2=$const.) does not hold. Prove that if $r^2 f(r)$ $=$const., then $p/\rho+q^2/2=$const , and state what is special about the motion corresponding to this form of $f(r)$.

24. A mass of liquid of density ρ whose external surface is a long circular cylinder of radius a, which is subjected to a constant pressure Π surrounds a co axial long circular cylinder of radius b. The internal cylinder is suddenly destroyed. Show that if q is the velocity at the internal surface when the radius is r, then

$$q^2=2\Pi(b^2-r^2)/[\rho\, r^2\log\{(r^2+a^2+b^2)/r^2\}].$$ [*Bom 1957*]

25. Liquid is contained between two parallel planes; the free surface is an elliptic cylinder whose axiz is perpendicular to the planes, and the semi-axes of whose section are a_1, b_1. All the liquid within a confocal elliptic cylinder, the semi-axes of whose section are a_2, b_2 is suddenly annihilated. Prove that if Π be the pressure at the outer surface, the initial pressure at any point of the liquid is

$$\Pi[\log(a+b)-\log(a_2+b_2)]/[\log(a_1+b_1)-\log(a_2+b_2)].$$

where a and b are the semi-axes of a confocal cylinder through the point.

26. A source of strength m is situated in a stream which at large distances from the source has uniform velocity U and pressure Π. Show that this combination represents irrotational flow past a certain semi-infinite surface and find the equation of the surface. Determine the part of the surface on which the pressure exceeds Π and find the force exerted by the fluid on this part.

27. (a) Explain on general grounds why two pulsating spheres in a liquid attract each other, if they are always in the same phase.

(b) A spherical shell of homogeneous gravitating liquid, having no initial motion, is left to itself. Find the pressure at any point during the collapse.

28. A vessel in the form of a hollow circular cone with axis vertical and vertex downwards, the top being open, is filled with water. A circular hole whose diameter is $(1/n)$ that of the top (n being large) is opened at the vertex. Show that the time taken for the depth of the water to fall to one half of its original value (h), cannot be less than

$$(4\sqrt{2}-1)n^2h/20\sqrt{g}. \qquad \text{[Mad 1953]}$$

29. A mass of homogeneous liquid, whose bounding surfaces are concentric spheres, is at rest under the action of no forces in a gas of uniform pressure. If the pressure of the external gas be suddenly increased, determine the instantaneous pressure in the liquid, and investigate the differential equation for the subsequent motion of the liquid and the pressure inside the shell at any time.

30. Water flows steadily with a free surface under the action of gravity along a long straight channel bounded by plane vertical wall, y=const. The bed of the channel is in the form of the surface $z=f(x)$ where z is measured in the direction of the upward vertical and $f(\pm\infty)\to 0$. The gradient of the bed is everywhere small and the velocity may be taken as horizontal and uniform over any section x=const. If the depth and velocity of water at a great distance upstream are respectively h and $\sqrt{\alpha^3 gh}$, show that the depth H at any x satisfies

$$(\alpha^3h^3/2H^2)+H+f(x)=(\tfrac{1}{2}\alpha^3+1)h.$$

Hence show that the maximum possible value of $f(x)$ is

$$\tfrac{1}{2}h(\alpha-1)^2(\alpha+2).$$

Show also that, if the bed of the channel attains its maximum level at one point only, the depth of water at a great distance down stream is nh where n is the positive root of

$$2n^2-\alpha^3n-\alpha^3=0$$

provided that there is no discontinuity in the gradient of the free surface.

31. If $f(x, y, z)$=const., $g(x,y,z)$=const., are the equations of a curve, show that the tangent to the curve has the direction $\nabla f\times\nabla g$. Hence show that if the above family of curves are vortex lines of a velocity field $\mathbf{q}$, then $\omega=curl\ \mathbf{q}=F\nabla f\times\nabla g$, where F is a scalar function.

Show further, that if the motion is irrotational, the Jacobian

$$\partial(F,f,g)/\partial(x,y,z)=0.$$

Deduce that if the scalar function $h(f,g)$ is so chosen that $(\partial h/\partial f)=F$, then, $\mathbf{q}=h(f,g)\nabla g$ is a solution of

$$\omega=curl\ \mathbf{q} \text{ and that } \omega=(\partial h/\partial f)f\nabla\times\nabla g.$$

Hence show the general solution of ω=curl $\mathbf{q}$ is

$$\mathbf{q}=-\nabla\phi+h(f,g)\nabla g, \text{ where } f=\text{const.}, g=\text{const.}$$

are two systems of surfaces which pass through the vortex lines, and ϕ solution of Laplace's equation. [Mad

32. If the motion is referred to a moving frame w[illegible] velocity angular velocity ω, prove that the vorticity satisfies

$$(\partial\zeta/\partial t)+\omega\times\zeta+(\mathbf{q}_r.\nabla)\zeta=(\zeta. \text{[illegible]}$$

where $\mathbf{q}_r=\mathbf{q}-\mathbf{u}-\omega\times\mathbf{r}$.

33. An infinite mass of liquid is moving irrotationally and steadily under the influence of a source m and an equal sink at a distance $2a$ from it. Prove that the kinetic energy of the liquid which passes in unit time across the plane which bisects at right angles the line joining the source and sink is

$$8\pi\rho m^3/7a^4,$$

where ρ is the density of the liquid. [*Del 1961 ; Pb 64*]

2.50. *Some hydrodynamical applications of Green's theorem.* The general or non-symmetric form of the theorem for single-valued functions ϕ_1 and ϕ_2 is

$$\int_V (\nabla\phi_1).(\nabla\phi_2)dv = -\int_V \phi_1\nabla^2\phi_2 dv - \int_S \phi_1 \frac{\partial\phi_2}{\partial n} dS \qquad [\S 0.40, \text{p.}6]$$

(*1*) If ϕ_2 is constant and $\nabla^2\phi_1=0$, we get

$$\int \rho \frac{\partial\phi}{\partial n} dS = 0 \qquad \text{where } \phi_1=\phi$$

which represents that the total flow of liquid into any closed region at any instant is zero.

(2) ***Kinetic energy of finite liquid.*** The kinetic energy is given by

$$T=\int_V \tfrac{1}{2}\rho q^2 dv$$

taken throughout the volume V occupied by the fluid. For *irrotational* motion $\mathbf{q}=-\nabla\phi$, $\nabla^2\phi=0$

$$\therefore \qquad T=\tfrac{1}{2}\rho\int (\nabla\phi)\cdot(\nabla\phi)dv = -\tfrac{1}{2}\rho\int_S \phi\frac{\partial\phi}{\partial n}dS.$$

[by *Green's Theorem*]

taken over the bounding surface of the liquid, *dn* denoting an element of *inward-drawn* normal.

Physical interpretation. We know that if $\mathbf{q}$ is the velocity and ρ the density of the liquid, then *K. E.* of the liquid within S is

$$T=\int_V \tfrac{1}{2}\rho q^2 dv = -\tfrac{1}{2}\rho\int \phi\frac{\partial\phi}{\partial n} dS$$

Since $\rho\phi$ is the impulsive pressure and $-(\partial\phi/\partial n)$ the inward velocity, it follows that the *K.E.* set up by impulses, in a system starting from rest, is the sum of the products of each impulse and half the velocity of its point of application. It also follows that the *K.E.* of a given mass of liquid moving irrotationally in a simply connected region depends on *the motion of its boundaries.* Clearly the surface integral

$$-\tfrac{1}{2}\rho\int_S \phi\frac{\partial\phi}{\partial n} dS$$

represents the *work done by the impulsive pressure in starting the motion from rest.*

(3) If the boundaries are *at rest*, it follows from (3) that $\partial\phi/\partial n=0$, so that

$$\tfrac{1}{2}\rho\int_V q^2 dv=0, \text{ i.e. } q=0 \text{ at every point.}$$

Hence if the boundaries are fixed, irrotational motion is impossible in a closed simply-connected region.

Ex. 1. Obtain the formula for calculating kinetic energy of a mass of homogeneous liquid moving irrotationally in a finite simply-connected space in the form

$$T=-\tfrac{1}{2}\rho\int \phi\frac{\partial\phi}{\partial n}dS.$$

Deduce that irrotational motion is impossible in a closed singly-connected region with fixed boundaries. [*Del 1958*]

Ex. 2. State and prove Green's theorem. Show that the kinetic energy of a given mass of liquid moving irrotationally in simply-connected space depends on the motion of its boundaries. [*Ban 1953*]

2.51. *Green's formula.* We shall now prove that *the potential ϕ_P at a point P in a mass of liquid moving irrotationally within a certain boundary is given by*

$$4\pi\phi_P=\int_S\left[\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)-\frac{1}{r}\frac{\partial\phi}{\partial n}\right]dS$$

where r is the distance of P from the element of area dS.

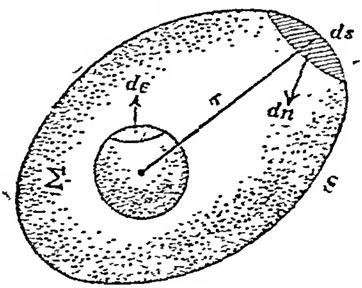
Proof. If ϕ_1, ϕ_2 both satisfy Laplace's equation, i.e. $\nabla^2\phi_1=0$; $\nabla^2\phi_2=0$, within a region bounded by S, we have, by *Green's Reciprocal Theorem*,

$$\int_S\left(\phi_1\frac{\partial\phi_2}{\partial n}-\phi_2\frac{\partial\phi_1}{\partial n}\right)dS=0. \tag{1}$$

Let P be any point within the region and put $\phi_1=\phi$ and $\phi_2=1/r$, where r is the distance of P from the element dS. It is easily verified that

$$\nabla^2(1/r)=0.$$

Since $\phi_2\to\infty$ as $r\to 0$ (i.e. at P), we must exclude P from the domain of integration of reciprocal Theorem (1). Hence we enclose P by a sphere Σ, radius ε so small that Σ is entirely within S.



Reciprocal Theorem applied to the region between Σ and S gives

$$\int_S\left[\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)-\frac{1}{r}\frac{\partial\phi}{\partial n}\right]dS=-\int_\Sigma\left[\phi\frac{\partial}{\partial\varepsilon}\left(\frac{1}{\varepsilon}\right)-\frac{1}{\varepsilon}\frac{\partial\phi}{\partial\varepsilon}\right]d\varepsilon$$

$$=\frac{1}{\varepsilon^2}\int_\Sigma\phi\, d\varepsilon; \tag{2}$$

because, $\int_{\Sigma}\left(\frac{\partial \phi}{\partial \varepsilon}\right)d\Sigma=\int_{V}\nabla^2\phi dv=0$ [by Gauss's Theorem].

Now the left hand side of (2) is independent of ε, and hence so also is the right-hand side, which is, consequently, equal to its limit as $\varepsilon\to 0$.

If ε is small enough to take $\phi=\phi_P$ on the whole of the surface Σ, we get approximately,

$$\frac{1}{\varepsilon^2}\int_{\Sigma}\phi d\Sigma=\frac{1}{\varepsilon^2}\phi_P\int_{\Sigma}d\Sigma=\frac{\phi_P}{\varepsilon^2}.\ 4\pi\varepsilon^2=4\pi\phi_P.$$

Hence proceeding to the limit when $\varepsilon\to 0$, we get from (2)

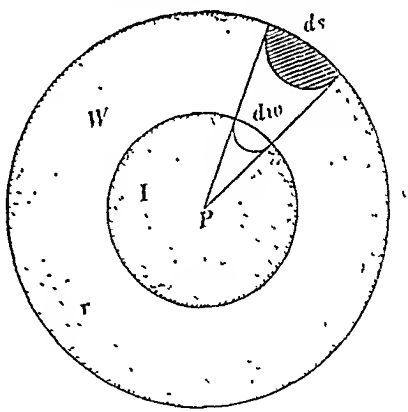
$$4\pi\phi_P=\int_V\left[\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)-\frac{1}{r}\frac{\partial\phi}{\partial n}\right]dS. \tag{3}$$

Ex. Prove that the potential ϕ_P at a point P in a mass of liquid moving irrotationally within a certain boundary S is given by

$$\phi_P=\frac{1}{4\pi}\int_S\phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dS-\frac{1}{4\pi}\int_S\frac{1}{r}\frac{\partial\phi}{\partial n}dS.$$

Modify this result when the liquid extends to infinity while certain boundaries S remain at finite distances. [*Bom 1952 ; Del 63*]

2.52. *Mean potential over spherical surface. If a region lying wholly in the liquid be bounded by a spherical surface throughout whose interior* $\nabla^2\phi=0$, *the mean value of the velocity potential* ϕ *over the surface is equal to its value at the centre of the sphere.*



Let ϕ_S and ϕ_m denote the value of ϕ at P and the mean value of ϕ over a sphere S, centre P, radius r. Draw another concentric sphere of unit radius. Then a cone with vertex P which intercepts dS from the sphere S intercepts $d\omega$ (the solid angle) from the sphere W, such that

$$dS/d\omega=r^2/(1)^2,$$
$$\Rightarrow dS=r^2d\omega. \quad (1)$$

Now $\phi_m=\frac{1}{4\pi r^2}\int_S\phi dS=\frac{1}{4\pi r^2}\int_W\phi r^2 d\omega=\frac{1}{4\pi}\int_W\phi d\omega$

so that
$$\frac{\partial\phi_m}{\partial r}=\frac{1}{4\pi}\int_W\frac{\partial\phi}{\partial r}d\omega=\frac{1}{4\pi r^2}\int_S\frac{\partial\phi}{\partial r}dS. \tag{2}$$

Since
$$\int_S\frac{\partial\phi}{\partial r}dS=\int_S\frac{\partial\phi}{\partial n}dS=\int_V \mathbf{n}.\nabla\phi dS=\int_V\nabla^2\phi dv$$

[by Divergence theorem]

and $\nabla^2\phi=0$ (by hypothesis), it follows that $\int(\partial\phi/\partial r)dS=0$ and hence (2) yields

$$\frac{\partial\phi_m}{\partial r}=0, \Rightarrow \phi_m=\text{constant (indcpendent of } r).$$

Hence, when S shrinks to a point P, $\phi_m=\phi_P$

Cor. 1. ϕ cannot be a maximum or minimum in the interior of any region throughout which $\nabla^2\phi=0$.

If possible let ϕ_P be a maximum value of ϕ at a point P. Describe a sphere with P as centre and radius ε where ε is small. Then (ϕ_m) : thc mean value of the velocity potcntial : must be less then ϕ_P which contradicts the theorem just proved. Similarly ϕ_P cannot be a minimum.

Cor. 2. In irrotational motion, the maximum value of the speed must occur on the boundary.

Let a point P *interior* to the fluid be taken as origin and take the x-axis in the direction of motion at P. Let Q be any point near to P; then if q, q' are the speeds at P and Q,

$$q^2=\left(\frac{\partial\phi}{\partial x}\right)_P^2 ; \; q'^2=\left(\frac{\partial\phi}{\partial x}\right)_Q^2+\left(\frac{\partial\phi}{\partial y}\right)_Q^2+\left(\frac{\partial\phi}{\partial z}\right)_Q^2 \cdot$$

Since
$$\nabla^2\left(\frac{\partial\phi}{\partial x}\right)=\frac{\partial}{\partial x}(\nabla^2\phi)=0$$

$(\partial\phi/\partial x)$ satisfies *Laplace's equation* and consequently there cannot be a maximum or minimum at P. Thus at points such as Q very near to P,

$$(\partial\phi/\partial x)^2 >(\partial\phi/\partial x)_P^2 \text{ whence } q'^2>q^2.$$

Hence q cannot be a maximum inside the fluid and its maximum value *if any,* must occur on the boundary.

NOTE : q^2 may be minimum in the interior of the fluid, for $q=0$ at a stagnation point.

Cor. 3. In steady irrotational motion the hydrodynamical pressure has its minimum value on the boundary.

The result follows from *Bernoulli's* Theorem

$$(p/\rho)+\tfrac{1}{2}q^2=\text{const.}$$

Since when q^2 is greatest, p must be least. But q^2 is greatest only on the boundary ; so p will also be minimum only on the boundary.

2.60. *Kelvin's minimum energy theorem. If a mass of liquid be set in motion by giving prescribed velocities to its boundaries, the kinetic energy in the actual motion is less than that in any other motion consistent with the same motion of the boundaries.*

Proof. Let T_1 be the K.E. of the actual motion of which ϕ is the velocity potential and q_1 the velocity. Let T_2 be the kinetic energy

and $\mathbf{q}_2$ the velocity *in any other possible state of motion*. As the equations of continuity must be satisfied, we must have

$$\nabla\cdot\mathbf{q}_1=0\ ;\ \nabla\cdot\mathbf{q}_2=0. \quad (1)$$

Since the boundary has the same normal velocity in either case, the condition to be satisfied is expressed by the relation

$$\mathbf{n}\cdot\mathbf{q}_1=\mathbf{n}\cdot\mathbf{q}_2 \quad (2)$$

Now $T_2-T_1=\frac{1}{2}\rho\int_V(\mathbf{q}_2^2-\mathbf{q}_1^2)dv$

$$=\tfrac{1}{2}\rho\int_V\left\{2\mathbf{q}_1\cdot(\mathbf{q}_2-\mathbf{q}_1)+(\mathbf{q}_2-\mathbf{q}_1)^2\right\}dv$$

$$=\rho\int_V\mathbf{q}_1\cdot(\mathbf{q}_2-\mathbf{q}_1)dv+\tfrac{1}{2}\rho\int_V(\mathbf{q}_2-\mathbf{q}_1)^2\,dv$$

$$=-\rho\int_V(\nabla\phi)\cdot(\mathbf{q}_2-\mathbf{q}_1)dv+\tfrac{1}{2}\rho\int_V(\mathbf{q}_2-\mathbf{q}_1)^2dv \quad (3)$$

From vector calculus

$\nabla\cdot[\phi(\mathbf{q}_2-\mathbf{q}_1)]=\phi[\nabla\cdot(\mathbf{q}_2-\mathbf{q}_1)]+(\nabla\phi)\cdot(\mathbf{q}_2-\mathbf{q}_1)=(\nabla\phi)\cdot(\mathbf{q}_2-\mathbf{q}_1)$ by (1)

This gives : $\int_V(\nabla\phi)\cdot(\mathbf{q}_2-\mathbf{q}_1)dv=\int_V\nabla\cdot[\phi(\mathbf{q}_2-\mathbf{q}_1)]\,dv$

$=\int_S\phi\mathbf{n}\cdot(\mathbf{q}_2-\mathbf{q}_1)$ [by Divergence theorem]

Since the integral on the extreme end is zero by (2), we get from (3)

$$T_2-T_1=\tfrac{1}{2}\rho\int_V(\mathbf{q}_2-\mathbf{q}_1)^2dv>0\ ;\ \text{i.e. } T_2>T_1.$$

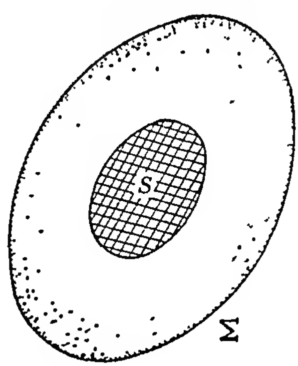
Thus, the irrotational motion of a liquid occupying a simply connected region has less kinetic energy than any other motion consistent with the same normal velocity of the boundary (but for which vortices are present inside).

Ex. State and prove Kelvin's theorem of minimum kinetic energy in hydrodynamics. [*Del 1957 ; Gtı 62, 58, 56, 54, 53 ; I.A.S. 55*]

2.61. Kinetic energy of infinite liquid. We shall prove that the *kinetic energy of an infinite mass of liquid of density* ρ, *moving irrotatianally is given by*

$$-\tfrac{1}{2}\rho\int_S\phi\,\frac{\partial\phi}{\partial n}\,dS,$$

where ϕ *denotes the single-valued velocity potential.*



Consider the liquid bounded internally by a solid S and externally by a large surface Σ, then the *K. E.* of the finite liquid contained into this region is given by

$$T_1 = -\tfrac{1}{2}\rho \int_S \phi \frac{\partial \phi}{\partial n} dS - \tfrac{1}{2}\rho \int_\Sigma \phi \frac{\partial \phi}{\partial n} d\Sigma. \tag{1}$$

Now the rate R at which mass flows into *any* surface γ through the boundary is, by definition

$$R = \int_\gamma \rho \mathbf{q}.\mathbf{n}\, d\gamma, \text{ where } \mathbf{n} \text{ is unit normal drawn } \textit{into} \text{ the surface.}$$

$$= -\int_\gamma \rho \frac{\partial \phi}{\partial n} d\gamma \qquad \text{for } \mathbf{q} = -\nabla\phi = -\mathbf{n}\frac{\partial \phi}{\partial n}.$$

Since there is no flow into the region across S, we must have, for the form of continuity,

$$-\int_S \rho \frac{\partial \phi}{\partial n} dS - \int_\Sigma \rho \frac{\partial \phi}{\partial n} d\Sigma = 0. \tag{2}$$

Multiplying (2) by $\frac{1}{2}c$, a constant, and subtracting from (1), we get

$$T_1 = -\tfrac{1}{2}\rho \int_S (\phi - c) \frac{\partial \phi}{\partial n} dS - \tfrac{1}{2}\rho \int_\Sigma (\phi - c)\frac{\partial \phi}{\partial n} d\Sigma. \tag{3}$$

Since for a *solid* body, $\int_S \frac{\partial \phi}{\partial n} dS = 0$, it follows from (2) that

$$\int_\Sigma \rho \frac{\partial \phi}{\partial n} d\Sigma = 0, \text{ i.e. } \int_\Sigma \frac{\partial \phi}{\partial n} d\Sigma \text{ is independent of } \Sigma.$$

If. at infinity, $\phi \to c$, we enlarge the surface Σ indefinitely in all directions whence we get

$$\int_\Sigma (\phi - c)\frac{\partial \phi}{\partial n} d\Sigma = 0.$$

Thus (3) simplifies to

$$T_1 = -\tfrac{1}{2}\rho \int_S (\phi - c) \frac{\partial \phi}{\partial n} dS = -\tfrac{1}{2}\rho \int_S \phi \frac{\partial \phi}{\partial n} dS. \tag{4}$$

2.70. ***Permanence of vorticity via Cauchy's integrals.*** We start with the Helmholtz vorticity equation, viz.

$$\frac{d}{dt}\left(\frac{\omega}{\rho}\right) = \left(\frac{\omega}{\rho} \cdot \nabla\right) \mathbf{q} \tag{1}$$

valid only for barotropic inviscid fluids under conservative body forces and proceed to find its solution, called Cauchy's integrals.

Now setting $\nabla = \frac{\partial}{\partial \mathbf{r}} = \frac{\partial\, ;\, \mathbf{r}_0}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}_0}$ and putting in (1) we get

$$\frac{d}{dt}\left(\frac{\omega}{\rho}\right) = \left(\frac{\omega}{\rho} \cdot \frac{\partial\, ;\, \mathbf{r}_0}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}_0}\right) \mathbf{q} = \frac{\omega}{\rho} \cdot \frac{\partial\, ;\, \mathbf{r}_0}{\partial \mathbf{r}} \frac{\partial\, ;\, \mathbf{q}}{\partial \mathbf{r}_0} \tag{2}$$

where $\mathbf{r}_0$ indicates the *initial* position of the particle at time t_0.

Now we differentiate the identity relation

$$\cdot \frac{\partial\, ;\, \mathbf{r}_0}{\partial \mathbf{r}} \frac{\partial\, ;\, \mathbf{r}}{\partial \mathbf{r}_0} = I \text{ (indemfactor)} \qquad (\S\ 0.60, \text{ p. } 8) \tag{3}$$

with respect to scalar t and get

$$\frac{d}{dt}\left(\frac{\partial ; \mathbf{r}_0}{\partial \mathbf{r}}\right)\frac{\partial ; \mathbf{r}}{\partial \mathbf{r}_0}+\frac{\partial : \mathbf{r}_0}{\partial \mathbf{r}}\frac{d}{dt}\left(\frac{\partial ; \mathbf{r}}{\partial \mathbf{r}_0}\right)=0$$

or

$$\frac{d}{dt}\left(\frac{\partial : \mathbf{r}_0}{\partial \mathbf{r}}\right)\frac{\partial : \mathbf{r}}{\partial \mathbf{r}_0}+\frac{\partial ; \mathbf{r}_0}{\partial \mathbf{r}}\frac{\partial ; \mathbf{q}}{\partial \mathbf{r}_0}=0 \tag{4}$$

because $d\mathbf{r}/dt=\mathbf{q}$. From (2) and (4) we get

$$\frac{d}{dt}\left(\frac{\omega}{\rho}\right)+\frac{\omega}{\rho}\cdot\frac{d}{dt}\left(\frac{\partial ; \mathbf{r}_0}{\partial \mathbf{r}}\right)\frac{\partial ; \mathbf{r}}{\partial \mathbf{r}_0}=0. \tag{5}$$

Post-multiplying (5) by $(\partial ; \mathbf{r}_0/\partial \mathbf{r})$ and using (3) yields

$$\frac{d}{dt}\left(\frac{\omega}{\rho}\right)\frac{\partial ; \mathbf{r}_0}{\partial \mathbf{r}}+\frac{\omega}{\rho}\frac{d}{dt}\left(\frac{\partial : \mathbf{r}_0}{\partial \mathbf{r}}\right)=0$$

or

$$\frac{d}{dt}\left(\frac{\omega}{\rho}\frac{\partial : \mathbf{r}_0}{\partial \mathbf{r}}\right)=0$$

Hence

$$\frac{\omega}{\rho}\cdot\frac{\partial ; \mathbf{r}_0}{\partial \mathbf{r}}=\text{const.}=\frac{\omega_0}{\rho_0}\cdot\frac{\partial : \mathbf{r}_0}{\partial \mathbf{r}_0}=\frac{\omega_0}{\rho_0} \tag{6}$$

because $(\partial ; \mathbf{r}_0/\partial \mathbf{r}_0)=I$ and ω_0 and ρ_0 are the values of ω and ρ at time t_0. Using (3) we can rewrite (6) as

$$(\omega/\rho)=(\omega_0/\rho_0)\,(\partial ; \mathbf{r}/\partial \mathbf{r}_0) \quad (\textit{Cauchy's integrals}) \tag{7}$$

From (7), it at once follows that if $\omega_0=0$ *then* $\omega=0$ *so that irrotational motion is permanent. Also if* $\omega\neq 0$ *then* ω *can never vanish so that rotational motion is also permanent.*

NOTE : The cartesian equivalents of (7) are

$$\frac{\xi}{\rho}=\frac{\xi_0}{\rho_0}\frac{\partial x}{\partial a}+\frac{\eta_0}{\rho_0}\frac{\partial x}{\partial b}+\frac{\zeta_0}{\rho_0}\frac{\partial x}{\partial c}$$

with two more similar relations.

Ex. 1. If $\mathbf{q}$ be the velocity (u, v, w) and rot $\mathbf{q}=(2\xi, 2\eta, 2\zeta)$, prove the following *Helmholtz equation of an inviscid fluid*

$$\frac{D}{Dt}\left(\frac{\xi}{\rho}\right)=\frac{\xi}{\rho}\frac{\partial u}{\partial x}+\frac{\eta}{\rho}\frac{\partial u}{\partial y}+\frac{\zeta}{\rho}\frac{\partial u}{\partial z}, \ldots \text{ etc.}$$

and hence prove that the elements of fluid which are free from vorticity at any time will remain so at all subsequent times. [*Bom 1950*]

Ex. 2. Establish Lagrange's Hydro-dynamical equations of motion, assuming the external forces to belong to the conservative system. Hence deduce or prove otherwise that if a velocity potential exists at one instant, for any finite portion of a perfect fluid in motion under the action of conservative system of forces, then, provided the density of the fluid be either constant or a function of the pressure only, a velocity potential exists for the same portion of the fluid at all instants before or after. [*Del 1957*]

2.80. *Some uniqueness theorems related to acyclic irrotational motion.* In the following we shall consider *acyclic irrotational motion only*, i.e. *motion in which the velocity potential is one valued*, it being understood that *in a simply connected region the only possible irrotational motion is acyclic.* We shall invariably use the expression for *K.E.* in the forms

$$T=\tfrac{1}{2}\rho\int_V q^2\,dv=-\tfrac{1}{2}\rho\int_S \phi\frac{\partial \phi}{\partial n}\,dS. \tag{A}$$

(1) There cannot be two different forms of acyclic irrotational motion for a given confined mass of liquid whose boundaries have prescribed velocities or are subject to given impulses.

If possible, let ϕ_1, ϕ_2 be the velocity potentials of two different motions subject to the conditions

$$\text{(i)}\ (\partial\phi_1/\partial n)=(\partial\phi_2/\partial n)\ ;\ \text{(ii)}\ \rho\phi_1=\rho\phi_2$$

Let $\phi=\phi_1-\phi_2$, then $\nabla^2\phi=\nabla^2\phi_1-\nabla^2\phi_2=0.$

Thus ϕ is a solution of Laplace's equation and therefore represents a possible irrotational motion in which

$$\frac{\partial\phi}{\partial n}=\frac{\partial\phi_1}{\partial n}-\frac{\partial\phi_2}{\partial n}=0.$$

Therefore, $q=0$ by (A) at every point. It follows that $\phi=\phi_1-\phi_2=$ constant, so that motions are essentially the same.

The same result applies to ensure that $\rho\phi_1-\rho\phi_2=$ constant, and thereby follows the result for impulsive pressures.

(2) Acyclic irrotational motion is impossible in a liquid bounded entirely by fixed rigid walls or it will cease instantly if the boundaries are brought to rest.

Since at every point of the *rigid* boundary, $(\partial\phi/\partial n)=0$, it follows by (A) that $q=0$ everywhere. Thus the liquid will be always at rest ; and hence motion will be impossible.

(3) Acyclic irrottaional motion is impossible in a liquid which is at rest at infinity and is bounded internally by fixed rigid walls ; or such a motion will cease instantly if the boundaries are brought to rest.

As there is no flow over the internal boundaries and the liquid is given to be at rest at infinity, the *K.E.* is still given by (A).

Hence $\partial\phi/\partial n=0$ gives $q=0$ everywhere. Consequently the liquid is at rest.

(4) The acyclic irrotational motion of a liquid, at rest at infinity, due to the prescribed motion of an immersed solid, is uniquely determined by the motion of the solid.

If possible, let ϕ_1, ϕ_2 be the velocity potentials of two different motions. The boundary conditions to be satisfied are

(i) $(\partial\phi_1/\partial n)=(\partial\phi_2/\partial n)$ at the surface of the solid (ii) $q_1=q_2=0$, at infinity.

Thus $\phi=\phi_1-\phi_2$ is the velocity potential of a possible motion because $\nabla^2\phi=\nabla^2\phi_1-\nabla^2\phi_2=0$, and that $(\partial\phi/\partial n)=(\partial\phi_1/\partial n)-(\partial\phi_2/\partial n)=0$, at the surface of the solid, and $q=0$, at infinity.

From $(\partial\phi/\partial n)=0$ at every point of the boundary, it follows that $q=0$ everywhere, so that $\phi_1-\phi_2=$constant and the motions are necessarily the same.

(5) If the liquid is in motion at infinity with uniform velocity, the acyclic irrotational motion, due to the prescribed motion of an immersed solid, is uniquely determined by the motion of the solid.

Superimpose on the whole system a velocity equal in magnitude and opposite in direction to the velocity at infinity; the relative kinematical conditions remain unaltered and the liquid is reduced to rest at infinity. Now apply (4); the result follows.

Ex. Sketch a proof of Green's Theorem and deduce the following:

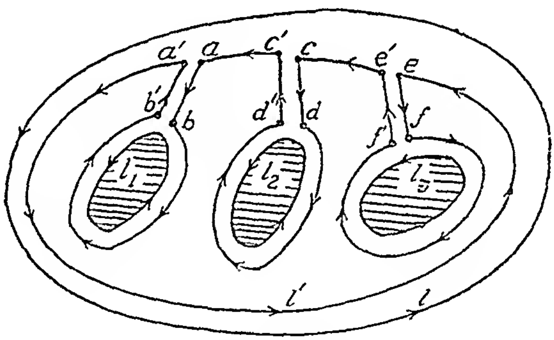
(a) A rigid envelope is filled with homogeneous frictionless liquid. It is not possible by any movements applied to the envelope to set its contents into motion which will persist after the envelope has come to rest.

[*Bom 1956*; *Del 55*; *Gti. 59*,; *Mad 58, 56*]

(b) There cannot be two different forms of irrotational motion for a given confined mass of liquid whose boundaries have prescribed velocities or subject to given impulses. [*Ban 1961*; *Bom 56*; *Cal 55*; *Mad 60*; *Sag 56*]

2.90. *Cyclic constants.* Consider an *irreducible circuit* l containing several (say three) holes (shown by shades) in the fluid which pierce any surface S which spans the circuit l. These holes might be occupied by solid obstacles. The circuit l' (vide Fig.) unlike l is reducible so that

$$\int_{l'} \mathbf{q}.d\mathbf{r} = \int_{S'} \boldsymbol{\omega}.d\mathbf{S}. \tag{1}$$



In the limiting case, when ab coincides with $a'b'$, cd coincides with $c'd'$ and ef coincides with $e'f'$ and the arcs bb', dd', ff' coincide respectively with the boundaries l_1. l_2, l_3 of the holes, the right hand side of (1) is over the surface spanning l which lies in the fluid. Further, in this limiting case,

$$\int_{l'} = \int_{a'e} + \int_{ef} + \int_{ff'} + \int_{f'e'} + \int_{e'c} + \int_{cd} + \int_{dd'} + \int_{d'c'} + \int_{c'a} + \int_{ab} + \int_{bb'} + \int_{b'a'}$$

$$= \left\{\int_{a'e} + \int_{e'e} + \int_{c'a} \right\} - \int_{f'f} - \int_{d'd} - \int_{b'b} + \left(\int_{ef} - \int_{e'f'}\right) + \left(\int_{cd} - \int_{c'd'}\right) + \left(\int_{ab} - \int_{a'b'}\right)$$

$$= \int_{l} - \int_{l_2} - \int_{l_3} - \int_{l_1} + 0 + 0 + 0$$

Thus, by transposition,

$$\int_l \mathbf{q}.d\mathbf{r}=\int_{l_1}\mathbf{q}.d\mathbf{r}+\int_{l_2}\mathbf{q}.d\mathbf{r}+\int_{l_3}\mathbf{q}.d\mathbf{r}+\int_S \boldsymbol{\omega}.d\mathbf{S} \quad \left[\because \int_l \mathbf{q}.d\mathbf{r}\to\int_S \boldsymbol{\omega}.d\mathbf{S}\right]$$

If the motion is irrotational, $\omega=0$, so that

$$\int_l \mathbf{q}.d\mathbf{r}=\Gamma_1+\Gamma_2+\Gamma_3 \qquad \text{where } \Gamma_i=\int_{l_i}\mathbf{q}.d\mathbf{r}$$

The quantity Γ_i is called the *cyclic* constant of the ith hole.

Cor. If there are n holes threading l, the ith hole being bounded by the simple closed curve l_i, then

$$\int_l \mathbf{q}.d\mathbf{r}=\Sigma p_i\,\Gamma_i$$

where p_i is the excess of the number of times l goes round the ith hole (in the same sense as l_i) over the number of times it goes in the opposite direction. The constants p_i are integers ; possitive, negative or may be zero.

Cor. 2. For irrotational motion, $\mathbf{q}=-\nabla\phi$, so that

$$\Gamma=\int_C \mathbf{q}.d\mathbf{r}=-\int_C \nabla\phi\,.\,d\mathbf{r}=-\int_C d\phi=-|\phi|_C.$$

Thus, $\Gamma=$ decrease in ϕ in going round the *irreducible* circuit.

Hence $\Gamma=\Sigma p_i\Gamma_i$, in a multipli-connected region. Thus, ϕ is many-valued function and is known as *cyclic potential.*

In a reducible circuit, $\Gamma=0$, i.e. the decrease in ϕ going once round any circuit is zero.

NOTE : Motion in which the circulation in every circuit does not vanish is known as *cyclic motion.*

2.91. Green's theorem for multipliconnected regions. For simple-valued functions ϕ, ϕ', the Green's theorem is

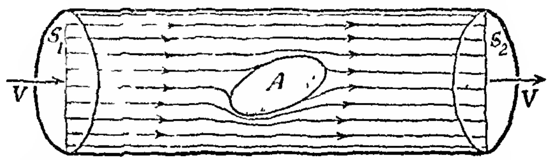
$$\int_V (\nabla\phi).(\nabla\phi')\,dv=-\int_S \phi\,\frac{\partial\phi'}{\partial n}\,dS-\int_V \phi\nabla^2\phi\, dv \tag{1}$$

If ϕ be many-valued function (i.e. *cyclic*), then the region must be multipli-connected. Consequently we insert the necessary barriers and the region is reduced to simply-connected ; rendering ϕ to be a single-valued function. The corresponding corrections to be made consist in including in the range of the surface integral both sides of each of the barriers. Now, if $+$ and $-$ indicate the positive and negative side of the barrier, then $\phi^+-\phi^-=\Gamma_i$, $(\partial\phi'/\partial n)^+=-(\partial\phi'/\partial n)^-$; and the contribution of this barrier to the surface integral

$$\int \phi(\partial\phi'/\partial n)da \;\text{ is }\; \Gamma_i\int (\partial\phi'/\partial n)\,da$$

loss $(\rho q_1 S_1 \delta t)\times q_1$ at AB. These rates of positive and negative amounts, viz. $\rho q_2{}^2 S_2$ and $\rho q_1{}^2 S_1$, are produced entirely by the thrusts acting on the walls and ends of the filament. Reverting the direction of q_1 at AB, we conclude that these thrusts must be equivalent to the forces $\rho S_1 q_1{}^2$ and $\rho S_2 q_2{}^2$ normally outwards at AB and CD respectively.

D'Alembert's paradox. Consider a long straight tube through non-viscous liquid flow with a constant speed V, suppose that the ends of the tube are bounded by *equal* cross-sectional areas S_1 and S_2 (i.e. $S_1=S_2$). If an obstacle A is placed in the middle of the tube, the flow in the immediate neighbourhood of A will be deranged, but at a great distance either up-stream or



down-stream the flow will be undisturbed. To keep the obstacle A at rest will, in general, require a force F and a couple G. Neglecting external force (e.g. gravity), F is the resultant force in the direction of flow, of the pressure thrusts on the boundary of A. If the fluid is divided into current filaments starting on S_1 and ending on S_2, then obstacle A acts on these filaments in contact with A by a force $-F$ in the direction of flow, and the pressure thrusts on the outer filaments bounded by the tube-walls being normal to the direction of flow cancel out. Hence, by Euler's momentum theorem, applied to each current filament in the tube, the resultant thrust on the liquid in the tube is $=-\rho S_1 V^2+\rho S_2 V^2=0$ (as $S_1=S_2$).

By Bernoulli's theorem : p_1 over $S_1=p_2$ over S_2. Thus

$$p_1 S_1-F-p_2 S_2=0, \Rightarrow F=0.$$

The cross-sectional area S can be taken very large, and hence the result $(F=0)$ holds for an infinite liquid as well.

Finally, superimposing a velocity V in the opposite direction on both the liquid and the obstacle does not alter the dynamical conditions. Hence, *a body moving with uniform velocity through an unbounded inviscid fluid otherwise at rest encounters no resistance. This theoretical conclusion (D' Alemberts paradox) is valid for any general body of reasonable shape.*

NOTE : This result is not valid for an accelerated body. The resistance is due to the *drag* on the surface.

Exp. 2. If a fluid be in motion with a velocity potential $\phi=z\log r$, and if the density at a point fixed in space be independent of the time, show that the surfaces of equal density are of the forms

$$r^2(\log r-\tfrac{1}{2})-z^2=f(\theta,\rho)$$

where ρ is the density at (z, r, θ) : the cylindrical co-ordinates. (Del 1966)

Sol. For surfaces of equal density, $\rho=$const., so that $d\rho/dt=0$.

Thus $$\frac{d\rho}{dt}=\frac{\partial\rho}{\partial t}+(\mathbf{q}.\nabla)\rho=0.$$

Since $\mathbf{q}=-\nabla\phi$ and $\partial\rho/\partial t=0$, the above reduces to

$$\left(-\frac{\partial\phi}{\partial r},-\frac{1}{r}\frac{\partial\phi}{\partial\theta},-\frac{\partial\phi}{\partial z}\right).\left(\frac{\partial\rho}{\partial r},\frac{1}{r}\frac{\partial\rho}{\partial\theta},\frac{\partial\rho}{\partial z}\right)=0$$

or $$\frac{\partial\phi}{\partial r}\frac{\partial\rho}{\partial r}+\frac{1}{r^2}\frac{\partial\phi}{\partial\theta}\frac{\partial\rho}{\partial\theta}+\frac{\partial\phi}{\partial z}\frac{\partial\rho}{\partial z}=0.$$

Now, $\phi=z\log r$, $\partial\phi/\partial r=z/r$, $\partial\phi/\partial\theta=0$, $\partial\phi/\partial z=\log r$

$$\frac{z}{r}\frac{\partial\rho}{\partial r}+\log r\frac{\partial\rho}{\partial z}=0, \Rightarrow z\frac{\partial\rho}{\partial r}+r\log r\frac{\partial\rho}{\partial z}=0.$$

Comparing with Lagrange's differential equation

$$P(\partial z/\partial x)+Q(\partial z/\partial y)=R, \text{ with solutions } dx/P=dy/Q=dz/R$$

we get,
$$\frac{dr}{z}=\frac{dz}{r\log r}=\frac{d\rho}{0}.$$

The last member yields ρ=const.$=\rho_0$ (say) and the first two provide

$$r\log r\, dr-z\,dz=0$$

Integrating $\frac{1}{2}r^2\log r-\frac{1}{4}r^2-\frac{1}{2}z^2$=const. (independent of r and z but may be function of θ and ρ).

Thus
$$r^2(\log r-\tfrac{1}{2})-z^2=f(\theta,\rho).$$

Exp. 3. Prove that if

$$\phi=-\tfrac{1}{2}(ax^2+by^2+cz^2)\ ;\ V=\tfrac{1}{2}(lx^2+my^2+nz^2)$$

where a, b, c, l, m and n are functions of the time and $(a+b+c)=0$, *irrotational motion is possible with a free surface of equipressure if*

$$(l+a^2+a')\,e^{2\int a\,dt}\ ;\ (m+b^2+b')e^{2\int b\,dt}\ ;\ (n+c^2+c')\,e^{2\int c\,dt}$$

are constants and $a'=\partial a/dt$, etc. [*Del 1966, 60*]

Sol. Since $\phi=-\frac{1}{2}(ax^2+by^2+cz^2)$, we get

$$-\partial\phi/\partial x=ax, -\partial^2\phi/\partial x^2=a, \text{ etc., so that } \nabla^2\phi=-(a+b+c)=0, \text{ (by given)}$$

Thus, the irrotational motion is possible. The pressure equation for non-steady irrotational fluid motion under conservative body forces $\mathbf{F}=-\nabla V$; is

$$(p/\rho)-(\partial\phi/\partial t)+\tfrac{1}{2}q^2+V=C(t) \tag{1}$$

Here
$$\partial\phi/\partial t=-\tfrac{1}{2}(a'x^2+b'y^2+c'z^2)\ ,\ q^2=\Sigma(\partial\phi/\partial x)^2=\Sigma a^2x^2.$$

Substitutions in (1) provide the expression for pressure p :

$$p/\rho=-\tfrac{1}{2}(\Sigma a'x^2)-\tfrac{1}{2}(\Sigma a^2x^2)-\tfrac{1}{2}(\Sigma lx^2)+C(t). \tag{2}$$

For a free surface of equipressure, p=const.$=\Pi$; $dp/dt=0$, i.e.

$$\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}=0. \tag{3}$$

Carrying out the necessary partial differentiations in (3) yields

$$\partial p/\partial t=-\tfrac{1}{2}\rho\,[\Sigma a''x^2+2\Sigma aa'x^2+\Sigma l'x^2]+C'(t)\rho$$
$$-u\partial p/\partial x=(\partial\phi/\partial x)\partial p/\partial x=ax.\rho(a'+a^2+l)x$$
$$-v\,\partial p/\partial y=\rho b(b'+b^2+m)y^2\ ;$$
$$-w\,\partial p/\partial z=\rho c\,(c'+c^2+n)\,z^2.$$

Substituting in (3), the condition for free boundary surface is

$$-\tfrac{1}{2}\rho\,\Sigma(a''+2aa'+l')x^2+C'(t)\rho-\rho\,\Sigma a(a'+a^2+l)x^2=0.$$

This equation is true for *all t* ; hence coefficients of x^2, y^2, z^2 and the constant must vanish separately. This gives

$$\left.\begin{aligned}\tfrac{1}{2}(a''+2aa'+l')+a(a'+a^2+l)&=0\\ \tfrac{1}{2}(b''+2bb'+m')+b(b'+b^2+m)&=0\\ \tfrac{1}{2}(c''+2cc'+n')+c(c'+c^2+n)&=0\\ C'(t)&=0\end{aligned}\right\} \tag{4}$$

The last equation implies $C(t)$=const. for all t, i.e. absolute constant.

The first of (4) yields

$$D(a'+a^2+l)+2a\,(a'+a^2+l)=0,\ (D=d/dt)$$

Integrating :
$$\int\frac{D(a'+a^2+l)}{a'+a^2+l}\,dt+\int 2a\,dt=\text{const}$$

or $\log (a'+a^2+l)\, e^{2\int a dt} = \text{const.}, \Rightarrow (a'+a^2+l)\, e^{2\int a dt} = \text{const. (different)}.$

The other expressions follow by the cyclic permutations.

2.92. General fluid motion. A continuous (velocity) vector field q *can be decomposed into irrotational vector* (i.e. whose *curl*=0) *and solenoidal vector* (i e. whose divergence=0) *fields.* Thus $\mathbf{q}=\mathbf{q}_i+\mathbf{q}_s$, where the subscripts i and s refer to the irrotational and solenoidal components respectively. Now

$$\text{div } \mathbf{q}_s=0, \Rightarrow \mathbf{q}_s=\textit{curl } \mathbf{A}\ ;\ \textit{curl } \mathbf{q}_i=0, \Rightarrow \mathbf{q}_i=-\textit{grad } \phi.$$

Further, since *curl* $(\mathbf{A}+\textit{grad } \psi)=\textit{curl } \mathbf{A}$, the vector A is not uniquely determined ; and as such we impose on it a further restriction, viz. div A=0. Hence,

$$\mathbf{q}=-\textit{grad } \phi+\textit{curl } \mathbf{A}, \text{ where div } \mathbf{A}=0. \tag{1}$$

Obviously, div $\mathbf{q}=-\nabla^2\phi=\Theta$ (say), *curl* $\mathbf{q}=\textit{curl curl } \mathbf{A}$.

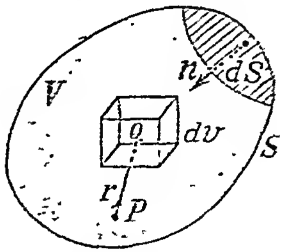
The latter equation gives *curl* $\mathbf{q}=\omega=\textit{grad}\,(\text{div } \mathbf{A})-\nabla^2\mathbf{A}=-\nabla^2\mathbf{A}$.

Thus,
$$\nabla^2\phi=-\Theta,\ \nabla^2\mathbf{A}=-\omega \tag{2}$$

which are analogous equations, called Poisson's equations. The functions ϕ and A are called *velocity potential* and *vector* (or *solenoidal*) *potential.*

Ex. Define the *divergence* and *curl* of a vector field in a manner independent of the co-ordinate system, and explain how they are connected with the motion of a fluid element. Given these two vector fields, show how to obtain (i) the circulation round a closed curve, (ii) the flux over a closed surface. [*Mad 1953*]

2.93. Vector potential. Figure shows liquid enclosed within a *fixed* envelope S and let the vorticity ω be given at every point of the liquid. If n is the unit *inward* normal at the element *dS* of S, then *on* S, the boundary condition is $\mathbf{n}.\mathbf{q}=0$.



Let us *fix* a point *P* within the liquid, the velocity there (at *P*) being q, that at any other point *Q*, the velocity be q′ and let *dv* be the element of volume at *Q*. Further, let

$$PQ=r,\ (\partial/\partial P)=\nabla,\ (\partial/\partial Q)=\nabla',\ \text{obviously } \nabla(r^{-1})=-\nabla'(r^{-1}) \tag{1}$$

We now consider the vector $\mathbf{A}_P=\mathbf{A}$ (say), being the solution of Poisson's equation : $\nabla^2\mathbf{A}=-\mathbf{q}$, and given by

$$\mathbf{A}=\frac{1}{4\pi}\int_V \frac{\mathbf{q}'}{r}\,dv \tag{2}$$

where the integral is taken through the volume *V* enclosed by *S*, point *P* remaining *fixed*. Now, since *curl curl* A=*grad* div **A** Poisson's equation leads to

$$\mathbf{q}=\nabla\times(\nabla\times\mathbf{A})-\nabla(\nabla.\mathbf{A})$$

As q' is independent of the position of P, we obtain from (2) and (1),

$$\nabla.A=\frac{1}{4\pi}\int_{\tau} q'.\nabla\left(\frac{1}{r}\right)dv=-\frac{1}{4\pi}\int_{\tau} q'.\nabla'\left(\frac{1}{r}\right)dv \qquad (4)$$

Also, $\quad \nabla'.(q'r^{-1})=r^{-1}(\nabla'.q')+q'.\nabla'(r^{-1})=q'.\nabla'(r^{-1})$

because $\nabla'.q'=0$, by continuity condition. Thus, (4) reduces to

$$\nabla.A=-\frac{1}{4\pi}\int_{\tau} \nabla'.(q'r^{-1})dv=\frac{1}{4\pi}\int_{\tau} \frac{n.q'}{r}dS=0,$$

where third term results from second by Gauss's divergence theorem and then it vanishes on account of the boundary condition $n.q'=0$. Hence, (3) yields

$$q=curl\ (curl\ A)=curl\ B\ ;\quad B=curl\ A \qquad (5)$$

The vector B, defined by $B=curl\ A$, is called the *vector potential* of the velocity q (compare with definition $q=-grad\ \phi$). To find B explicitly, we have from (2) and (5)

$$4\pi B=\nabla\times\int_{\tau}\frac{q'}{r}dv=-\int_{\tau} q'\times\nabla\left(\frac{1}{r}\right)dv$$

$$=\int_{\tau} q'\times\nabla'\left(\frac{1}{r}\right)dv \quad \text{by (1)}$$

$$=\int_{\tau}\left\{\frac{1}{r}(\nabla'\times q')-\nabla'\times\left(\frac{q'}{r}\right)\right\}dv$$

$$\left[\because\ \nabla'\times\left(\frac{q'}{r}\right)=\frac{1}{r}(\nabla'\times q')+\nabla'\left(\frac{1}{r}\right)\times q'\right]$$

$$=\int_{\tau}\frac{\omega'}{r}dv+\int_{S}\frac{n\times q'}{r}dS$$

$$\left\{\because\ \int_{\tau}(\nabla\times a)\,dv=-\int_{S}(n\times a)\,dS,\ \text{Gauss's theorem}\right\}$$

where $\nabla'\times q'=\omega'$. Thus B in terms of ω' and q' at the boundary of S is given by

$$B=\frac{1}{4\pi}\int_{\tau}\frac{\omega'}{r}dv-\frac{1}{4\pi}\int\frac{n\times q'}{r}dS \qquad (6)$$

Cor. Unbounded fluid. When the fluid is unbounded, and $q=O(r^{-2})$ at a great distance, the surface integral in (6) tends to zero. Therefore,

$$B=\frac{1}{4\pi}\int\frac{\omega'}{r}dv \qquad (7)$$

Now $\quad q=\nabla\times B=-\frac{1}{4\pi}\int_{\tau}\omega'\times\nabla\left(\frac{1}{r}\right)dv=\frac{1}{4\pi}\int_{\tau}\frac{\omega\times r}{r^3}dv \qquad (8)$

$(r=QP)$

Thus the velocity at P may be regarded as the vector sum of elementary velocities, that corresponding to the vorticity in the volume element dv at Q being

$$d\mathbf{q}=(dv/4\pi r^3)(\boldsymbol{\omega}'\times \mathbf{QP})=(dv/4\pi r^3)(\omega_Q \sin \alpha)\,\hat{\mathbf{u}}$$

This fictitious elementary velocity is *induced* at P by dv at Q.

2.94. ***Kinetic energy of a system of vortices.*** This is given by

$$2T=\rho\int_V \mathbf{q}.\mathbf{q}\, dv=\rho\int_V \mathbf{q}.\,(curl\,\mathbf{B})\, dv \tag{1}$$

Since $\operatorname{div}(\mathbf{B}\times\mathbf{q})=\mathbf{q}.\,curl\,\mathbf{B}-\mathbf{B}.\,curl\,\mathbf{q}$, (1) may be set as

$$2T=\rho\int_V \operatorname{div}(\mathbf{B}\times\mathbf{q})dv+\rho\int_V \mathbf{B}.\,(curl\,\mathbf{q})\, dv$$

or

$$T=\tfrac{1}{2}\rho\int_S \mathbf{B}.\,(\mathbf{n}\times\mathbf{q})\, dS+\tfrac{1}{2}\rho\int_V \mathbf{B}.\boldsymbol{\omega}\, dv \tag{2}$$

[by Gauss's theorem, n *inward*]

If we suppose that the liquid extends to infinity and is at rest there, and that the vortices are all within a finite distance of the origin, then at a great distance, $q \doteq 0(r^{-3})$ B ; $\doteq 0(r^{-2})$ and hence the surface integral in (2) vanishes. Thus,

$$T=\tfrac{1}{2}\rho\int \boldsymbol{\omega}.\mathbf{B}\, dv \tag{3}$$

But for such a case as under consideration

$$B=\frac{1}{4\pi}\int \frac{\omega'}{r}dv' \qquad \text{[§ 2.93 (7), p. 130]}$$

hence

$$T=\frac{1}{8\pi}\,\rho\iint\frac{\boldsymbol{\omega}.\boldsymbol{\omega}'}{r}\,dv\,dv' \tag{4}$$

where ω, ω' are the vorticities at P and Q, and dv, dv' elements of volume at these points, and each volume integral extends through the whole space occupied by the vortices.

Cor. Integration in (4) *over vortex filaments*. Let ds, ds' be the elements of length of two filaments, a, a' their cross-sections, ω, ω' the corresponding vorticities, and ε the angle between ds and ds'. Then, since $ads=dv$, $\omega a=k$, etc., we get from (4)

$$T=\frac{1}{8\pi}\,\rho\Sigma\, kk'\iint\frac{\cos\varepsilon}{r}\,ds\,ds' \tag{5}$$

where integration is along filaments and the summation includes each pair of filaments once.

Ex. A mass of homogeneous liquid is supposed to extend to infinity in all directions and to be at rest there, all the vortices being supposed to be within a finite region. (The motion of the liquid is due to vortices only.) Prove [illegible] kinetic energy of the liquid can be expressed in the form

$$T=\frac{1}{2\pi}\,\rho\iint\frac{\boldsymbol{\omega}.\boldsymbol{\omega}'}{r}\,dv\,dv'$$

where r is the mutual distance between the elements dv, dv' at which the vorticities are ω, ω' respectively. Obtain also the corresponding expression for kinetic energy when the motion is due to vortex filaments only. [Del 1960]

Problems for solution

1. Explain the meaning of the term *rotational* as applied to fluid motion ; and determine the character of the circulatory motion of fluid, round a straight axis, which is not rotational.

Show that, in such a case, minute bubbles of air in the circulating fluid will be sucked in towards the axis. [Mad 1959]

2. Establish Lagrange's hydrodynamical equations for the motion of fluid particles, and obtain Cauchy's integrals. Deduce that the motion of a fluid under conservative forces, if once irrotational, is always irrotational.

Give a physical meaning for the velocity potential, when it exists.

When a body immersed in a fluid executes periodic vibrations, it appears to exert an attraction on other bodies at rest in the fluid. Give a general explanation of this phenomenon. [Gti 1959 ; Mad 53]

3. Show that if a heterogeneous incompressible liquid moves irrotationally under the action of conservative forces, the surfaces of equal pressure and equal density coincide, and that a homogeneous liquid cannot move irrotationally under the action of nonconservative forces.

4. Prove that if the velocity potential at any instant be λxyz, the velocity at any point $(x+\xi,\ y+\eta,\ z+\zeta)$ relative to the fluid at the point (x, y, z), where (ξ, η, ζ) are small, is normal to the quadric

$$x\eta\zeta+y\zeta\xi+z\xi\eta=\text{const.}$$

with centre at (x, y, z).

5. If Γ is the circulation around any closed circuit moving with the fluid, prove that

$$\frac{d\Gamma}{dt}=\int p\, d\left(\frac{1}{\rho}\right)$$

if the external forces have a potential, and the pressure is a function of density alone.

6. Liquid moves irrotationally in two dimensions under the action of conservative forces whose potential Ω satisfies $\nabla^2\Omega=0$. Prove that the pressure satisfies the equation

$$\nabla^2(\log \nabla^2 p)=0.$$

7. A body moves in a given manner, without change of volume, in an inviscid liquid. T_0 denotes the K.E. of the liquid when it has no external boundary and is at rest at infinite distance ; T'_0 denotes the K.E. of that part of the fluid which is outside a closed surface S_0 which is external to the body ; T denotes the K.E. of the fluid when S_0 is its external boundary and is fixed. Prove that, if the regions occupied by the fluid are simply-connected,

$$T>T_0+T'_0.$$

8. Deduce from the principle that the K.E. set up is a minimum that, if a mass of incompressible liquid be given at rest, completely filling a closed vessel of any shape and if any motion of the liquid be produced suddenly by giving arbitrarily prescribed normal velocities to all the points of its bounding surface subject to the condition of constant volume, the motion produced is irrotational.

9. Prove that in acyclic irrotational motion of a homogeneous fluid the total momentum of the fluid contained within a sphere of any radius is equivalent to a single vector through the centre of the sphere.

10. (*a*) If ϕ is constant over the boundary of any simply-connected region occupied by liquid in irrotational motion, prove that ϕ has the same constant value throughout the interior.

(b) Prove that, if the normal velocity is zero at every point of the boundary of liquid occupying a simply connected region, and moving irrotationally, ϕ is constant throughout the interior of that region.

(c) Liquid moving irrotationally occupies a simply connected region bounded partly by surfaces over which ϕ is constant, and partly by surfaces over which the normal velocity is zero. Prove that ϕ has the same constant value throughout the region.

11. How would you determine whether a given function $\phi=f(x, y, t)$ is or is not a possible form for the velocity potential of a homogeneous incompressible liquid. Is

$$\phi=a\, e^{2\pi y/\lambda} \,.\, \cos\,[2\pi(x-vt)/\lambda]$$

a possible form.

Prove that in any irrotational motion of a liquid whose velocity at infinity is zero, any cause which reduces the bounding surfaces to rest will reduce the liquid to rest. [*Del 1939*]

12. Obtain the equation of continuity of a fluid in spherical polar co-ordinates.

Show that in the case of the irrotational motion of a thin layer of liquid on the surface of a sphere, the velocity potential is of the form

$$f(\log \tan \tfrac{1}{2}\theta+i\varphi)+F(\log \tan \tfrac{1}{2}\theta-i\varphi)$$

where θ and φ are the polar and azimuthal angles and f, F are arbitrary functions. [*Born 1953*]

13. Prove that if a thin stratum of homogeneous liquid move irrotationally on the surface of a sphere of radius r, the velocity potential ϕ satisfies the equation

$$\frac{\partial^2\phi}{\partial\omega^2}+\frac{\partial^2\phi}{\partial\chi^2}=0, \text{ where } \chi=\log \tan \theta/2.$$

Hence show that the velocity potential at any point on the sphere can be put in the form

$$\phi=f\left(\frac{x+iy}{r+z}\right)+f\left(\frac{x-iy}{r+z}\right)$$

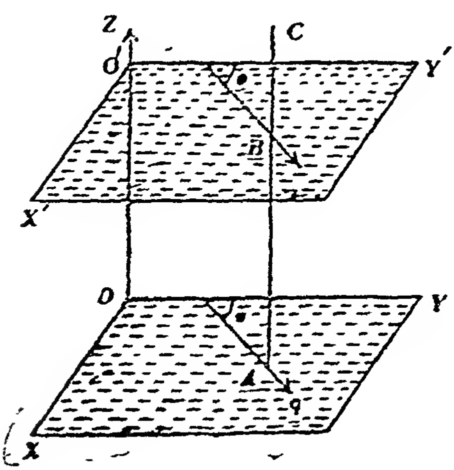
where x, y, z are the co-ordinates of a point referred to rectangular axes through the centre of the sphere, and r, θ, ω are the spherical polar co-ordinates of the point. [*Del 1958*]

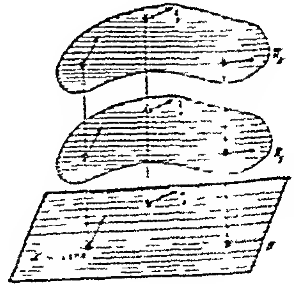
3 : *Special Methods for Non-viscous Liquids*

3.00. Introductory remarks. In this chapter, we consider the simplest case of plane flow ; i.e. we will refer all dynamic computations for the hydrodynamic pressure forces, their moments and kinetic energy to a layer of unit height cut by two planes parallel to the plane of flow. In considering the plane problem for an irrotational flow of an incompressible fluid, we direct our attention, first of all, on the construction of the kinematic flow pattern around a body fixed in a flow or for the motion of a body in a fluid at rest. This construction leads us to the determination of the complex potential, i.e. to the study of hydrodynamical singularities ; sources and vortices, in the entire plane of the flow which, in the absence of the body, would lead to the same kinematic flow pattern as would be observed after insertion of the body into the flow. Once the kinematic flow pattern is constructed, the pressure forces on the body in the flow are determined by Bernoulli's integral for steady motion and by Cauchy's integral for unsteady motion.

Accordingly, we have dealt with basic hydrodynamical singularities, their complex potentials, their image systems as well as conformal invariance. The interesting case of axis-symmetric flow and Stokes's stream function is also dealt with. The chapter is closed with an account of images in three dimensions.

3.01. Motion in two-dimensions. When the lines of motion are all parallel to a fixed plane (say XOY) and velocity at the corresponding





points of all planes parallel to that of XOY-plane has the same magnitude and direction, the motion is said to be two-dimensional.

Let, from a point A in the XOY planc a perpendicular be erected to meet a parallel plane $X'O'Y'$ (lying in the fluid) in the point B, *the corresponding point*. Then if the velocity at A is q making an angle θ with OY, the velocity at B is equal in magnitude and parallel in direction to the velocity at P. Thus the velocity shall be a function of x, y and t only but not of z.

It is quite useful to suppose the fluid in the two-dimensional motion to be confined between two barrier planes parallel to the plane of motion and at unit distance apart. The *reference plane of motion* is parallel to and mid-way between the hypothetical fixed planes.

Thus any closed curve drawn in the reference plane will represent a cross-section of cylindrical surface of unit thickness, (i.e. the dimension perpendicular to the plane of motion) bounded by the barrier planes.

3.02. Lagrange stream function ψ. In two-dimensional motion of incompressible fluid, the velocity q is a function of x, y, t but not of z, so that the differential equation of the stream lines is given by

$$(dx/u)=(dy/v) \qquad \text{or} \qquad vdx-udy=0 \tag{1}$$

The equation of continuity is

$$\nabla \cdot \mathbf{q}=0, \text{ i.e. } (\partial u/\partial x)+(\partial v/\partial y)=0 \tag{2}$$

But (2) is the condition that the differential equation (1) should be exact; it follows that (1) must be a perfect differential, $d\psi$ (say). Thus

$$vdx-udy=d\psi=(\partial\psi/\partial x)dx+(\partial\psi/\partial y)dy$$

so that $$u=-(\partial\psi/\partial y)\ ;\ v=(\partial\psi/\partial x). \tag{3}$$

We call the *function* ψ the *Lagrange stream function* or *current function*.

Obviously the stream lines are given by the solution of (1), i.e. ψ=const. Thus *the stream function is constant along a stream line.* It is clear from the foregoing considerations that the *existence of stream function is merely a consequence of the continuity and incompressibility of the fluid.* The current function always exists in all types of two-dimensional motion whether *rotational* or *irrotational*.

3.03. Cauchy-Riemann equations. From $\mathbf{q}=-\text{grad}\ \phi$, [i.e. $u=-(\partial\phi/\partial x)$ and $v=-(\partial\phi/\partial y)$] and (3) above we get

$$u=-(\partial\phi/\partial x)=-(\partial\psi/\partial y)\ ;\ v=-(\partial\phi/\partial y)=(\partial\psi/\partial x) \tag{1}$$

A further differentiation provides

$$(\partial^2\phi/\partial y\partial x)=(\partial^2\psi/\partial y^2)\ ;\ (\partial^2\phi/\partial x\partial y)=-(\partial^2\psi/\partial x^2).$$

Assuming the validity of $(\partial^2\phi/\partial y\partial x)=(\partial^2\phi/\partial x\partial y)$, we are led to

$$(\partial^2\psi/\partial x^2)+(\partial^2\psi/\partial y^2)=0 \qquad \text{or} \qquad \nabla^2\psi=0 \tag{2}$$

We may similarly obtain

$$(\partial^2\phi/\partial x^2)+(\partial^2\phi/\partial y^2)=0 \qquad \text{or} \qquad \nabla^2\phi=0 \tag{3}$$

The relations embodied in (1), viz.

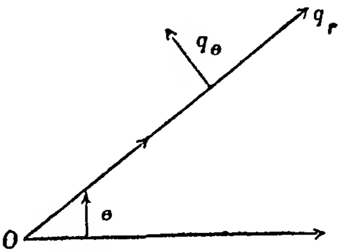
$$(\partial\phi/\partial x)=(\partial\psi/\partial y)\ ;\ (\partial\phi/\partial y)=-(\partial\psi/\partial x) \tag{1}$$

or their vector equivalents :

$$\textit{grad}\ \phi=(\textit{grad}\ \psi)\times\mathbf{k}\ ;\ \text{or}\ \textit{grad}\ \psi=\mathbf{k}\times(\textit{grad}\ \phi)\ [\mathbf{k}=(0,0,1)] \tag{1'}$$

are the usual *Cauchy-Riemann* partial differential equations. The equation (2) or (3) is Laplace's equation.

Cor Radial and cross-radial velocities in terms of ψ. If the motion be referred to polar coordinates, we have $x=r\cos\theta$, $y=r\sin\theta$. Now transforming the rectangular co-ordinates to polar coordinates the Cauchy-Riemann partial differential equation, viz. $\textit{grad}\ \phi=(\textit{grad}\ \psi)\times\mathbf{k}$, provides



$$[(\partial\phi/\partial r),\ (1/r)(\partial\phi/\partial\theta),\ 0]=[(\partial\psi/\partial r),\ (1/r)(\partial\psi/\partial\theta),\ 0]\times(0,0,1) \tag{1}$$

Remembering that $\mathbf{q}=-\textit{grad}\ \phi$, (1) yields the results

$$q_r=-(\partial\phi/\partial r)=-(\partial\psi/r\partial\theta)\ ;\ q_\theta=-(\partial\phi/r\partial\theta)=(\partial\psi/\partial r) \tag{2}$$

These are the required velocity components.

NOTE : If D is a *simply-connected* domain of steady plane flow of an inviscid incompressible fluid, and D contains no sources or vortices (i.e. $\textit{curl}\ \mathbf{q}=0$), then at all points of this domain

$$(\partial u/\partial x)+(\partial v/\partial y)=0,\ (\partial v/\partial x)-(\partial u/\partial y)=0\ ; \tag{1}$$

being the conditions of continuity and irrotationality. However, these are the conditions that the differential equations

$$vdx-udy=0,\ udx+vdy=0 \tag{2}$$

are the *total* differentials of certain functions ψ and ϕ of variables x, y in D. Thus

$$vdx-udy=d\psi\ ;\ udx+vdy=-d\phi \tag{3}$$

These yield the above results : $\mathbf{q}=-\textit{grad}\ \phi=\mathbf{k}\times\textit{grad}\ \psi$; and if l is any curve (path) in the simply-connected domain D, connecting some fixed point z_0 with the variable point z, then

$$-\phi(x,y)=\int_l (udx+vdy)\ ;\ \psi=\int_l (vdx-udy). \tag{4}$$

Obviously, the level lines of ψ in D give the trajectories of the moving particles, i.e. the stream lines of the flow.

In the general case of steady flow in two-dimensions, of an inviscid incompressible fluid, the sources and vortices producing the flow are distributed over a set of points S ; this set together with the moving fluid, occupies some region R of the open plane. The domains D of the source-free irrotational flow are the (disjoint) components of the open set which remains when we shunt the points of $\bar{S}$ from the interior of R by cutting suitable holes. The general domain D so

obtained is *multiply-connected* and the functions ϕ and ψ defined in D by integral formulae in (4) are, in general, multi-valued. It will be clear from the above that *continuously-differentiable branches* of ϕ and ψ, which satisfy (3), (4) and Cauchy-Riemann equations, can be separated in any simply-connected subdomain D_0 of D, and enable us to define the integrals

$$\int (\partial\phi/\partial r)dr.\int_C(\partial\psi/\partial r)dr \tag{5}$$

around any positively-oriented closed contour C in D. The values obtained for integrals in (5) will not depend on the particular continuous branches chosen :

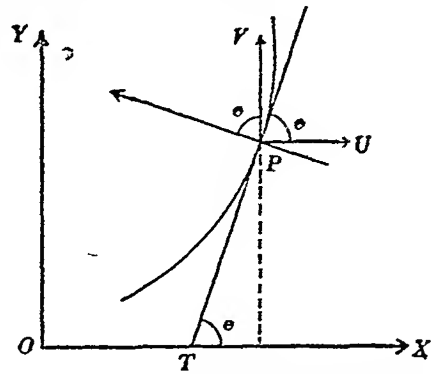
$$\Gamma=\int\left(-\frac{\partial\phi}{\partial r}\right)dr=\int \mathbf{q}.d\mathbf{r}\ ;\ \ Q=\int\left(\frac{\partial\psi}{\partial r}\,dr\right)=\int_C \mathbf{q}.\mathbf{n}dr$$

where Γ (the total strength of the vortices enclosed by C) and Q (the flow across C into the surrounding domain) are the *cyclical constants* for the functions ϕ and ψ.

Ex. Show that in steady two-dimensional irrotational motion both ϕ and ψ satisfy Laplace's equation, and that

$$u=-(\partial\phi/\partial x)=-(\partial\psi/\partial y)\ ;\ v=-(\partial\phi/\partial y)=(\partial\psi/\partial x) \qquad [Pb\ 1954]$$

3.04. Physical meaning of Lagrange stream function. Consider a curve l in the xy-plane. If the tangent at any point P of the element dS makes an angle θ with x-axis, the direction cosines of the normal there-at, (directed from right to left) shall be $(-\sin\theta, \cos\theta, 0)$. The flow, Q, across the curve l from right to left is



$$Q=\int_l \rho\mathbf{q}.\mathbf{n}\,ds$$

$$=\rho\int_l[(\partial\psi/\partial y)\sin\theta]+(\partial\psi/\partial x)\cos\theta]ds$$

$$\{\mathbf{q}=\mathbf{k}\times\nabla\psi=[-(\partial\psi/\partial y),(\partial\psi/\partial x),0]\}$$

$$=\rho\int_l\left(\frac{\partial\psi}{\partial y}dy+\frac{\partial\psi}{\partial x}dx\right)=\rho\int_l d\psi=(\psi_2-\psi_1)\rho$$

where ψ_2, ψ_1 are the values of ψ at the final and initial points of the curve. Thus, the difference between the values of stream function at any two points of a curve equals the flow across that curve.

Cor. Let AB be an *infinitesimal* arc of a curve whose length is δs. Then flow across it is $Q=\rho q\delta s$ as well as $Q=\rho(\psi_2-\psi_1)=\rho\delta\psi$. Thus, $q=d\psi/ds$, the velocity in terms of the stream function.

Exp. 1. Prove the following properties of a liquid in two dimensions:

(i) *There is always a stream function whether the motion is irrotational or rototional.*

(ii) *A stream line cuts itself at a point of stagnation and the two branches are at right angles when the motion is irrotational.*

(iii) *If the speed is everywhere the same, the stream lines are straight.*

[*Del 1955, 39 ; Gti 64, 63 ; Osm 59 ; Pna 64*]

Sol. (i) The equation of continuity is valid whether the motion is rotational or irrotational. And div $\mathbf{q}=0$, i.e. $(\partial u/\partial x)+(\partial v/\partial y)=0$, is the necessary condition that the stream lines in two-dimensions, viz. $dx/u=dy/v$, i.e. $v dx-u dy=0$. should admit a perfect differential say $d\psi$. This function ψ is Lagrange's stream function and clearly exists for irrotational as well as rotational fluid motion.

(ii) For a two-dimensional flow, a stagnation point is, by definition, characterized by the relations

$$\mathbf{q}=0, \quad \partial\psi/\partial x=0, \; \partial\psi/\partial y=0.$$

There is no loss of generality if we set $\psi=0$ at the stagnation point and let it be at the origin of coordinates. We now apply the Maclaurin expansion formula for the immediate vicinity of (0, 0) : This yields

$$\psi=\tfrac{1}{2}[x^2(\partial^2\psi/\partial x^2)_0+2xy(\partial^2\psi/\partial x\partial y)_0+y^2(\partial^2\psi/\partial y^2)_0]+\cdots \tag{1}$$

But in the neighbourhood of the origin, (1) represents a pair of straight lines, e.g.

$$ax^2+by^2+2hxy=0 \tag{2}$$

where $\qquad (\partial^2\psi/\partial x^2)_0=a, \; (\partial^2\psi/\partial y^2)_0=b, \; (\partial^2\psi/\partial x\partial y)_0=h.$

The two lines given by (2) are at right angles if $a+b=0$. This implies

$$(\partial^2\psi/\partial x^2)_0+(\partial^2\psi/\partial y^2)_0=(\nabla^2\psi)_0=0,$$

which is certainly true for irrotational fluid motion.

(iii) The stream lines with constant velocity components are

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$$

The solutions are $\qquad vx-uy=\text{const.} ; \; vz-wy=\text{const.}$

The intersection of these planes are necessarily straight lines.

Exp. 2. If λ denotes a variable parameter, and f a given function ; find the condition that $f(x, y, \lambda)=0$ should be a possible system of stream lines for steady irrotational motion in two dimensions. [*Bom 1959 ; Pb 54*]

Sol. If $f(x, y, \lambda)=0$ represents a possible system of stream lines for $\lambda=a_1$, a_2, ... ; then these should correspond to the relations $\psi=b_1, b_2, \ldots$ say ; consequently ψ's must be dependent on λ's only. Hence, ψ must be a function of λ, i.e. $\psi=\psi(\lambda)$. Now

$$\nabla\psi=(\partial\psi/\partial\lambda)\nabla\lambda \; ; \; \nabla^2\psi=(\partial\psi/\partial\lambda)\,\nabla^2\lambda+(\nabla\lambda).[\partial/\partial\lambda)\nabla\psi]$$

Using the first of relations into the second, and noting that for irrotational two-dimensional motion $\nabla^2\psi=0$, we obtain

$$(\partial\psi/\partial\lambda)\nabla^2\lambda+(\nabla\lambda)^2(\partial^2\psi/\partial\lambda^2)=0$$

or
$$\frac{\nabla^2\lambda}{|\nabla\lambda|^2}=\frac{(\partial^2\lambda/\partial x^2)+(\partial^2\lambda/\partial y^2)}{(\partial\lambda/\partial x)^2+(\partial\lambda/\partial y)^2}=-\frac{(\partial^2\psi/\partial\lambda^2)}{(\partial\psi/\partial\lambda)}. \tag{1}$$

Thus, $\nabla^2\lambda/|\nabla\lambda|^2$ is a function of λ only which is the required condition.

Exp. 3. Incompressible fluid of density ρ is contained between two co-axial circular cylinders, of radii a and b ($a<b$), and between two rigid planes perpendicular to the axis at a distance l apart. The cylinders are at rest and the fluid is circulating in irrotational motion, its velocity being V at the surface of the inner cylinder. Prove that the kinetic energy is $\pi\rho a^2 l V^2 \log(b/a)$.

[*Bom 1963 ; Del 62 ; Gti 55 ; Pb 52, 49*]

Sol. For irrotational two-dimensional fluid motion

$$\nabla^2\psi=(\partial^2\psi/\partial r^2)+(1/r)(\partial\psi/\partial r)+(1/r^2)(\partial^2\psi/\partial\theta^2)=0 \quad (1)$$

However, the circumstances of the motion require ψ to be only function of r, so that $(\partial^2\psi/\partial\theta^2)=0$. We then have the first integral of (1) as

$$(\partial\psi/\partial r)=C/r=\text{Cross-radial velocity}. \quad (2)$$

Obviously, the radial velocity is zero (as $\partial\psi/\partial\theta=0$). Thus (2) gives the resultant velocity. Since $\partial\psi/\partial r=V$ when $r=a$, the constant $C=aV$. Thus, $q=(aV/r)$. Now

$$\text{K.E. of the fluid}=\int_a^b \tfrac{1}{2}\,(2\pi r dr\, l\rho).(aV/r)^2$$

$$=\pi\rho la^2V^2 \log\,(b/a).$$

NOTE : Motion here is cyclic and could also be discussed via

$$T=-\tfrac{1}{2}\rho\int_S \phi\frac{\partial\psi}{\partial n}dS-\tfrac{1}{2}\rho K\int_S\frac{\partial\phi}{\partial n}dS$$

where the first integral here is zero, K=circulation$=2\pi aV$ and

$$(\partial\phi/\partial n)\,dS=l\,(C/r)dr.$$

Ex. 1. (a) Determine the condition that

$$U=ax+by\;;\;V=cx+dy$$

may give the velocity components of an incompressible fluid. Show that the stream lines of this motion are conic sections in general, and rectangular hyperbolas when the motion is irrotational. [*Kr 1960*]

(b) Show that $U=2cxy$, $V=c(a^2+x^2-y^2)$ are the velocity components of a possible fluid motion. Determine the stream function and sketch the stream lines.

Ex. 2. Show that the equation of continuity is satisfied by

$$U=akr^n\,e^{-k(n+1)\theta} \text{ and } V=ar^n\,e^{-k(n+1)\theta}$$

where U, V are the velocities in the directions of r and θ increasing (respectively). Determine the stream function and show also that the fluid speed at any point is

$$(n+1)\,\psi\sqrt{1+k^2}/r,$$

where ψ is the stream function. [Motion is supposed to be two-dimensional]. [*Del 1951 ; Sag 57*]

Ex. 3. Define the notion of stream function in the case of two-dimensional motion of a liquid, and derive the velocity components from the stream function. Also bring out clearly the relations between the stream function, and the velocity potential. [*Pna 1960*]

Ex. 4. Sketch the stream line which passes through the *stagnation point* of the motion given by

$$\psi=U\,[y-a\,\tan^{-1}(y/x)]$$

and determine the velocity at the points where this line crosses the axis of y.

Ex. 5. In a steady two-dimensional motion of an incompressible liquid, the stream lines are given by

$$x=f_1\,(k,c),\;\; y=f_2\,(k,c),$$

where c is a parameter defining a stream line and k is a parameter defining a point on a stream line. Show that the particle at the point given by (k_0, c_0) at time t_0 will be at the point given by (k, c_0) at time t, where

$$t-t_0=\left[\,C\int_{k_0}^{k}\frac{\partial\,(x,y)}{\partial\,(k,c)}dk\,\right]_{c=c_0}$$

and C is a function of c. [*Gti 1956*]

Ex. 6. Show that the curvature of a stream line in a steady motion is

$$\frac{1}{q^2}\frac{\partial}{\partial n}\left(\frac{p}{\rho}+V\right),$$

where p, ρ, q are the pressure, density and velocity of the liquid, V the potential of the external forces, and ∂n is an element of the principal normal to the stream line, and hence obtain the velocity potential of the two dimensional irrotational motion for which the stream lines are confocal ellipses.

3.05. ***Vorticity in two-dimensions.*** *In the two dimensional motion of an incompressible fluid, the vorticity of any particle remains constant.*

For an incompressible fluid in the xy-plane, $\nabla=(\partial/\partial x, \partial/\partial y, 0)$ $\mathbf{q}=(u, v, 0)$, $\boldsymbol{\omega}=\nabla\times\mathbf{q}=[0, 0, (\partial v/\partial x)-(\partial u/\partial y)]=\mathbf{k}\omega$ (say). Hence, for constant ρ, Helmholtz's vorticity equation (vide §1.90 p. 68)

$$(d\boldsymbol{\omega}/dt)=(\boldsymbol{\omega}.\nabla)\mathbf{q}, \Rightarrow (d\omega/dt)=0,$$

for $\boldsymbol{\omega}.\nabla=\omega(\mathbf{k}.\nabla)=0$. Hence $\boldsymbol{\omega}=$ *constant* $=\mathbf{k}\omega$. We may regard ω as a vortex strength per unit area.

Furthur, $\boldsymbol{\omega}=$ *curl* $\mathbf{q}=[(\partial v/\partial x)-(\partial u/\partial y)]\mathbf{k}$ reduces, when stream function exists ($v=\partial\psi/\partial x$, $u=-\partial\psi/\partial y$), to

$$\boldsymbol{\omega}=[(\partial^2\psi/\partial x^2)+(\partial^2\psi/\partial y^2)]\mathbf{k}=(\nabla^2\psi)\mathbf{k}. \qquad (1)$$

Exp. Show that ψ satisfies the equation

$$\mathbf{k}\frac{\partial}{\partial t}(\nabla^2\psi)+\nabla\psi\times\nabla(\nabla^2\psi)=0$$

k being a unit vector perpendicular to the plane of motion. (*Pra 1965, 64*)

Sol. Since for a two-dimensional fluid motion $\mathbf{q}=\mathbf{k}\times$ *grad* ψ ($=-$ *grad* ϕ), the definition of vorticity implies

$$\boldsymbol{\omega}=curl\ \mathbf{q}=curl\ (\mathbf{k}\times grad\ \psi)=(\nabla^2\psi)\ \mathbf{k}$$

Substituting for $\boldsymbol{\omega}$ in Helmholtz's vorticity equation: $(d/dt)(\boldsymbol{\omega}/\rho)=(\boldsymbol{\omega}.\nabla/\rho)\mathbf{q}$ and cancelling the constant ρ we obtain

$$\mathbf{k}(d/dt)(\nabla^2\psi)=(\nabla^2\psi\ \mathbf{k}.\nabla)(\mathbf{k}\times\nabla\psi).$$

The right hand member is identically zero ($\mathbf{k}.\nabla=0$), the expansion of the left hand member gives

$$\mathbf{k}\ (\partial\nabla^2\psi/\partial t)+\mathbf{k}(\mathbf{q}.\nabla)\ \nabla^2\psi=0 \qquad (i)$$

Now, $\mathbf{k}\ (\mathbf{q}_0.\nabla)=\mathbf{k}(\mathbf{k}\times\nabla\psi.\nabla)=\mathbf{k}(\mathbf{k}.\nabla\psi\times\nabla)=\mathbf{k}[\mathbf{k}.\mathbf{k}\ (\partial_x\psi\partial_y-\partial_y\psi\partial_x)]$

$$=\mathbf{k}\ (\partial_x\psi\partial_y-\partial_y\psi\partial_x)=\nabla\psi\times\nabla \qquad (\partial_x\psi=\partial\psi/\partial x, \text{ etc}) \quad (ii)$$

Substitutions from (ii) into (i) provide the result stated.

3.10. ***Basic hydrodynamical singularities in two-dimensional flows.*** Many potential problems are solved by combining simpler known solutions. The technique involves the use of 'principle of superposition' and is often called "method of singularities", since the simple potential solutions used all involve mathematically singular functions. Physically, this implies, that at the location of any hydrodynamical singularity the velocity is infinite.

Here, we shall discuss the four basic hydrodynamic singularities in 2-dim flow, viz. the source (and sink), the doublet, the vortex, and the parallel stream. Since the parallel stream may be generated by a source at $-\infty$ and an equal sink at $+\infty$ (vide § 3.24 p. 152), there are essentially three independent singularities.

(1) *Source. If the two-dimensional motion of a liquid consists of symmetrically distributed outward radial flow from a point, the point is called a simple source.*

But *we must remember* that our two-dimensional motion is the motion of a liquid occupying three dimensions, the simple source is necessarily a *line source* and it may be regarded as a straight axis of unit length between two fixed planes, which emits fluid radially and symmetrically.

Strength of the source. If $2\pi m\rho$ is the rate of emission of volume per unit time, m is called the strength of the source.

A source of *negative strength* or *inward radial flow* is called a *sink.*

If q_r is the radial velocity at a distance r from the source, the flux (rate of flow) out of the circle of radius r is $2\pi r\rho q_r$. Hence by definition

$$2\pi r\rho q_r=2\pi m\rho\,; \Rightarrow q_r=m/r.$$

(2) *Doublet. A combination of a source of strength m and a sink of strength $-m$ at an infinitesimal distance δs apart is called a doublet or a double source or a dipole.*

Strength of the doublet. If the product $m\,\delta s$, where m is infinitely great and δs infinitesimally small, remains finite and equal to μ (say), then μ is called the strength of the doublet.

The line δs is called the *axis of the doublet,* the positive direction along the axis being reckoned *from sink to source.*

(3) *Circular vortex. The section of a cylindrical vortex tube, (whose cross-section is a circle of radius a) by the plane of motion is a circle and the liquid inside such a tube is said to form a circular vortex.*

Strength of a circular vortex. If ω is the angular velocity and πa^2, the cross-sectional area of the tube, supposed small, then circulation

$$\Gamma=\int_C \mathbf{q}.d\mathbf{r}=\int_S curl\ \mathbf{q}.d\mathbf{S}=\omega\int_S dS=\omega\pi a^2.$$

This product of the cross-section and angular velocity at any point of the vortex tube is constant along the vortex and is known as the strength of the circular vortex.

Rectilinear or columnar vortex filament. The strength k of circular vortex is given by

$$k=\pi\omega a^2.$$

If we let $a\to 0$ and $\omega\to\infty$ such that the above product remains constant, we get a *rectilinear* vortex filament and represent it by a point in the plane of motion.

The strength of a vortex filament is *positive* when the circulation round it is counter-clockwise and *negative* when clockwise.

NOTES : In vortex motion the curvature of the stream lines introduces the action of centrifugal force which must be counter-balanced by a pressure gradient in the fluid.

We may divide vortices into four types :

(i) *Forced vortex.* The fluid rotates as a rigid body with constant angular velocity.

(ii) *Free cylindrical vortex.* The fluid moves along stream lines which are concentric circles in horizontal planes and there is no variation of total energy with radius.

(iii) *Free spiral vortex.* This is a combination of the free cylindrical vortex and radial flow.

(iv) *Compound vortex.* The fluid rotates as a forced vortex at the centre and as a free vortex outside.

3.20. *Complex potential.* The relation

$$w=f(z)=\phi+i\psi, \tag{1}$$

defined over a simply-connected domain D, where ϕ and ψ are velocity potential and stream function of the two-dimensional irrotational motion of a perfect liquid, is defined as the complex potential of the fluid motion.

Since, $$(\partial\phi/\partial x)=(\partial\psi/\partial y)\ ;\ (\partial\phi/\partial y)=-(\partial\psi/\partial x) \tag{2}$$

are satisfied, it follows that w is analytic function of $z=(x+iy)$ in any region where ϕ and ψ are single-valued funetions.

Conversely, *if w is analytic, its real and imaginary parts give the velocity potential and stream function of a possible irrotational two-dimensional fluid motion.*

Complex velocity. Differentiating partially with regard to x, the complex potential $w=\phi+i\psi$, we obtain

$$\frac{\partial w}{\partial x}=\frac{\partial\phi}{\partial x}+i\frac{\partial\psi}{\partial x}=\frac{\partial\phi}{\partial x}-i\frac{\partial\phi}{\partial y}\quad\text{by (2)}$$

But, $\dfrac{\partial w}{\partial x}=\dfrac{dw}{dz}\dfrac{\partial z}{\partial x}=\dfrac{dw}{dz}$ as $\dfrac{\partial z}{\partial x}=1$; and $\mathbf{q}=-\textit{grad}\ \phi$, we get

$$-(dw/dz)=u-iv=q\,(\cos\theta-i\sin\theta)=qe^{-i\theta}. \tag{3}$$

The combination $(u-iv)$ is called the *complex velocity**. The speed is the modulus of complex velocity, so that

$$q=|-(dw/dz)|\sqrt{(u^2+v^2)}$$

Speed may also be calculated from the fact that

$$q^2=u^2+v^2=(u+iv)(u-iv)=(dw/dz)\,(d\bar{w}/d\bar{z}).$$

The points where velocity is zero are called stagnation points. Thus for stagnation points, $(dw/dz)=0$.

NOTE : In the more general case when D is multiply-connected domain, and if C is a contour in D surrounding a particular component of the frontier of D the *cyclical constants* for ϕ and ψ [equation (5), p. 137] will not both vanish, Therefore, ϕ and ψ, defined in a

*$-(dw/dz)=u-iv$ represents the vector $\bar{\mathbf{q}}$ which is the mirror image of the velocity vector $\mathbf{q}$ with respect to the real axis ox.

multiply-connected domain D, given by (1) will be, in general, *many-valued*; and thus $f(z)$ is a multiform analytic function defined in D. We can now separate regular branches of $f(z)$ on any simply-connected sub-domain D_0, and every such branch which is regular at a point z_0 in D will have the *same* derivative $f'(z_0)$, so that $f'(z)$ is *uniform* in D.

3.201. Connection between plane hydrodynamical problems and theory of complex variables. The complex potential :

$$w=f(z) \text{ or } \phi+i\psi=f(x+iy)$$

shows that every definite choice of an analytic function $f(z)$ gives a definite system of stream lines $\psi=$ const. and the equipotential lines $\phi=$ const. ; hence this establishes a definite kinematic image of the velocity field q (in fact two images as ϕ and ψ are conjugate functions). Thus, the kinematic study of the plane fluid dynamics is closely linked to the complex-variable theory. Consequently. many propositions of this well-developed branch of Mathematics find their hydrodynamical interpretations. Some applications of this theory shall be given in this chapter and some in chapter 6.

Ex. 1. (a) Show how the two-dimensional irrotational motion of a liquid may be described by a complex potential w of the form $w=f(z)$. Find the velocity in terms of the complex potential. [*Bom 1950*]

(b) Prove that any relation of the form

$$w=f(z), \text{ where } w=\phi+i\psi,\ z=x+iy$$

represents a two-dimensional irrotational motion in which the magnitude of the velocity is given by $|(dw/dz)|$. [*Ag 1964, 54 ; Pb 66*]

(c) Show that ϕ and ψ are the velocity potential and stream function of a possible two-dimensional irrotational motion, provided $\phi+i\psi=w(z)$, where $z=x+iy$. Interpret $|(dw/dz)|$. [*Bom 1957*]

Ex. 2. Show that the any two-dimensional irrotational motion of a liquid may be transformed into any other by multiplying the velocity of each particle of the fluid by exp (P) and turning its direction round through an angle θ, where P, and $-\theta$ are suitably chosen conjugate functions of x, y.

Exp. 1. Find the lines of flow in the two-dimensional fluid motion given by

$$\phi+i\psi=-\tfrac{1}{2}n(x+iy)^2 e^{2int}$$

Prove or verify that the path of the particles of the fluid (in polar co-ordinates) may be obtained by eliminating t from the equations.

$$r\cos(nt+\theta)-x_0=r\sin(nt+\theta)-y_0=nt\,(x_0-y_0).$$

[*Ban 1953 ; Kr 60 ; Mad 56 ; Osm 63 ; Pb 60, 50*]

Sol. $\phi+i\psi=-\frac{1}{2}n(x+iy)^2 e^{2int}=-\frac{1}{2}nr^2 e^{2i\theta}\,e^{2int}=-\frac{1}{2}nr^2 e^{2i\lambda}$

where $\lambda=nt+\theta$. Equating the real and imaginary parts, we get

$$\phi=-\tfrac{1}{2}nr^2\cos(2\theta+2nt),\ \psi=-\tfrac{1}{2}nr^2\sin(2\theta+2nt).$$

Here the motion is non-steady and irrotational. The lines of flow are by $\psi=$const., viz.

$$nr^2\sin 2(\theta+nt)=\text{const}$$

To find the paths of the particles of fluid

$$\dot{r}=-\partial\phi/\partial r=nr\cos 2\lambda\ ;\ r\dot{\theta}=-$$

From $\lambda=nt+\theta$, we get $\dot{\lambda}=n+\dot{\theta}$ and thus

$$nr\cos 2\lambda=\frac{dr}{dt}=\frac{dr}{d\lambda}\frac{d\lambda}{dt}=\frac{dr}{d\lambda}(n+\dot{\theta})=\frac{dr}{d\lambda}n(1-\sin 2\lambda)$$

or $$dr/r=\cos 2\lambda\, d\lambda/(1-\sin 2\lambda)$$

Integration provides

$$r^2(1-\sin 2\lambda)=\text{const.} \quad \text{or} \quad r(\cos\lambda-\sin\lambda)=A \text{ (const.)}$$

Since when $t=0$, $\lambda=\theta_0$ and $r=r_0$; $A=r_0\cos\theta_0-r_0\sin\theta_0=x_0-y_0$.

Hence $$r[\cos(nt+\theta)-\sin(nt+\theta)]=x_0-y_0 \qquad (\because \lambda=nt+\theta) \tag{1}$$

or $$r\cos(nt+\theta)-x_0=r\sin(nt+\theta)-y_0. \tag{2}$$

From $$\frac{d\lambda}{dt}=n+\frac{d\theta}{dt}, \Rightarrow \frac{d\lambda}{dt}=n(1-\sin 2\lambda)$$

we get $$\int\frac{d\lambda}{1-\sin^2\lambda}=\int\frac{\sec^2\lambda\, d\lambda}{(1-\tan\lambda)^2}=nt+B \text{ (const.)}$$

This gives : $$\frac{1}{1-\tan\lambda}=\frac{\cos\lambda}{\cos\lambda-\sin\lambda}=nt+B$$

Initially, $t=0$, $\lambda=\theta_0$, so that $B=r_0\cos\theta_0/r_0(\cos\theta_0-\sin\theta_0)=x_0/(x_0-y_0)$.

Thus, $$\frac{r\cos\lambda}{r(\cos\lambda-\sin\lambda)}=\frac{r\cos\lambda}{x_0-y_0}=nt+\frac{x_0}{x_0-y_0}$$

[by (1), $r(\cos\lambda-\sin\lambda)=x_0-y_0$]

or $$r\cos\lambda=nt(x_0-y_0)+x_0$$

or $$r\cos(nt+\theta)-x_0=nt(x_0-y_0) \tag{3}$$

From (2) and (3) we finally get

$$r\cos(nt+\theta)-x_0=r\sin(nt+\theta)-y_0=nt(x_0-y_0).$$

Exp. 2. Liquid of density ρ is flowing in two dimensions between the oval curves $r_1r_2=a^2$ and $r_1r_2=b^2$, where r_1, r_2 are distances measured from two fixed points. If the motion is irrotational and quantity q per unit time crosses any line joining the bounding curves, prove that the K.E. is

$$\pi\rho q^2/\log(b/a).$$

[*Bom 1957, 50 ; Cal 55, 53 ; Del 57, 48 ; Gti 60 ; Sag 56*]

Sol. Two-dimensional irrotational motion takes place in a doubly connected region $r_1r_2=a^2$, ($\psi=$const) and $r_1r_2=b^2$, ($\psi=$const.) and these ψ-forms suggest the type of complex potential to be tried. Assume, therefore,

$$W=iA\log(z-z_1)(z-z_2), \text{ i.e. } \phi+i\psi=iA[\log r_1r_2+i(\theta_1+\theta_2)]$$

where $$z-z_1=r_1e^{i\theta_1},\ z-z_2=r_2e^{i\theta_2}.$$

Thus, $$\phi=-A(\theta_1+\theta_2),\ \psi=A\log r_1r_2$$

Now, $$q=\psi_b-\psi_a=A(\log b^2-\log a^2)=2A\log(b/a)$$

$\therefore$ $$A=q/2\log(b/a).$$

Since region is doubly-connected, the cyclic constant K needs also be determined and consequently

$K=$(circulation)$=$decrease in ϕ on describing the circuit completely once only

$=+A(2\pi+2\pi)=4\pi A$ (difference on two sides of barrier)

Now kinetic energy of cyclic irrotational motion is given by

$$T=-\tfrac{1}{2}\rho\int\phi\frac{\partial\phi}{\partial n}dS-\tfrac{1}{2}\rho K\int\frac{\partial\phi}{\partial n}dS \text{ (for one barrier)}$$

$$=2\pi A\rho\int d\psi=2\pi\rho A(\psi_b-\psi_a)=\pi\rho q^2/\log(b/a).$$

Note here $\int\phi(\partial\phi/\partial n)\,dS=0$, on a rigid boundary.

Ex. 1. (*a*) Discuss the motion represented by $w=Az^2$ and show that the speed is everywhere proportional to the distance from the origin.

(*b*) Discuss the motion represented by $w=(Ua^3/2z^2)$ and show that the stream lines are lemniscates.

Ex. 2. (*a*) Prove or verify that the complex potential defined by

$$\frac{z}{a}=\exp\left(-\frac{nw}{m}\right)+\exp\left(\frac{(1-n)\,w}{m}\right)$$

makes the stream line $\psi=\pm m\pi$, straight and radiating from the origin. Prove that the flow is inwards towards the origin in one of the angles thus formed and outwards from the origin in the other (re-entrant) angle.

(*b*) If $w^2=z^2-1$, prove that stream lines for which $\psi=1$ is $y^2(1+x^2)=x^2$. Regarding this as a fixed boundary, show that the motion is that of a uniform stream flowing past the boundary.

Ex. 3. (*a*) Show that the velocity potential

$$\phi=\frac{1}{2}\log\frac{(x+a)^2+y^2}{(x-a)^2+y^2}$$

gives a possible motion, and determine the form of the stream lines.

Show that curves of equal speed are the *ovals of Cassini* given by $r\,r'=$const.

(*b*) Prove that for liquid circulating irrotationally in the part of the plane between two-intersecting circles, the curves of constant velocity are *Cassini's ovals*. [*Ag 1964, 54*]

Ex. 4. Determine the stream lines for the liquid that is streaming steadily and irrotationally in two dimensions in the region bounded by one branch of a hyperbola and its minor axis. [*Lkn 1959*]

3.21. *Some standard complex potentials.* It will be found that it is often easier to work directly in terms of w rather than in terms of ϕ and ψ. Hence there is a necessity of formulating fields in terms of w for some special cases, which we now consider.

(1) *Uniform stream.* When a uniform stream advances with velocity q inclined at an angle α with the x-axis, then $u=q\cos\alpha$, $v=q\sin\alpha$ so that, $-dw/dz=u-iv=qe^{-i\alpha}$. Thus,

$$w=-qze^{-i\alpha},\ (q,\ \alpha \text{ real constants}). \tag{1}$$

This provides $\phi+i\psi=-q\,(x+iy)(\cos\alpha-i\sin\alpha)$, so that

$$\phi=-q\,(x\cos\alpha+y\sin\alpha)\ ;\ \ \psi=-q\,(y\cos\alpha-x\sin\alpha).$$

Thus, the stream lines are straight lines, and in particular, when $\psi=0$, these lines pass through the origin.

(2) *Simple source.* If a source of strength m at the origin $0(z=0)$ is alone in incompressible fluid with no boundaries, the flow will be purely radial. Hence if q_r is the radial velocity across a small circle C, of radius r and centre O, then by definition, the flow across C provides

$$2\pi rq_r=2\pi m,\ \Rightarrow\ q_r=m/r.$$

Thus, $dw/dz=-u+iv=-(m/r)[\cos\theta-i\sin\theta]=-m/(r\,e^{i\theta})$

or $$dw/dz=-m/z,\ \Rightarrow\ w=-m\log z. \tag{2}$$

Clearly, at $z=0$, the velocity is infinite.

Since w exists, the motion is irrotational; the velocity potential and stream function are

$$\phi=-m \log r\,;\ \psi=-m\theta.$$

If the source is situated at the point $z=z_0$, then by a change of origin

$$w=-m \log (z-z_0) \tag{2'}$$

If there are several sources of strengths $m_1, m_2, \ldots m_n$ situated at $z_1, z_2, \ldots z_n$; then the required complex potential is, by superimposition,

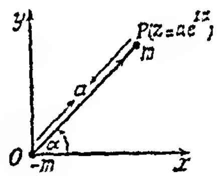
$$w=-m_1 \log (z-z_1)-m_2 \log (z-z_2) \ldots -m_n \log (z-z_n).$$

For a negative source (*sink*) of strength m situated at $z=z_0$, the complex potential shall be

$$w=m \log (z-z_0).$$

(3) *Doublet.* Consider a source of strength m at the point $P(z=ae^{i\alpha})$, and a sink of strength, $-m$ at the origion 0 $(z=0)$. The complex potential is

$$\begin{aligned} w&=m \log z-m \log (z-ae^{i\alpha}) \\ &=-m \log [1-(ae^{i\alpha}/z)] \\ &=(mae^{i\alpha}/z)+(ma^2e^{2i\alpha}/2z^2)+\ldots \text{ [log series]} \end{aligned}$$



If $ma\to\mu$, as $m\to\infty$ and $a\to 0$, then P tends to coincide with O along OP and there results a doublet of strength μ at O in the direction OP (axis of the doublet) with complex potential

$$w=\mu\, e^{i\alpha}/z \tag{3}$$

If the doublet is situated at the point $z=z_0$, and its axis its inclined at an angle α with x-axis, then

$$w=\mu\, e^{i\alpha}/(z-z_0) \tag{3'}$$

If there are several doublets each of strength $\mu_1, \mu_2, \ldots \mu_n$. situated at $z_1, z_2, \ldots z_n$ and their axes inclined at $\alpha_1, \alpha_2, \ldots \alpha_n$ with x-axis, then

$$w=\frac{\mu_1\, e^{i\alpha_1}}{z-z_1}+\frac{\mu_2\, e^{i\alpha_2}}{z-z_2}+\frac{\mu_3\, e^{i\alpha_3}}{z-z_3}+\ldots+\frac{\mu_n\, e^{i\alpha_n}}{z-z_n}.$$

(4) *Circulation about a circular cylinder.* In case of a doubly connected region, the possibility of *cyclic motion* does exist and as such we pro ceed to explain it presently in the case of a circle.

*When the circulation in a circuit is $2\pi k$, k is called the strength of the circulation.**

*The purpose of this definition is to avoid the constant occurrence of the factor 2π and to establish the correspondence between the definitions of two dimensional source and circulation *so far as* the strengths are concerned.

Consider the complex potential

$$w = ik \log z \qquad (4)$$

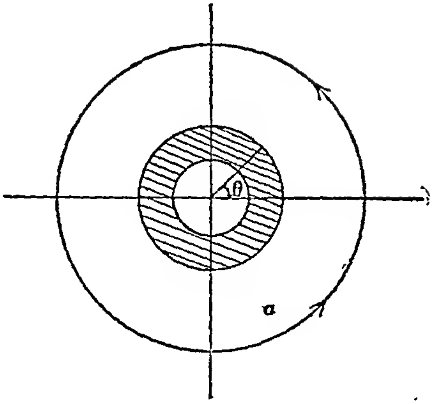
On the cylinder

$$|z| = a;\ z = a\,e^{i\theta}$$

$$\therefore \quad \phi + i\psi = ik \log (a\,e^{i\theta})$$
$$= -k\,\theta + ik \log a$$

Thus $\phi = -k\,\theta$;

$$\psi = k \log a \qquad \text{(const.)}$$



It follows that the cylinder $|z| = a$ is a stream line. By going once round the cylinder in the positive sense, θ increases by 2π and then

$$\phi = -k\,(\theta + 2\pi)$$

indicates decrease by $2\pi k$ in the value of ϕ. But *circulation = decrease in ϕ on describing the circuit once* $= 2\pi k$.

Hence there is a circulation of amount $2\pi k$ about the cylinder.

Obviously, $\quad |-(dw/dz)| = |k/z| = k/r.$

It follows that $q = (k/r)$ or *k is the speed at unit distance from the origin.*

As a useful deduction it may be observed that $w = ik \log z$ will also apply to the circulatory motion of liquid between the two concentric cylinders for $\psi = k \log r$ reduces to $k \log a$ and $k \log b$ on the cylinders $|z| = a$, and $|z| = b$, respectively.

(5) Rectilinear vortex. Consider a single cylindrical vortex tube whose cross-section is a circle of radius a, surrounded by unbounded fluid. We shall assume that the vorticity over the area of this circle is of constant value ω and zero everywhere outside.

If, as useful ψ is the stream function then $\zeta = \nabla^2\psi$ (§3.05 p 140), and hence

$$\zeta = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\,\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2\psi}{\partial\theta^2} \qquad (5)$$

The symmetry about the origin requires ψ to be a *function of r only* so that $\partial^2\psi/\partial\theta^2 = 0$. Thus

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\,\frac{\partial\psi}{\partial r} = \frac{1}{r}\,\frac{d}{dr}\left(r\,\frac{d\psi}{dr}\right) = \begin{cases}\zeta \text{ for } r < a \text{ inside vortex}\\ 0 \text{ for } r > a \text{ outside vortex}\end{cases}$$

Integration provides : $r(d\psi/dr) = \frac{1}{2}\,\zeta\, r^2 + A$ when $r < a$ and $r(d\psi/dr) = B$ (const.) when $r > a$. We are interested in the fluid outside $|z| = a$, so that integrating the preceding result we get

$$\psi = B \log r + C, \qquad (r > a).$$

The constant C may be chosen to be zero. To find B, we observe that the motion outside the vortex is irrotational, hence velocity potential exists and related to ψ by

$$(\partial\psi/\partial r) = -\partial\phi/r\partial\theta, \Rightarrow \phi = -B\theta + D, \quad (\because\ \partial\psi/\partial r = B/r)$$

We may neglect the constant D as before. Thus

$$\phi = -B\theta.$$

Now, the strength of the vortex is its circulation k (say). But *circulation=decrease in the value of ϕ on describing the circuit once only.*

$\therefore \quad k=-B[\theta-(\theta+2\pi)]=2\pi B\,;\ \Rightarrow B=k/2\pi=K^*$, say.

Thus, $\phi=-K\theta,\ \psi=K\log r\,;\ w=\phi+i\psi=iK(\log r+i\theta)$

or
$$w=iK\log z \tag{5}$$

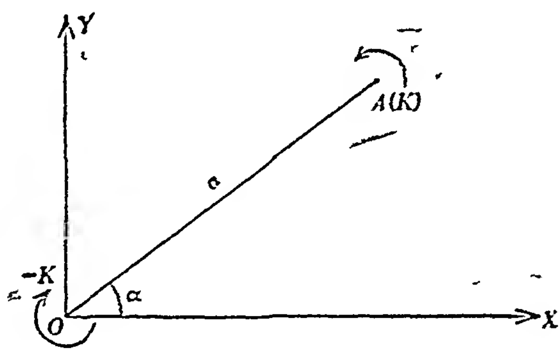
If the rectilinear vortex is situated at the point $z=z_0$, then by a change of origin

$$w=iK\log(z-z_0). \tag{5'}$$

If there are several vortices of strength $K_1, K_2, \ldots K_n$, situated at $z_1, z_2, \ldots z_n$, then the complex potential will be

$$w=iK_1\log(z-z_1)+iK_2\log(z-z_2)+\ldots+iK_n\log(z-z_n). \tag{5''}$$

(6) *Vortex doublet. The limiting case of two equal and opposite vortices as they coalesce is known as a vortex doublet.*



Consider a vortex of strength K at $z=a\,e^{i\alpha}$ and a vortex $-K$ at the origin. Then

$$w=iK\log(z-ae^{i\alpha})-iK\log z$$

$$=iK\log\left(1-\frac{ae^{i\alpha}}{z}\right)=-iK\left(\frac{ae^{i\alpha}}{z}+\frac{1}{2}\,\frac{a^2e^{2i\alpha}}{z^2}+\ldots\right)$$

If we let $a\to 0$ and $K\to\infty$ such that $aK=\mu$, then

(6)

This is the required complex potential for a vortex doublet at the origin. Also the complex potential of a double-source at the origin is $\mu e^{i\alpha}/z$. It follows that the complex potential of a vortex doublet is the same as that for a double source with its axes rotated through a right angle. In fact the two types of doublets are identical.

(7) *Spiral vortex. The combination of a source and a vortex is known as spiral vortex or vortex source.*

Consider a source of strength m and a vortex of strength K both at the origin. Then superimposing their corresponding complex potentials we get

$$w = -m \log z + iK \log z = (-m + iK) \log z. \tag{7}$$

This is the required complex potential for a spiral vortex at the origin and is supported by the fact that as we go once round the origin, w decreases by $2\pi(im + K)$. Thus ϕ $(= -m \log r - K\theta)$ decreases by $2\pi K$ and ψ $(= K \log r - m\theta)$ decreases by $2\pi m$. Therefore w satisfies the condition for a vortex and a source at the origin.

NOTE : The *circulation* Γ and the *flux* (flow) Q of the velocity vector $\mathbf{q}$, with respect to the closed contour C are defined by

$$\Gamma = \int_C \mathbf{q}.d\mathbf{r} = \int_C (u dx + v dy) = -\int_C d\phi,$$

$$Q = \int_C \mathbf{q}.\mathbf{n} ds = \int_C (u \sin\theta - v \cos\theta) ds = \int_C (u dy - v dx) = -\int_C d\psi$$

where $\mathbf{n} = (\sin\theta, -\cos\theta, 0)$ is the *outward* unit normal to the closed contour C. These immediately imply

$$\Gamma + iQ = -\int_C d(\phi + i\psi) = -\int_C (dw/dz)/dz$$

If $w'(z)$ $[= dw/dz]$ is defined within C and has a finite number of singularities there, then by Cauchy's residue theorem :

$$\Gamma + iQ = -2\pi i \text{ [sum of the residues of } w(z)].$$

If a is the pole of the function $w'(z)$, then $w(z)$ possesses in the neighbourhood of a an expansion of the form

$$w(z) = \frac{C_{-n}}{(z-a)^n} + \dots + \frac{\mu}{2\pi} \cdot \frac{1}{z-a} + \frac{\Gamma + iQ}{2\pi i} \log (z-a) + C_0 + C_1(z-a) + \dots$$

The term $\{(\Gamma + iQ) \log (z-a)/(2\pi i)\}$; (Γ, Q real numbers) defines *a vortex source* of strength Q and intensity Γ, often denoted by $(a ; Q, \Gamma)$. The term $\mu/2\pi (z-a)$, is doublet with moment μ, where μ is a complex number, arg $p = \alpha$; the direction of the axis of the doublet through a in the direction of the stream line. The remaining terms $C_{-r}/(z-a)^r$, define at the point a the *multiplets* of order $2r$. If at infinity

$$w(z) = (C_{-n}/z^n) + \dots + (C_{-1}/z) + \{(\Gamma + iQ)/2\pi i\} \log z + C_0 + \dots + C_n z^n$$

then the above definitions are valid *at* ∞, e.g. the term $\{(\Gamma + iQ)/2\pi i\}$ $\log z$ is the vortex source at ∞ etc.

Ex. 1. Find the complex potential due to (i) a two-dimensional point source, (ii) a two-dimensional doublet. [*Del 1965*]

Ex. 2. A single source is placed in an infinite perfectly *elastic* fluid, which is also a perfect conductor of heat. Show that if the motion be steady, the velocity V at a distance r from the source satisfies the equation

$$[V-(k/V)]\,(\partial V/\partial r)=(2k/r)$$

and hence that $$r=e^{V^2/4k}/\sqrt{V}$$

3.22. The circle theorem. If a solid cylinder $|z|\leqslant a$ is introduced (avoiding singularities) into a field of two-dimensional irrotational flow in incompressible inviscid fluid with no rigid boundaries, previously of complex potential $f(z)$, then the complex potential becomes

$$w=f(z)+\bar{f}(a^2/z).$$

Proof. Let C be the cross-section of the circular cylinder $|z|=a$ Then,

(i) on this circle we have $z\bar{z}=a^2$ or $\bar{z}=a^2/z$, so that for points on the circle $|z|=a$,

$$w=f(z)+\bar{f}(a^2/z)=f(z)+\bar{f}(\bar{z})$$

$$=f(z)+\overline{f(z)}: \text{ a purely real quantity.}$$

Thus for such points, w is a real quantity. But $w=\phi+i\,\psi$; it follows that $\psi=0$ on C, i.e. C is a *stream line* in the new flow.

(ii) Since *by hypothesis* all the singularities of $f(z)$ lie outside the circle $|z|=a$, all those of $f(a^2/z)$ and *therefore* all those of $\bar{f}(a^2/z)$ will lie within it, because for a point z outside C, the point (a^2/z) is inside C. Consequently, the additional term $\bar{f}(a^2/z)$ introduces no new singularities into the flow *outside C.*

Thus the function w will satisfy the conditions (Laplace's equation, etc.) for irrotational fluid motion with C inserted as does the function $f(z)$ in the absence of C.

NOTE: The above theorem proves *extremely useful* for calculating the image system whenever a circular cylinder is present in the field of sources, doublets or vortices.

Ex. There is a two-dimensional irrotational motion of a perfect liquid specified by the complex potential $w=f(z)$, such that there are no rigid boundaries and such that there are no singularities of the flow within the circle $|z|=a$. The flow is assumed to be due to a system of sources, vortices, doublets and possibly more complicated singularities, all of which are exterior to the circle $|z|=a$ Find the complex potential of the new flow when a solid cylinder $|z|=a$ is introduced into the above field of flow. [*Del 1963*]

3.23. Combination of sources and streams. The complex potentials of the motions due to a uniform stream and any number of sources are additive provided no boundaries are present in the liquid.

Consider the complex potential

$$w=-Uz-m_1\log z-m_2\log(z-a),$$

then $$-\frac{dw}{dz}=U+\frac{m_1}{z}+\frac{m_2}{z-a}.$$

Now when $z \to \infty$, $dw/dz \to U$, i.e. $u - iv = U$.

Thus $u = U$, $v = 0$, so that there is a uniform stream. Near $z = a$ we put $z - a = re^{i\theta}$, where r is small, then

$$-\frac{dw}{dz} = u - iv = U + \frac{m_1}{a + re^{i\theta}} + \frac{m_2}{r} e^{-i\theta}.$$

Since r is very small, m_2/r is very large and so the term $U + [(m_1/(a + re^{i\theta})]$ can be neglected as compared with $m_2 e^{-i\theta}/r$.

Hence $$u - iv = \frac{m_2}{r} e^{-i\theta} = \frac{m_2}{r}(\cos\theta - i\sin\theta),$$

which gives, $u = \dfrac{m_2}{r}\cos\theta$; $v = \dfrac{m_2}{r}\sin\theta$ and consequently

$$q = \sqrt{u^2 + v^2} = m_2/r.$$

It follows that there is an outward radial flow from $z = a$, where a source of strength m_2 is situated.

Again putting $z = re^{i\theta}$, where r is small, we get

$$-\frac{dw}{dz} = u - iv = U + \frac{m_1}{r} e^{-i\theta} + \frac{m_2}{re^{i\theta} - a}$$

As before, $U + \dfrac{m_2}{re^{i\theta} - a}$ is negligible compared with $\dfrac{m_1}{r} e^{-i\theta}$, and therefore

$$u - iv = m_1(\cos\theta - i\sin\theta)/r.$$

This yields, $\quad u = m_1 \cos\theta/r$; $v = m_1 \sin\theta/r$

so that there is a radial velocity m_1/r at $z = 0$ due to a source of strength m_1 at that point. Hence the potential

$$w = -Uz - m_1 \log z - m_2 \log(z - a)$$

gives a uniform stream U at infinity, and the sources of strength m_1 and m_2 at $z = 0$ and $z = a$ respectively.

Thus the *additive property* is established.

Cor. 1. The complex potential,

$$w = -Uz - m_1 \log(z - a_1) - m_2 \log(z - a_2) - \ldots m_r \log(z - a_r)$$

gives a uniform stream U at infinity, and the sources of strength $m_1, m_2, \ldots m_r$ at $z = a_1, a_2, \ldots, a_r$.

NOTE : The theorem fails if boundaries occur in the liquid. To prove this, consider the complex potential

$$w = Uz + (Ua^2/z) - m \log(z - a). \qquad (1)$$

(i) The first term Uz represents the stream whose complex potential is Uz.

(ii) The second term (Ua^2/z) represents the presence of the $|z| = a$ in the liquid (*Circle theorem*).

(iii) The third term $-m \log(z-a)$ represents the complex potential due to a source m at $z=a$.

Obviously (1) is the complex potential function of some motion, but it does not represent the motion of the stream past a cylinder in the presence of the source. To prove it, we put $z=ae^{i\theta}$ (on the circle), then

$$\psi=-m \tan^{-1}\left(\frac{a \sin \theta}{a \cos \theta-a}\right)+Ua \sin \theta+\frac{Ua^2}{a}(-\sin \theta)$$

$$=-m \tan^{-1}(-\cot \tfrac{1}{2}\theta)=-m \tan^{-1}[\tan(\tfrac{1}{2}\pi+\tfrac{1}{2}\theta)]$$

i.e. $$\psi=-\tfrac{1}{2}m(\pi+\theta).$$

Thus the stream function does not become constant on the circle $r=a$, so that the cylinder is not a stream line.

3.24. Uniform streaming deduced from a combination of a source and a sink. Let the liquid motion in the x-y plane be due to a source m at $z=a$ and a sink $-m$ at $z=-a$, where a is real and positive.

The complex potential of the liquid motion is

$$\begin{aligned} w&=m \log(z+a)-m \log(z-a) \\ &=m \log[1+(z/a)]-m \log[1-(z/a)]+\text{const.} \\ &=m(\xi-\tfrac{1}{2}\xi^2+\tfrac{1}{3}\xi^3-\ldots)+m(\xi+\tfrac{1}{2}\xi^2+\tfrac{1}{3}\xi^3+\ldots), \quad [z/a=\xi] \\ &=(2mz/a)+(2m/a^3)S, \end{aligned}$$

where we have *omitted* the immaterial constant and have assumed $|z|<a$, so that logarithms may be expanded, and S is the sum of an obviously convergent series.

Now, when $m\to\infty$, $a\to\infty$ (i.e. source and sink get indefinitely apart), i.e. $(1/a)\to 0$, but $(m)(1/a)\to\tfrac{1}{2}U$ (a finite quantity), the coefficient of S gets zero, so that the complex potential reduces to

$$w=Uz, \Rightarrow \phi=Ux, \psi=Uy.$$

If follows that stream lines are all parallel to the real axis and flow uniformly in the negative direction of this axis.

Thus the motion due to the limiting form of the combination of a source and a sink (i.e., *a doublet*) *is a uniform stream.*

3.25. Examples illustrative of complex potentials

Exp. 1. In a two dimensional liquid motion ϕ and ψ are the velocity potential and current function ; show that a second fluid motion exists in which ψ is the velocity potential and $-\phi$ the current function. Also prove that if the first motion be due to sources and sinks, the second motion can be built up by replacing a source and an equal sink by a line of doublets uniformly distributed along any curve joining them. [*Mad 1952*]

Sol. First Part. The complex potential of the fluid motion is defined by the relation

$$w=\phi+i\psi$$

where the real part ϕ denotes the velocity potential and the imaginary part ψ denotes the current function and these are connected by *Cauchy-Riemann partial differential equations*

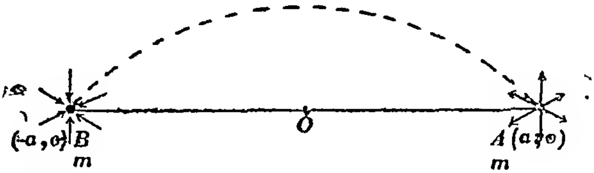
$$(\partial\phi/\partial x)=(\partial\psi/\partial y) \; ; \; (\partial\phi/\partial y)=-(\partial\psi/\partial x)$$

If, however, we consider the complex potential

$$w_1=-i\,w=-i\,\phi+\psi$$

then *Cauchy-Riemann partial differential equations* are again satisfied. Hence there exists a fluid motion in which $+\psi$ is the velocity potential and $-\phi$ the current function.

SECOND PART. Consider a source m at A and an equal sink $-m$ at B. Let the line AB be taken for x-axis and the mid-point of AB be the origin. If $AB=2a$,



then the complex potential due to these is

$$w=-m\log(z-a)=m\log(z+a)=m\log[(z+a)/(z-a)].$$

Now let any curve joining A and B be a line of doublets with axes inclined *at right angle* with OA. Then the complex potential w_1 due to this line of doublets is given by

$$w_1=\int_A^B \frac{me^{i\pi/2}}{z-t}\,dt=im\log\left(\frac{(z-a)}{(z+a)}\right).$$

This implies that $w_1=iw$ or $w=-iw_1$. Thus, the conditions of the problem in first part are satisfied and hence the conclusion for part second.

2. *Two sources, each of strength m, placed at the points* $(\pm a,\ 0)$ *and a sink o strength 2m is placed at the origin. Show that the stream lines are the curves*

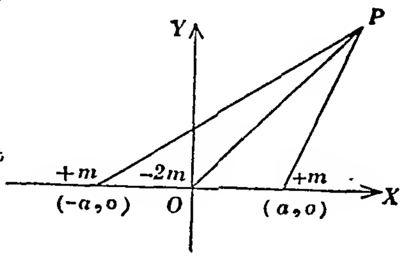
$$(x^2+y^2)^2=a^2(x^2-y^2+\lambda xy)$$

where λ *is a variable parameter.*

Show also that the fluid speed at any point is $2ma^2/r_1r_2r_3$ *where* r_1, r_2, r_3 *are respectively the distances of the point from the sources and the sink.*

[*Jad 1960 ; Osm 62 ; Pbi 66 ; Raj 63 ; I.A.S. 62*]

Sol. The complex potential of the superimposed system is given by



$$w=-m\log(z-a)-m\log(z+a)+2m\log z$$

$$=m[\log z^2-\log(z^2-a^2)]$$

or $\phi+i\psi=m[\log(x^2-y^2+2ixy)-\log(x^2-y^2-a^2+2ixy)]$

Equating the imaginary parts, we get

$$\psi=m\{\tan^{-1}[2xy/(x^2-y^2)]-\tan^{-1}[2xy/(x^2-y^2-a^2)]\}$$

$$=m\tan^{-1}\left(\frac{-2a^2xy}{(x^2+y^2)-a^2(x^2-y^2)}\right)$$

Hence the stream lines are given by

$$m\tan^{-1}(-2/\lambda)=m\tan^{-1}\{-2a^2xy/[(x^2+y^2)-a^2(x^2-y^2)]\}$$

or
$$(x^2+y^2)^2=a^2(x^2-y^2+\lambda xy)$$

by properly chosing the constant value of the stream function.

The fluid speed at any point is given by $|dw/dz|$.

But $\quad dw/dz=2m\{(1/z)-[z/(z^2-a^2)]\}=[-2a^2m/z(z-a)(z+a)]$

$\therefore\ q=|dw/dz|=(2ma^2/r_1.r_2.r_3)$, where $|z|=r_1,\ |z-a|=r_2,\ |z+a|=r_3$.

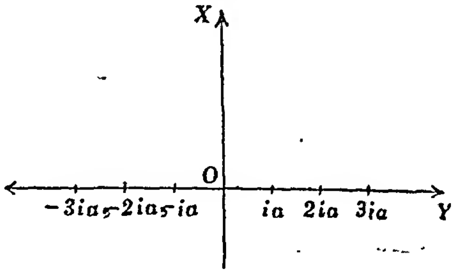
3. Parallel line-sources (perpendicular to the x-y plane) of equal strength m are placed at the points $z=nia$, when $n=\ldots-2,-1,0,1,2,3\ldots$; prove that the complex potential is

$$w=-m\log\sinh(\pi z/a).$$

Hence show that the complex potential for two-dimensional doublets (line doublets), with their axes parallel to the x-axis, of strength μ at the same points, is given by

$$w=\mu\coth(\pi z/a).$$

[Del 1953 ; Osm 62]



Sol. Let us consider $(2n+1)$ sources, taking the origin at the middle one, viz. $z=0$. The complex potential of these $(2n+1)$ sources is

$$w_{2n+1}=-m\log z-m\sum_{r=1}^{n}\log(z-ira)-m\sum_{r=1}^{n}\log(z+ira)$$

$$=-m\log z(z^2+a^2)(z^2+2^2a^2)(z^2+3^2a^2)\ldots(z^2+n^2a^2)$$

$$=-m\log(\pi z/a)(1+z^2/a^2)(1+z^2/2^2a^2)\ldots(1+z^2/n^2a^2)+\text{constant} \quad (1)$$

Now putting $\theta=\pi z/a$ in the infinite product of $\sinh\theta$, viz.

$$\sinh\theta=\theta(1+\theta^2/\pi^2)(1+\theta^2/2^2\pi^2)\ldots(1+\theta^2/n^2\pi^2)\ldots$$

and taking limit of (1) when $n\to\infty$, we get, when the *irrelevant constant* in (1) is neglected

$$w=-m\log\sinh(\pi z/a). \quad (2)$$

The complex potential for the doublets at the same points is the negative derivative of (2). Thus

$$w'=-\partial w/\partial z=(m\pi/a)\coth(\pi z/a)=\mu\coth(\pi z/a)$$

[Note : For a source m at a, $w=-m\log(z-a)$ & for a doublet, $w=m/(z-a)$].

4. A source of strength m and a vortex of strength K are placed at the origin of the two-dimensional motion of unbounded liquid. Prove that the pressure at infinity exceeds the pressure at distance r from the origin by $\frac{1}{2}(m^2+K^2)\rho/r^2$.

Prove further that the stream lines are equiangular spirals.

Sol. Superimposing the complex potentials of the flow due to a source and a vortex we get

$$w=-m\log z+iK\log z=(-m+iK)\log z$$

Thus, $\quad \phi=-m\log r-K\theta,\quad \psi=K\log r-m\theta.$

Obviously $\psi=$const. yields $r=A\,e^{a\theta}$: an equiangular spiral.

To find the pressure due to a spiral vortex, we use the Bernoulli-Euler's pressure equation

$$(p/\rho)+\tfrac{1}{2}q^2+\chi=C.$$

Now, velocity q of a spiral vortex at $z=re^{i\theta}$ is given by

$$q^2=(dw/dz)(d\bar{w}/d\bar{z})=[(-m+iK)/z][(-m-iK)/\bar{z}]=(m'^2/r^2). \qquad (m'^2=m^2+K^2)$$

Thus pressure at a distance r is given by

$$p=C-\rho\varkappa-(\rho m'^2/2r^2) \tag{1}$$

As $r\to\infty$, $p\to\Pi$, and (1) yields $\Pi=C-\rho\varkappa$ whence (1) can be rewritten as

$$p=\Pi-(\rho m'^2/2r^2) \quad \text{or} \quad \Pi-p=\rho m'^2/2r^2.$$

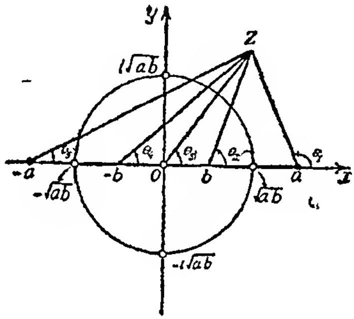
We may also observe from (1) that *the pressure due to a spiral vortex is the same as that due to a source of suitable strength.*

5. *Liquid extending to infinity is bounded internally by the fixed cylinder whose cross-section is the circle $x^2+y^2=ab$, where $a>b>0$. The liquid is in steady motion due to equal uniform line sources through the points $(\pm a, 0)$ parallel to the axis of the cylinder. Find the complex potential and show that the stream line which leaves the source at $(a, 0)$ in the direction making an angle θ with ox has an asymptote which makes an angle $\frac{1}{2}\theta$ with ox.*

Show further that, in the finite portion of the plane, there are four stagnation points.

Sol. The complex potential due to source m at $z=a$ and source m at $z=-a$ is obviously

$$-m \log(z-a)-m\log(z+a)$$
$$=-m\log(z^2-a^2).$$



When circular cylinder is inserted, the new complex potential, by virtue of 'circle theorem' becomes

$$w=-m\log(z^2-a^2)$$
$$-m\log[(a^2b^2/z^2)-a^2]$$
$$=2m\log z-m\log(z^2-a^2)-$$
$$m\log(z^2-b^2) \quad (1)$$

The stream function, given by the imaginary part of (1) is

$$\psi=2m\theta_3-m\theta_1-m\theta_2-m\theta_4-m\theta_5$$

and the stream lines are given by $\psi=\text{constant}=\lambda$ (say).

The stream line required which leaves the point A $(a, 0)$ at an angle θ with ox is obtained by giving λ the value which corresponds to a point on it very close to A $(a, 0)$. For such a point, which lies on the tangent to the curve at A, we have

$$\theta_1=\theta, \text{ (say) }; \ \theta_2=\theta_3=\theta_4=\theta_5=0, \text{ whence } \lambda=-m\theta.$$

For a point on the stream line at a far off distance from O,

$$\theta_1=\theta_2=\theta_3=\theta_4=\theta_5, \text{ whence } -2m\theta_1=-m\theta \text{ or } \theta_1=\tfrac{1}{2}\theta.$$

Thus, the angle between ox and the asymptote to the curve is $\frac{1}{2}\theta$.

For stagnation points $dw/dz=0$, which yields

$$(2m/z)-[2mz/(z^2-a^2)]-[2mz/(z^2-b^2)]=0, \text{ or } z^4=a^2b^2$$

Thus, $z=\pm\sqrt{(ab)}$, $\pm i\sqrt{(ab)}$; these being the intersection of the circle with the coordinate axes.

Ex. 1. What arrangement of sources and sinks in two dimensions will give rise to the function

$$w=\log[z-(a^2/z)].$$

Draw a rough sketch of the stream lines in this case, and prove that two of them subdivide into the circle $r=a$, and the axis of y.

[*Ag 1958 ; Jad 59 ; Kr 59*]

Ex. 2. Explain what hydrodynamical problem can be solved by means of the transformation

$$w=-m \log [(z-a)/(z+a)].$$ [*Ag 1953*]

Ex. 3. In two-dimensional motion, there is a uniform source along the real axis to total output $2\pi\, m$ stretching from $x=0$ to $x=a$. Show that the complex potential is

$$w=-\frac{m}{a}\int_0^a \log\ (z-\zeta)d\zeta=-m\left[\frac{z}{a}\log z-\frac{z-a}{a}\log (z-a)\right].$$

Combine this with a uniform stream U parallel to the x-axis and show that the dividing stream line is

$$Uy+(m/a)[x(\theta_2-\theta_1)+a\theta_2+y\log (r_1/r_2)-(\pi/a)]=0$$

where r_1, r_2 are the distances and θ_1, θ_2 the corresponding angles from a point on the line to the ends of the source. Trace the form of this line. [*Osm 1962*]

Ex. 4. Explain the significance of singular points, like sources and vortices in hydrodynamical problems. Show that any irrotational motion may be produced by a suitable distribution of them. [*Mad 1960*]

Ex. 5. Show that the velocity components given by

$$u=U\left[1-\frac{ay}{x^2+y^2}+\frac{b^2(x^2-y^2)}{(x^2+y^2)^2}\right],\quad v=U\left[\frac{ax}{x^2+y^2}+\frac{2b^2xy}{(x^2+y^2)^2}\right]$$

represent a possible fluid motion in two dimensions.

Show that the motion is irrotational, and interpret the meaning of the terms in complex potential.

Ex. 6. If there is a source m at A and a sink $-m$ at B and a uniform stream in the direction BA, find the *stagnation points*, and prove that they lie on AB or the perpendicular bisector according as the stream is relatively strong or feeble. [*Del 1948 (Adv.)*]

If there is a source at $(a, 0)$ and $(-a, 0)$ and sink at $(0, a)$, $(0, -a)$ all of equal strength, show that the circle through these four points is a stream line.

Ex. 7. Along the x-axis there exists for each stretch from $x=2na$ to $x=(2n+1)a$, a two-dimensional source of strength k per unit length, and from $x=(2n-1)a$ to $x=2na$, a two-dimensional sink of equal strength when n takes all positive and negative integral values. If w is the complex potential, find $-(dw/dz)$.

If in a channel bounded by walls at $x=a$ and $x=-a$, a line source stretches from $x=0$ to $x=a$ and an equal line sink from $x=0$ to $x=-a$; find the velocity at any point along the walls.

Ex. 8. Doublets of equal strength μ are placed at the points $z=nai$ when $n=\ldots -2, -1, 0, 1, 2, 3, \ldots$ in a uniform stream $-U$ parallel to the axis of x; prove that the stream line $\psi=0$ is

$$\frac{ay}{\pi b^2}=\frac{\sin (2\pi y/a)}{\cos (2\pi x/a)-\cos (2\pi y/a)}$$

and show that this consists in part of the x-axis and in part of an oval curve which is nearly circular (diameter $2b$) if b is small compared with a. Show that this solves the problem of a stream flowing through a set of parallel equidistant rails of approximately circular section.

Ex. 9. Two sources of equal strength are situated respectively at the points $(\pm a, 0)$ in an unbounded fluid. Show that at any point on the circle $x^2+y^2=a^2$, the fluid velocity is parallel to the axis of y, and inversely as the ordinate of the point. Determine also the point in the axis of y at which the velocity is greatest.

Hence show that, if a uniform stream parallel to the axis of y be combined with the two sources, there are necessarily two points at which the velocity vanishes.

Ex. 10. A thin sheet of incompressible fluid moves on the surface of a sphere of unit radius. Show that the velocity potential and stream function are conjugate functions of the Cartesian co-ordinates of the stereographic projection of any point; and that if the boundary moves as a rigid curve on the sphere and its axis of instantaneous rotation cut the sphere in O, the stream function at any point P of the boundary differs from $\omega \cos OP$ by a constant, where ω is the instantaneous angular velocity of the boundary.

3.26. Theorem of Blasius. In a steady two-dimensional irrotational motion given by the complex potential $w=f(z)$, if the pressure thrusts on the fixed cylindrical obstacle of any shape are represented by a force (X, Y) and a couple of moment M about the origin of co-ordinates, then, neglecting the external forces,

$$X-iY=\tfrac{1}{2}\, i\rho\int_C \left(\frac{dw}{dz}\right)^2 dz,$$

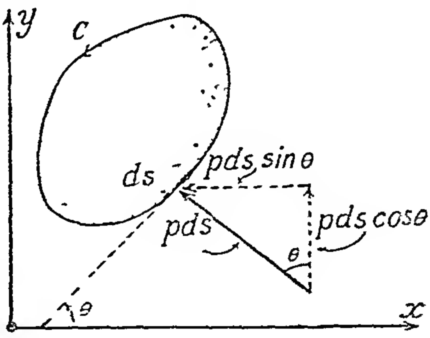
$$M=\text{Real part of, } -\tfrac{1}{2}\rho\int_C z\left(\frac{dw}{dz}\right)^2 dz$$

where ρ is the fluid density and integrals are taken round the contour C of the cylinder.

Proof. Let us consider an element of arc ds surrounding a point $P(x, y)$ of the fixed cylindrical obstacle (of unit height), the tangent to which makes an angle θ with x-axis. The fluid thrust at P (x, y), of magnitude pds, will act along the *inward* normal to the cylindrical obstacle and its components, parallel to the coordinate axes are

$-p \sin\theta\, ds$, $+p\cos\theta\, ds$

respectively. Thus, the force acting on element ds is given by



$$dF=dX+i\,dY=-p\sin\theta\, ds+ip\cos\theta\, ds$$
$$=ip(\cos\theta+i\sin\theta)\, ds=ip\, e^{i\theta}\, ds=ip\, dz \qquad (1)$$

because, $\quad dz=dx+i\,dy=ds\cos\theta+i\,ds\sin\theta=ds\, e^{i\theta}. \qquad (2)$

Since, C (the boundary of the obstacle) represents a stream line, we have by Bernoulli-Cauchy integral, $p+\frac{1}{2}\rho q^2=K$ (const.) or $p=K-\frac{1}{2}\rho q^2$, where q is the fluid speed on the stream line. Also,

$$dw/dz=-u+iv=-q(\cos\theta-i\sin\theta)=-q\, e^{-i\theta} \qquad (3)$$

Then, integrating over C, we find

$$F=X+iY=\int_C ipdz=i\int_C (K-\tfrac{1}{2}\rho q^2)dz$$

$$=-\tfrac{1}{2}i\rho\int_C q^2 dz \qquad \left[K\int_C dz=0\right]$$

$$=-\tfrac{1}{2}i\rho \int_C q^2 e^{i\theta}\, ds \qquad [\because\ dz=e^{i\theta}\, ds]$$

$$=-\tfrac{1}{2}i\rho \int_C (q^2 e^{2i\theta})(e^{-i\theta}\, ds)$$

or $$\bar{F}=X-iY=\tfrac{1}{2}i\rho\int_C (q^2 e^{-2i\theta})(e^{i\theta}\, ds)=\tfrac{1}{2}i\rho \int_C\left(\frac{dw}{dz}\right)^2 dz \tag{4}$$

We consider clockwise moments as positive. The moment about the origin of the fluid thrust acting on element ds (vide Fig) is $\mathbf{r}\times d\mathbf{F}$, i.e.

$$dM=(pds \sin \theta)y+(pds \cos \theta)x=p(ydy+xdx)$$

because $ds \sin \theta=dy$ and $ds \cos \theta=dx$. Then on using Bernoulli-Cauchy integral, the total moment is

$$M=\int_C p(ydy+xdx)=\int_C (K-\tfrac{1}{2}\rho q^2)(ydy+xdx)$$

$$=K\int_C (ydy+xdx)-\tfrac{1}{2}\rho \int_C q^2(ydy+xdx). \tag{5}$$

$$=0-\tfrac{1}{2}\rho\int_C q^2(x \cos \theta+y \sin \theta)\, ds$$

where we have used the fact that the first integral in (5) is zero, since $ydy+xdx$ is an exact differential. Hence

$$M=-\tfrac{1}{2}\rho \int_C q^2(x \cos \theta+y \sin \theta)\, ds$$

$$=\text{Real part of}\left\{-\tfrac{1}{2}\rho \int_C q^2(x+iy)(\cos \theta-i \sin \theta)ds\right\}$$

$$=\text{Real part of}\left\{-\tfrac{1}{2}\rho \int_C q^2\, ze^{-i\theta}\, ds\right\}$$

$$=Re\left\{-\tfrac{1}{2}\rho \int_C z(qe^{-i\theta})^2(e^{i\theta}\, ds)\right\}$$

$$=Re\left\{-\tfrac{1}{2}\rho \int_C z\left(\frac{dw}{dz}\right)^2 dz\right\}.$$

Sometimes we write this result in the form

$$M+iN=-\tfrac{1}{2}\rho\int_C z\left(\frac{dw}{dz}\right)^2 dz$$

where N has no simple physical significance.

Cor. Cauchy-Blasius theorem. We combine the two important theorems, viz.

$$\int_C f(z)dz=2\pi i\ [\textit{sum of residues of } f(z)], \qquad [\textit{Cauchy's Residue theorem}]$$

and $$X-iY=\tfrac{1}{2}i\rho\int_C\left(\frac{dw}{dz}\right)^2 dz \qquad [\textit{Blasius theorem}]$$

to yield $$X-iY=-\pi\rho\,[\textit{sum of residues of } (dw/dz)^2 \textit{ within } C]$$

This result may be called* '*Cauchy-Blasius theorem*', and quoted in this form, shall be found more useful.

Ex. 1. Show that the force per unit length on a fixed cylinder surrounded by incompressible fluid, of density ρ, in steady two-dimensional irrotational motion in planes perpendicular to the generators is of magnitude

$$\tfrac{1}{2}\rho i e^{i\alpha}\int (dw/dz)^2\, dz,$$

where w is the complex potential, α is the inclination of the force to the x-axis, and the integral is taken positively round the boundary of the cylinder.

Ex. 2. Discuss the two-dimensional liquid motion described by the potential function

$$w=V_0[z+(a^2/z)]-(i\,\Gamma/2\pi)\,\log z,$$

Γ being a constant. Draw a rough sketch of the stream lines. Find the stagnation points and discuss the cases $\Gamma <, =, > 4\pi V_0 a$.

Calculate the resultant force on the body in the liquid. [*Bom 1950*]

Exp. *Find the complex potential for the motion due to a system consisting of a coincident line-source and line-vortex in the presence of a circular cylinder of radius a, whose axis is parallel to and at a distance b (> a) from the line of the source and vortex. Show that the cylinder is attracted by a force of magnitude*

$$2\pi\rho a^2\,(m^2+K^2)/b(b^2-a^2)$$

per unit length.

Sol. Superimposing the complex potentials due to a source m and vortex K at $z=0$, we obtain

$$w=-m\log z+iK\log z=(iK-m)\log z. \tag{1}$$

When the circular cylinder $|z-b|=a$, $(b>a)$ is inserted, the complex potential (1), by virtue of circle theorem reduces to

$$w=(iK-m)\log z+(-iK-m)\log\{[a^2/(z-b)]+b\} \tag{2}$$

The force components on the cylinder C are given by Cauchy-Blasius theorem

$$X-iY=-\pi\rho\,\{\text{sum of the residues of } (dw/dz)^2 \textit{ within } C\} \tag{3}$$

Now, $$\left(\frac{dw}{dz}\right)^2=\left\{\frac{iK-m}{z}-\frac{(iK+m)}{a^2+b(z-b)}\,\frac{a^2}{(z-b)}\right\}^2 \tag{4}$$

The only singularities of dw/dz *within* C are, from (4), at $z=b$ and at $z=b-(a^2/b)$, because $z=0$ is *not inside* C. $\{b>a$, so $a>a^2/b$, $\{b-(a^2/b)\}>b-a\}$. Now if $R[x]$ stands for residue at x, then

$$R[b]=\{2(iK-m)/z\}\{(iK+m)a^2/(bz+a^2-b^2)\} \text{ at } z=b$$
$$=-2\,(K^2+m^2)/b$$
$$R[b-(a^2/b)]=\{2(iK-m)/z\}\{iK+m)a^2/b(z-b)\} \text{ at } z=b-(a^2/b)$$
$$=2\,(K^2+m^2)b/(b^2-a^2).$$

Substituting in (3) we obtain

$$X-iY=-2\pi\rho\,(K^2+m^2)\{b/(b^2-a^2)-1/b\}=-2\pi\rho a^2\,(K^2+m^2)/b(b^2-a^2).$$

Thus $Y=0$, $X=-2\pi\rho a^2\,(K^2+m^2)/b(b^2-a^2)$, the negative sign implies that the cylinder is attracted towards the origin where the vortex spiral is situated.

*The author firmly believes that this name is quite appropriate and that hydrodynamicists will approve it.

3.30. Motion of a solitary vortex filament. To find the velocity of the point $P(z)$ due to a vortex filament K at $z=z_0$, we have

$$q=-\frac{dw}{dz}=-\frac{iK}{(z-z_0)} \qquad (1)$$

for $w=iK \log (z-z_0)$.

Now put : $z-z_0=R\,e^{i\theta}$,

so that (vide Fig.)

$AP=R$, arg $(z-z_0)=\theta$

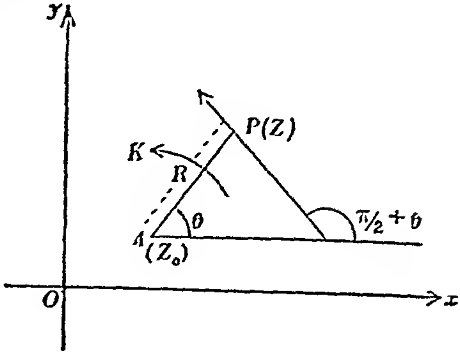
By (1), we then have

$u-iv=-i\,(K/R)\,e^{-i\theta}$.

Thus, $u=-(K/R)\sin\theta$

$=-(K/R^2)\,(y-y_0)$

$v=(K/R)\cos\theta=(K\,R^2)\,(x-x_0)$



where $z=x+iy$, $z_0=x_0+iy_0$. From these we get

$$\sqrt{u^2+v^2}=(K|R) \text{ and } (v/u)=-\cot\theta=\tan(90+\theta).$$

Thus the direction of motion at P is perpendicular to AP with speed (K/R) in the sense given by the rotation of the vortex at A.

Cor. Case of several vortices. If the motion is due to n vortex filaments of strengths K_s at the points z_s $(s=1, 2, 3,\ldots n)$, the complex potential at the point $P(z)$, *outside the vortex filaments* is given by

$$w=\Sigma\, iK_s \log (z-z_s), \quad s=1, 2,\ldots n$$

and the components of velocity at this point are given by

$$u-iv=-(dw/dz)=-\Sigma\, i\{K_s/(z-z_s)\}, \quad s=1, 2, \ldots n.$$

Alternatively, the preceding result is obtained by adding the velocity components due to the separate vortices in the form

$$u=-\Sigma\, K_s\,\{(y-y_s)/r_s^2\},\ v=\Sigma\, K_s\,\{(x-x_s)/r_s^2\},\ s=1, 2, \ldots n$$

where $\qquad z_s=x_s+iy_s.$

3.31. Motion of an individual vortex in a 'vortex-field. The stream function ψ at a distance $r < a$ (the radius of a particular cylindrical vortex) is determined by $\nabla^2\psi=\zeta$ (the vorticity) : using polar coordinates, ψ is clearly a function of r *only*, hence

$$\nabla^2\psi=\frac{d^2\psi}{dx^2}+\frac{1}{r}\,\frac{d\psi}{dr}=\frac{1}{r}\,\frac{d}{dr}\left(r\,\frac{d\psi}{dr}\right)=\zeta\,;\ (\text{for } r < a) \qquad (1)$$

Twice integrating (1) and noting that ζ is constant we get

(i) $(d\psi/dr)=\frac{1}{2}r\,\zeta+(A/r)$, (ii) $\zeta=\frac{1}{4}r^2\zeta+A\log r+B$,

where A and B are arbitrary constants.

Now the velocity at right angles to the radius vector is given by $(d\psi/dr)$ which is $\zeta r+(A/r)$. The velocity is obviously not infinite at the origin so that A must be zero. Hence the transverse velocity in

the vortex is ζr which vanishes at the origin (centre). *Thus the centre of a circular vortex and thereby the vortex filament, alone in the otherwise undisturbed fluid will not tend to move ; that is, the vortex filament induces no velocity at its centre. It follows that if such a vortex is in the presence of other vortices, it will not move of itself, but its motion through the liquid will be entirely due to the velocities induced by the remaining vortices.* Hence the complex velocity of the vortex k_r, which is produced *solely* by the other vortices is

$$u_r - iv_r = -\sum_{s=1}^{n} i\{K_s/(z_r - z_s)\} \; ; \; s \neq r. \tag{2}$$

The preceding result (2) may also be obtained as

$$w' = w - i\,K_r \log(z - z_r)$$

so that $\quad u_r - iv_r = -(dw'/dz) = -(dw/dz) + i\,\{K_r/(z - z_r)\}$

where *after differentiation, z_r is to be written for z.*

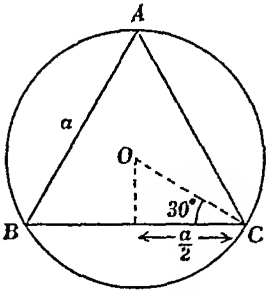
NOTE. The *induced velocity* at z_r may at once be written by differentiating the function $w = i\Sigma K_r K_s \log(z_r - z_s) : r \neq s$, with respect to z_r. Thus, considering the case of three vortices

$$w = i\,[K_1K_2 \log(z_1 - z_2) + K_2K_3 \log(z_2 - z_3) + K_3K_1 \log(z_3 - z_1)]$$

$$\therefore\; u_1 - iv_1 = -(1/K_1)(\partial w/\partial z_1) = -i[\{K_2/(z_1 - z_2)\} + \{K_3/(z_1 - z_3)\}] \; ; \text{ etc.}$$

Exp. 1. Three parallel rectilinear vortices of the same strenght K and in the same sense meet any plane perpendicular to them in an equilateral triangle of side a. Show that the vortices all move round the same cylinder with uniform speed in time $2\pi a^2/3K$. [*Bom 1961*]

Sol. If r be the radius of the circumcircle of the equilateral triangle ABC, then $r = a/\sqrt{3}$. The complex potential of the vortices situated at the points $[z = r\,e^{2p\pi i/3}, p = 1, 2, 3]$ is



$$w = iK\,[\log(z - r\,e^{2\pi i/3}) + \log(z - r\,e^{4\pi i/3}) + \log(z - r\,e^{6\pi i/3})]$$

$$= iK \log(z^3 - r^3).$$

The velocity *induced* at $z = r\,e^{2\pi i} = r$, by *others* is

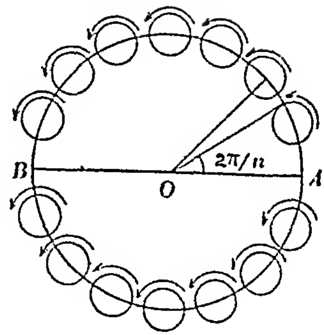
$$U_1 - iV_1 = -\frac{d}{dz}\{iK \log(z^3 - r^3) - iK \log(z - r)\} = -iK\left\{\frac{2z + r}{z^2 + zr + r^2}\right\}$$

Thus $\quad q_1 = |U_1 - iV_1| = K\,|(2z + r)/(z^2 + zr + r^2)|_{z=r} = K/r.$

The required time : $\quad T = (2\pi a/\sqrt{3}) \div (K/r) = 2\pi a^2/3K.$

Exp. 2. If n rectilinear vortices of the same strength k are symmetrically arranged as generators of a circular cylinder of radius a in an infinite liquid, prove that the vortices will move round the cylinder uniformly in time $8\pi^2a^2/(n-1)k$, and find the velocity of any point of the liquid.

[*Bom 1955 ; Del 47, 45, 37, 33 ; Jad 60 ; Pna 65, 64, 63 ; Pb*



Sol. Let the n rectilinear vortices each of strength k be situated at the points, $z_0, z_1, z_2, z_3 \ldots z_{n-1}$; of the circle so that

$$z_r = a\, e^{2\pi r i/n}, \qquad r=0, 1, \ldots n-1$$

Thus, since $a\, e^{2\pi r i/n}$ are the n distinct nth roots of a^n, $(k/2\pi = K)$

$$w = iK \sum_{r=0}^{n-1} \log (z - a\, e^{2\pi r i/n})$$

$$= iK \log \prod_{r=0}^{n-1} (z - a\, e^{2\pi r i/n})$$

$$= iK \log (z^n - a^n) \qquad (1)$$

The fluid velocity at any point, not occupied by any vortex, is given by $|dw/dz| = Kn\, |z^{n-1}| \,/\, |(z^n - a^n)|$.

We shall now find the velocity experienced by any one of the vortices, say that which is situated at $A(z=a)$. Then

$$w_0 = iK \log (z^n - a^n) - iK \log (z-a) = iK \log [(z^n - a^n)/(z-a)]$$

$$= iK \log (z^{n-1} + z^{n-2} a + \ldots + z\, a^{n-2} + a^{n-1})$$

Thus, $$\frac{dw_0}{dz} = iK \, \frac{(n-1)\, z^{n-2} + (n-2)\, z^{n-3} a + \ldots + a^{n-2}}{z^{n-1} + z^{n-2}\, a + \ldots + z\, a^{n-2} + a^{n-1}}.$$

And $$\left(\frac{dw_0}{dz}\right)_{z=a} = \frac{iK}{a} \, \frac{(n-1)+(n-2)+\ldots+2+1}{n} = \frac{iK}{a} \, \frac{n-1}{2}.$$

This implies that $q_x = 0$, $q_y = K(n-1)/2a$. Consequently, the radial and cross-radial velocities of the vortex at $z=a$, are $q_r = 0$, $q_\theta = K(n-1)/2a$. And by symmetry, each of the vortex has the same transverse velocity.

The required time $= 2\pi a / \{(n-1)\, K/2a\} = 8\pi^2 a^2/(n-1)\, k$.

Note. From (1): $\phi + i\psi = iK \log [(r^n \cos n\theta - a^n) + i r^n \sin n\theta]$,

$$\therefore \qquad \psi = \tfrac{1}{2} K \log [(r^n \cos n\theta - a^n)^2 + (r^n \sin n\theta)^2]$$

$$= \tfrac{1}{2} K \log (r^{2n} - 2a^n r^n \cos n\theta + a^{2n})$$

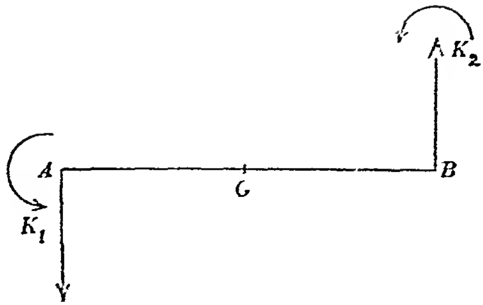
The relation $\mathbf{q} = \boldsymbol{\omega} \times \mathbf{r}$, $\Rightarrow$ $\omega = K(n-1)/2a^2$.

Ex. A rectilinear vortex of strength K is at a distance f from, and is parallel to, the axis of a solid cylinder of radius a and infinite length: the circulation about the cylinder is such that the vortex remains stationary. Show that the magnitude of this circulation is $Kf^2/(f^2 - a^2)$.

3.32. System of two vortex filaments. Let the vortex filaments of strength K_1 and K_2 be situated at the points $A\,(z=z_1)$ and $B\,(z=z_2)$, then

$$w = iK_1 \log (z - z_1) + iK_2 \log (z - z_2).$$

The velocity $(\dot{x}_r, \dot{y}_r)$ of a particular vortex K_r is produced solely by the other vortices, since it cannot affect its own motion, therefore



$$\begin{aligned}\dot{\bar{z}}_r &= \dot{x}_r - i\dot{y}_r \\ &= u_r - iv_r \\ &= -(dw_r/dz)_{z=z_r} \\ &= \sum_{s \neq r} [-iK_s/(z_r - z_s)]\end{aligned}$$

Thus, $\qquad \dot{\bar{z}}_1 = iK_2/(z_1 - z_2), \quad \dot{\bar{z}}_2 = -iK_1/(z_2 - z_1)$

These yield $K_1\dot{\bar{z}}_1 + K_2\dot{\bar{z}}_2 = 0$, and since K_1 and K_2 are constants, this can be further written as

$$\frac{d}{dt}\left\{\frac{K_1z_1+K_2z_2}{K_1+K_2}\right\}=0, \Rightarrow \frac{K_1z_1+K_2z_2}{K_1+K_2}=\text{const.} \tag{1}$$

The point $(K_1z_1+K_2z_2)/(K_1+K_2)$ may be called the *centre of vorticity* by analogy with the centre of gravity : the strengths of vortices replacing the masses. Thus, if $K_1+K_2 \neq 0$, *the centre of vorticity is fixed* (*this point is not necessarily a stagnation point*).

Now since $K_1.AG = K_2.GB$, i.e. $AG/GB = K_2/K_1$ we easily get

$$AG = K_2.AB/(K_1+K_2).$$

And to put velocity of A in terms of AG we see that

$$|u_1 - iv_1| = \frac{K_2}{AB} = \left(\frac{K_2.AB}{K_1+K_2}\right)\left(\frac{K_1+K_2}{(AB)^2}\right) = AG.\omega \quad \left(\because \frac{d\mathbf{r}}{dt} = \omega \times \mathbf{r}\right)$$

where $\qquad \omega = (K_1+K_2)/(AB)^2.$

Thus the line AB rotates with this angular velocity ω. Further, neither vortex has a component of velocity along AB, it follows that AB remains constant in length.

Ex. 1. Two parallel line vortices of strengths $K_1, K_2, (K_1+K_2 \neq 0)$, in unlimited liquid across the z plane at right angles at points A, B respectively, the centre of mass of masses K_1 at A and K_2 at B is G. Show that if the motion of the liquid is due solely to these vortices, G is a fixed point about which A and B move in circles with angular velocity $(K_1+K_2)/AB^2$. Show also that the speed at any point P in the z-plane is $(K_1+K_2)\,CP/AP.BP$, where C is the centre of masses K_2 at A and K_1 at B. [*Bom 1963 (Old)*]

[$w = iK_1 \log(z-z_1) + iK_2 \log(z-z_2)$ yields

$$\frac{dw}{dz} = \frac{iK_1}{z-z_1} + \frac{iK_2}{z-z_2} = \frac{i\,(K_1+K_2)[z-\{K_1z_2+K_2z_1)/(K_1+K_2)\}]}{(z-z_1)(z-z_2)}.$$

Now $z - \{(K_1z_2+K_2z_1)/(K_1+K_2)\} = z - z_c$ and thus

$$|z-z_c| = CP, \quad |z-z_1| = AP, \quad |z-z_2| = BP.$$

Hence $\qquad |dw/dz| = (K_1+K_2).CP/AP.BP$].

Ex. 2. (*a*) Find the stream function due to a single line-vortex of strength k.

When an infinite liquid contains two parallel and equal rectilinear vortices at a distance $2b$ apart, prove that the stream lines relative to the vortices are given by the equation

$$x^2+y^2-2b^2 \log [(x-b)^2+y^2][(x+b)^2+y^2]=c,$$

the origin being the middle point of the join which is taken for the axis of x.

[*Bom 1952*]

(*b*) When an infinite liquid contains two parallel and equal vortices of the same strength, and the spin is in the same sense in both, show that the relative stream lines are given by

$$\log (r^4+b^4-2b^2r^2 \cos 2\theta)-(r^2/2b^2)=\text{constant},$$

θ being measured from the join of vortices, the origin being its middle point.

Show also that the surfaces of equipressure at any instant are given by

$$r^4+b^4-2b^2r^2 \cos 2\theta=\lambda\ (r^2 \cos 2\theta+a^2).$$

3.33. *Vortex pair. A pair of vortices each of strength K but of opposite rotations is called a vortex pair.*

Let the vortex filaments of strength K and $-K$ be situated at A $(z=z_1)$ and B $(z=z_2)$ at time $t=0$. The complex potential at this instant (i.e. $t=0$) is given by

$$w=iK \log (z-z_1)-iK \log (z-z_2).$$

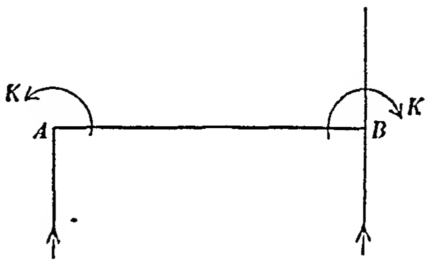


Fig. (i)

The velocity (u_r, v_r) of a particular vortex K_r is produced *solely by the other* vortices, since it cannot affect its own motion, therefore

$$u_r-iv_r=-(dw_r/dr)_{z=z_r}$$

where $w_r=\sum_{s\neq r} iK_s \log (z-z_s)$.

Thus, $u_1-iv_1=$

$$-iK\left[\frac{d}{dz}\log (z-z_2)\right]_{z=z_1}$$

$$=-iK/(z_1-z_2)$$

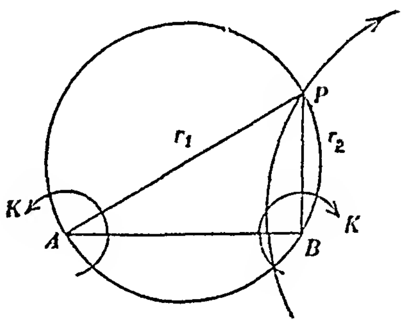


Fig. (ii)

Similarly, $u_2-iv_2=-iK/(z_2-z_1)$.

Hence $q_1=K/|z_1-z_2|=K/AB=q_2$. Thus, *both the vortices have a velocity K/AB at right angles to AB, in the negative y-direction,* each moving due to the other.

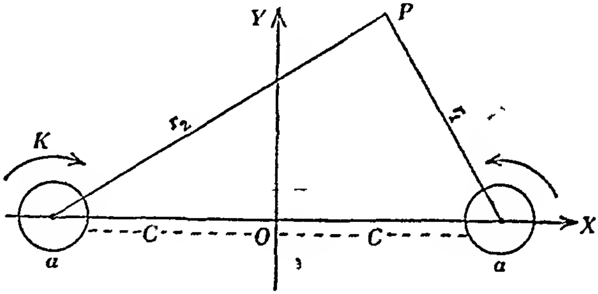
The stream function at $t=0$, is given by

$$\psi=K \log \{|(z-z_1)|/|(z-z_2)|\}=K \log (r_1/r_2)$$

where $AP=r_1$ and $BP=r_2$. This means that the stream lines are given by the coaxal circles $r_1/r_2=$const. which have A and B as limiting points. The equipotential lines ($\phi=$const.) are the conjugate coaxal circles which pass through the limiting points z_1 and z_2 (Fig. ii, p. 164).

3.34. Energy due to a pair of vortices. Consider a circular vortex pair each of radius a distant $2c$ apart.

If K is the strength of each vortex, the complex potential is given by



$$w=\phi+i\psi=iK\log(z-c)-iK\log(z+c)$$

whence, $\quad \psi=K\log\{|z-c|/|z+c|\}=K\log(r_1/r_2)$

where $\quad |z-c|=r_1, \quad |z+c|=r_2.$

Obviously, the value of ψ *on* the circle is obtained by putting $r_1=a$, $r_2 \doteq 2c$ and thus $\psi c_1=K\log(a/2c)=-K\log(2c/a)$.

The kinetic energy of the fluid *outside* the vortex at $z=a$ is

$$T_1=\tfrac{1}{2}\rho\iint \mathbf{q}.\mathbf{q}\,dxdy=\tfrac{1}{2}\rho\iint \mathbf{k}\times \textit{grad}\,\psi.\mathbf{q}\,dxdy \qquad [\mathbf{k}=(0,0,1)]$$

$$=\tfrac{1}{2}\rho\iint \textit{grad}\,\psi\times\mathbf{q}.d\mathbf{S} \qquad (d\mathbf{S}=\mathbf{k}\,dxdy)$$

Now, *curl* $(\psi\mathbf{q})=\psi$ *curl* $\mathbf{q}+$*grad* $\psi\times\mathbf{q}=$*grad* $\psi\times\mathbf{q}$ as *curl* $\mathbf{q}=0$ (for irrotational motion), hence

$$T_1=\tfrac{1}{2}\rho\iint \textit{curl}\,(\psi\mathbf{q}).d\mathbf{S}$$

$$=-\tfrac{1}{2}\rho\int_{C1}\psi\mathbf{q}.d\mathbf{r} \text{ (by Stokes' theorem)} \tag{1}$$

$$=\tfrac{1}{2}\rho\,\psi C_1(-\Gamma), \text{ where } \int\mathbf{q}.d\mathbf{r}=\Gamma \text{ is circulation, and } \psi \text{ is const.}$$

$$=\tfrac{1}{2}\rho\,[-K\log(2c/a)][-2\pi K]=\pi\rho K^2\log(2c/a). \tag{2}$$

The negative sign with the integral in (1) is accounted for by the fact that C_1 is an internal boundary with regard to the surrounding fluid. An equal amount of kinetic energy is contributed by the other vortex also, so the total kinetic energy external to the vortices is

$$T_0=2T_1=2\pi\rho K^2\log(2c/a).$$

Now the fluid *inside* the vortex rotates with angular velocity (K/a^2) and moves as a whole with velocity $(K/2c)$ induced by the other vortex. Thus, its kinetic energy is

$$T'=\pi a^2\rho\left[\tfrac{1}{2}\frac{K^2}{4c^2}+\tfrac{1}{2}\frac{a^2}{2}\frac{K^2}{a^4}\right]=\tfrac{1}{4}\pi\rho K^2$$

where we have neglected (a^2/c^2). Obviously, total *internal* energy is $2T'=T_i$.

Thus, to this order of approximation, the total energy is

$$T=T_i+T_0=2\pi\rho K^2 \left[\tfrac{1}{4}+\log (2c/a)\right].$$

Exp. (a) *When an infinite liquid contains two parallel, equal and opposite rectilinear vortices at a distance 2b, prove that the stream lines relative to the vortices are given by the equation*

$$\log \frac{x^2+(y-b)^2}{x^2+(y+b)^2}+\frac{y}{b}=c$$

the origin being the middle point of the join, which is taken for axis of y.
[*Kr 1961, 59 ; Ll.n 63 ; Osm 60 ; Pb 58, 50 ; Ut 64*]

(b) *An infinite liquid contains two parallel, equal and opposite rectilinear vortex filaments at a distance 2b. Show that the paths of the fluid particles relative to the vortices can be represented by the equation*

$$\log \frac{r^2+b^2-2rb\cos\theta}{r^2+b^2+2rb\cos\theta}+\frac{r\cos\theta}{b}=\text{const.}$$ [*Bom 1953*]

(c) *Show that for a vortex pair the relative stream lines are given by*

$$K\{(y/2a)+\log (r_1/r_2)\}=\text{const.}$$

where 2a is the distance between the vortices and r_1, r_2 *are the distances of any point from them.* [*Roj 1964*]

Sol. (a) The velocities of each of the vortices is $(K/AB)=(K/2b)$ and this is directed towards the x-axis. *To find the stream lines relative to the vortices, we are to impose a velocity on the whole system equal and opposite to the velocity of advance of the vortex.* Thus, to the complex potential of the vortex pair, we are to add a term $(K/2b)\,z$, because, $-dw/dz=-K/2b$ along XO supplies this information. Hence, for the case under consideration

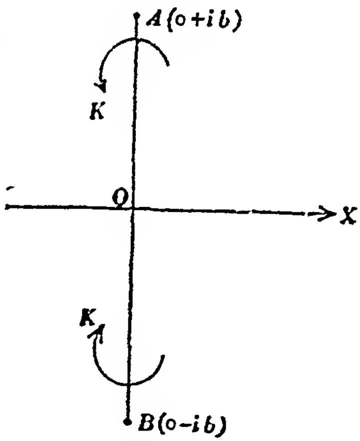
$$w=iK\log\{(z-ib)/(z+ib)\}+(K/2b)\,z$$

Equating the imaginary parts we get

$$\psi=\tfrac{1}{2}K\log[x^2+(y-b)^2]-\tfrac{1}{2}K\log[x^2+(y+b)^2]+(K/2b)\,y.$$

Thus the *relative stream lines* are given by

$$\log\frac{x^2+(y-b)^2}{x^2+(y+b)^2}+\frac{y}{b}=c.$$



(b) It is evident that the vortex pair lies along x-axis, so that by interchanging x and y in part (a), and using polar co-ordinates we get

$$\log\frac{r^2+b^2-2br\cos\theta}{r^2+b^2+2br\cos\theta}+\frac{r\cos\theta}{b}=c \text{ (const.)}$$

(c) This is merely $\psi=$const, where b is replaced by a.

Ex. Obtain the stream function due to a vortex pair. Deduce the equation of stream lines relative to the vortices.

If the vortex pair is replaced by two vortices having the same strength and direction of spin, what will be the equation of the relative stream lines ? [*Pb 1954*]

3.35. Kirchhoff vortex theorem: general system of vortex filaments.

If (r_1, θ_1) (r_2, θ_2),...*be the polar coordinates at any time t of a system of rectilinear vortices of strength* K_1, K_2, . *then*

$$\Sigma K_p x_p = A,\ \ \Sigma K_p y_p = B\ ;\quad \Sigma K_p r_p^2 = C,\quad \Sigma K_p r_p^2 \dot{\theta}_p = \Sigma K_p K_q = D$$
where A, B, C and D are constants.

The fluid motion due to n vortex filaments of strengths K_p at the points $z_p = x_p + iy_p$, where $p = 1, 2, 3 \ldots\ldots n$ is given by the complex potential

$$w = \Sigma_p iK_p \log (z - z_p). \tag{1}$$

The velocity at any point of the fluid, not occupied by any vortex, is given by

$$-u + iv = dw/dz = \textstyle\sum_p iK_p/(z - z_p).$$

The velocity (u_p, v_p) of a particular vortex K_p is produced *solely* by the *other* vortices as it will not move on its own account. Its velocity is therefore given by

$$-u_p + iv_p = \left(\frac{dw_p}{dz}\right)_{z=z_p} = \left[\frac{d}{dz} \sum_{q \neq p} iK_q \log (z - z_q)\right]_{z=z_p}$$

$$= \sum_{q \neq p} \frac{i\,K_q}{(z_p - z_q)} \tag{2}$$

Multiply both sides of (2) by K_p and sum up from $p = 1$ to n; this gives

$$\Sigma K_p (-u_p + iv_p) = \sum_p \sum_{q \neq p} \left[i\,K_p\,K_q/(z_p - z_q) \right] = 0\ ; \tag{3}$$

the double summation vanishes because the terms cancel in pairs, e.g. $i\,K_p\,K_q\,(z_p - z_q)$ cancels $i\,K_q\,K_p\,(z_q - z_p)$ and there are no terms in K_p^2, etc. Thus (3) yields

$$\textstyle\sum K_p u_p = 0,\quad \Sigma K_p v_p = 0 \tag{4}$$

or

$$\Sigma K_p x_p = A,\quad \Sigma K_p y_p = B \tag{4'}$$

because $u_p = dx_p/dt$, etc. and A, B are constants of integration. Further,

$$\Sigma_p K_p z_p \left(dw_p/dz\right)_{z=z_p} = \Sigma K_p z_p \left[\sum_{q \neq p} i\,K_q/(z_p - z_q)\right].$$

This gives

$$\Sigma_p K_p(x_p + iy_p)(-u_p + iv_p) = i\ \Sigma K_p K_q = i \times \text{real quantity}\ ; \tag{5}$$

since such pairs of terms as $K_p K_q z_p/(z_p - z_q) + K_q K_p z_q/(z_q - z_p) = K_p K_q$ and again there are no terms in K_p^2. Now (5) yields.

$$\Sigma K_p (x_p v_p - y_p u_p) = \Sigma K_p K_q = D \text{ (const)}\ ;$$
$$\Sigma K_p (x_p u_p + y_p v_p) = 0 \tag{6}$$

Using $x\dot{y} - y\dot{x} = r^2\dot{\theta}$ in the first equation of (6) and integrating the second equation of (6), we get

$$\Sigma K_p r_p^2 \dot{\theta}_p = \Sigma K_p K_q\ ;\ \Sigma K_p r_p^2 = C \text{ (const.)},\quad [r^2 = x^2 + y^2].$$

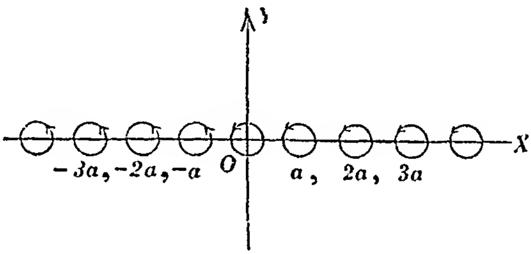
Ex. *If* $(r_1, \theta_1), (r_2, \theta_2) \dots$ *be the polar co-ordinates at time t of a system of rectilinear vortices of strength* $k_1, k_2 \dots$, *prove that*

$$\Sigma kr^2 = \text{const. and } \Sigma kr^2\dot{\theta} = (\tfrac{1}{2}\pi)\ \Sigma k_1 k_2.$$

[*Alig 1964* ; *Ald 63, 59* ; *Ban 47* ; *Bom 64* (old), *63 56* ; *Cal 53* ; *Del 56* ; *Jad 59* ; *Lkn 62* ; *Mad 60* ; *Osm 61, 59* ; *Pb 53* (*Sup* ; *Raj 65*)]

3.36 Single infinite row of vortices. We shall now *find the complex potential of an infinite row of parallel rectilinear vortices of the same strength K at a distance a apart.*

Let us consider $(2n+1)$ vortices, taking the origin at the middle one and axis of x through the centres of their sections so that these are situated at $0, \pm a, \pm 2a, \dots \pm na$.



The complex potential of these $(2n+1)$ vortices is

$$w_{2n+1} = iK \log z + iK \log (z-a) + \dots + iK \log (z-na) +$$
$$iK \log (z+a) + iK \log (z+2a) + \dots + iK \log (z+na).$$
$$= iK \log z\, (z^2+a^2)(z^2-2^2a^2)\dots(z^2-n^2a^2)$$
$$= iK \log \frac{\pi z}{a}\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{z^2}{2^2a^2}\right)\dots\left(1-\frac{z^2}{n^2a^2}\right) +$$
$$iK \log (-1)^n (a/\pi)(a^2 \,.\, 2^2a^2 \dots n^2a^2).$$

Omitting the *irrelevant* constant, the same can be expressed as

$$w_{2n+1} = iK \log \frac{\pi z}{a}\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{z^2}{2^2a^2}\right)\dots\left(1-\frac{z^2}{n^2a^2}\right) \tag{1}$$

We now put $\theta = (\pi z/a)$ in the infinite product :

$$\sin \theta = \theta\, (1-\theta^2/\pi^2)(1-\theta^2/2^2\pi^2) \dots (1-\theta^2/n^2\pi^2)\dots$$

to obtain

$$\sin (\pi z/a) = (\pi z/a)(1-z^2/a^2)(1-z^2/2^2a^2)\dots(1-z^2/n^2a^2)\dots \tag{2}$$

Now let $n \to \infty$ in (1), then from (1) and (2) we get

$$w = iK \log \sin (\pi z/a). \tag{3}$$

The velocity of the *vortex at the origin* is given by

$$q_0 = -\frac{d}{dz}\left[\ iK \log \sin \frac{\pi z}{a} - iK \log z\ \right]_{z=0}$$

$$= -iK\left[\frac{\pi}{a} \cot \frac{\pi z}{a} - \frac{1}{z}\right] \to 0, \text{ as } z \to 0.$$

Thus it is at rest ; similarly every other vortex is at rest so that *the vortex row induces no velocity in itself.*

The velocity components at any point of the fluid not occupied by any vortex filament are obtained from

$$u-iv=-(dw/dz)=-iK(\pi/a)\cot(\pi z/a)$$

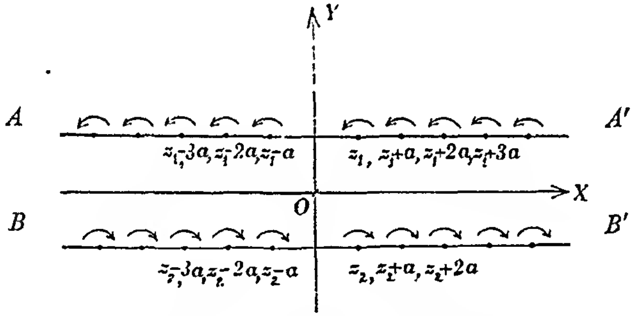
$$=-iK\lambda.\ \frac{\cos\lambda(x+iy)\sin\lambda(x-iy)}{\sin\lambda(x+iy)\sin\lambda(x-iy)},\quad \left(\lambda=\frac{\pi}{a}\right)$$

$$\therefore\qquad u=-\frac{K\lambda\sinh 2\lambda y}{\cosh 2\lambda y-\cos 2\lambda x},\quad v=\frac{K\lambda\sin 2\lambda x}{\cosh 2\lambda y-\cos 2\lambda x}$$

(where $\lambda=\pi/a$)

3.37. ***Double infinite row of vortices.*** Let us suppose that we have a system consisting of an infinite number of vortices, each of strength K, evenly placed along a line, and another system also consisting of infinite number of vortices, each of strength $-K$, spaced similarly, along a parallel line. Let the line mid-way between the two lines of vortices be taken as x-axis.

Let one vortex on infinite row AA' be at $z=z_1$ and one vortex on



infinite row BB' be at $z=z_2$, so that the *system consists of vortices K at $z=z_1\pm na$* and *vortices $-K$ at $z=z_2\pm na$*, where $n=0, 1, 2, 3\ldots$ At any point z outside the vortices, the complex potential due to the two vortices K at $z=z_1\pm na$ and two vortices $-K$ at $z=z_2\pm na$ is

$$w_0=iK\log\frac{(z-z_1-na)(z-z_1+na)}{(z-z_2-na)(z-z_2+na)}=iK\log\frac{(z-z_1)^2-n^2a^2}{(z-z_2)^2-n^2a^2}$$

Clearly, $iK\log[(z-z_1)/(z-z_2)]$ is the complex potential due to K at $z=z_1$ and $-K$ at $z=z_2$. Hence, the complex potential for the *whole* system is

$$w=iK\log\frac{(z-z_1)}{(z-z_2)}+iK\sum_{n=1}^{\infty}\log\frac{(z-z_1)^2-n^2a^2}{(z-z_2)^2-n^2a^2}$$

$$=iK\log\frac{(z-z_1)}{(z-z_2)}\prod_{n=1}^{\infty}\frac{1-[(z-z_1)^2/n^2a^2]}{1-[(z-z_2)^2/n^2a^2]}\qquad(1)$$

Since $\sin\theta=\theta\prod_{n=1}^{\infty}(1-\theta^2/n^2\pi^2)$, for all values of θ, real or complex, we get on setting θ equal to $\pi(z-z_1)/a$ and $\pi(z-z_2)/a$ in turn

$$\left.\begin{aligned}\sin[\pi(z-z_1)/a]&=[\pi(z-z_1)/a]\prod\{1-(z-z_1)^2/n^2a^2\}\\ \sin[\pi(z-z_2)/a]&=[\pi(z-z_2)/a]\prod\{1-(z-z_2)^2/n^2a^2\}\end{aligned}\right\}\ n=1,\ldots\infty$$

Hence, (1) reduces to

$$w=iK\log\frac{\sin[\pi(z-z_1)/a]}{\sin[\pi(z-z_2)/a]}. \qquad (2)$$

The velocity components at any point of the fluid not occupied by any vortex filament are obtained (putting $\pi/a=\lambda$) from

$$\begin{aligned}u-iv&=-(dw/dz)=-iK\lambda\{\cot\lambda(z-z_1)-\cot\lambda(z-z_2)\}\\ &=2iK\lambda.\sin\lambda(z_2-z_1)/[\cos\lambda(z-z_1)-\cos\lambda.(2z-z_1-z_2)]. \qquad (3)\end{aligned}$$

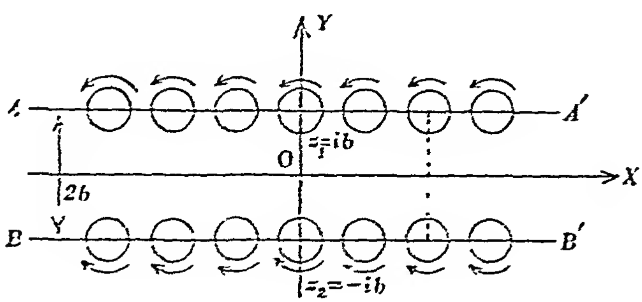
To find the velocity (u_1, v_1) of the vortex K at z_1 (say), we have,

$$\begin{aligned}u_1-iv_1&=-\{D[w-iK\log(z-z_1)]\}\text{ when } z=z_1 \text{ and } D=d/dz\\ &=iK\{\lambda\cot\lambda(z-z_2)-\lambda.\cot\lambda(z-z_1)+(z-z_1)^{-1}\}_{z=z_1}\end{aligned}$$

Since $\{\cot\lambda(z-z_1)-[1/\lambda(z-z_1)]\}\to 0$ as $z\to z_1$, we have

$$u_1-iv_1=iK\lambda\cot\lambda(z_1-z_2), \qquad (\lambda=\pi/a). \qquad (4)$$

Cor. 1. Let the two infinite rows AA' and BB' of the vortices at a distance $2b$ apart symmetrically placed with regard to the real axis have rotations directed in opposite directions, so that the *vortices are directly opposite in pairs.*



Here we have $z_1=ib$, $z_2=-ib$, so that (2) yields

$$w=iK\log\frac{\sin[\pi(z-ib)/a]}{\sin[\pi(z+ib)/a]}.$$

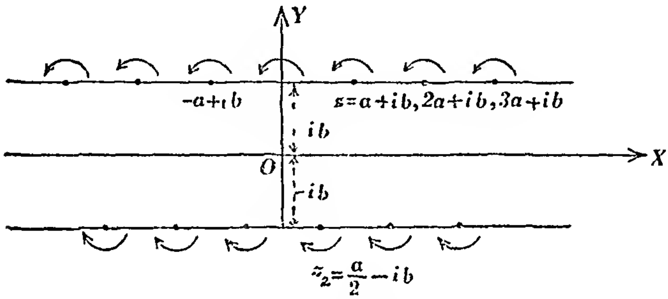
The velocity (u_1, v_1) of the vortex K at $z=ib$ is, by (4)

$$u_1-iv_1=iK\lambda.\cot\lambda(2ib)=K\lambda\coth(2\lambda b).$$

Thus $$v_1=0 \text{ and } u_1=K\lambda\coth(2\lambda b).$$

Since the choice of the vortex taken at $z=z_1$ is perfectly arbitrary, every vortex in AA' has the same velocity and, from symmetry, every vortex in BB' has also the same velocity. Thus the street moves as a whole in the direction of its length, with speed $(K\pi/a)\coth(2\pi b/a)$.

Cor. 2. Karman vortex street. This consists of two parallel infinite rows AA' and BB' of the same spacing a, so arranged that each vortex K of AA' is directly above the mid-point of the join of two vortices, each $-K$, of BB'.*



The vortices in the upper row are at $z=ma+ib$ and in the lower at $z=(m+\frac{1}{2})a-ib$, where $m=0, \pm1, \pm2, \ldots$ Thus the complex potential is

$$w=iK\log\frac{\sin\pi(z-ib)/a}{\sin\pi(z+ib-\frac{1}{2}a)/a}$$

which is obtained by putting $z_1=ib$ and $z_2=\frac{1}{2}a-ib$ in § 3.37(2), p. 170.

The velocity of the vortex K at $z_1=ib$ is, by § 3.37(4), p. 170

$$u_1-iv_1=\frac{iK\pi}{a}\cot\pi(ib-\tfrac{1}{2}a+ib)/a$$

$$=-\frac{iK\pi}{a}\tan\left(\frac{2\pi ib}{a}\right)=\frac{K\pi}{a}\tanh\left(\frac{2\pi b}{a}\right)$$

whence $\qquad v_1=0,\ u_1=(K\pi/a)\tanh(2\pi b/a)$.

Thus, the street now moves through the liquid with velocity

$$(K\pi/a)\tanh(2\pi b/a).$$

Ex. 1. (a) Prove Kelvin's theorem relating to circulation in a closed circuit.

If an infinite row of parallel straight vortices, each of strength K, be situated at the points

$$0, \pm a, \pm 2a, \ldots \pm na, \ldots$$

show that the complex potential is given by

$$w=iK\log\sin(\pi z/a);$$

find the velocity components at any point of the fluid. [*Cal 1954*]

*The student is recommended to construct an independent proof of this important theorem with the help of § 3.37 p.169.

(b) Prove that the product of cross-section and angular velocity at any point on a vortex filament is constant all along the vortex filament and for all time.

Find the complex potential of an infinite row of parallel rectilinear vortices of the same strength k at a distance a apart. [*Pb 1956 (S)*]

Ex. 2. (a) A *vortex street* consists of an infinite set of vortices of strengths k at the points

$$y=b, x=na\,;\ (n=0, \pm 1, \pm 2, \ldots)$$

together with an infinite set of strengths $-k$ at the points $y=-b$, $x=na$. Prove that the street induces in itself a translation with speed

$$(k/2a) \coth (2\pi b/a)$$

along its length. [*London School of Mathematics B.A. (Hons.) 1951*]

(b) Define strength of a vortex filament and prove that it is constant and further that the vortex lines move with the fluid.

Two rows of (straight) vortices exist on two infinite parallel lines with the same spacing, the vorticities in the two lines being equal but opposite. How do the lines move ? Find the condition that the system may be in relative equilibrium with a motion of translation parallel to the lines. [*Del 1948*]

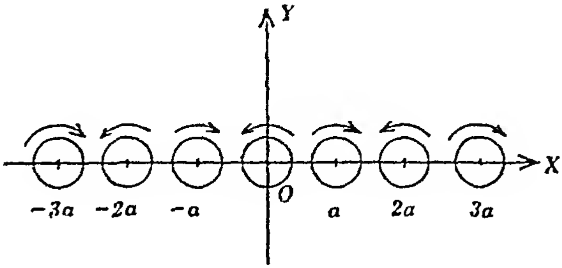
Exp. *An infinite row of equidistant rectilinear vortices are at a distance a apart. The vortices are of the same numerical strength k but they are alternately of opposite signs. Find the complex function that determines the velocity potential and stream function. Show that the vortices remain at rest and draw the stream lines. Show also that, if λ be the radius of a vortex, the amount of flow between any vortex and the next is*

$$(k/\pi) \log \cot (\pi\lambda/2a).$$ [*Ald 1956 ; Jad 58 ; Pb 49 ; Pbi 66*]

Sol. The system consists of $(k/2\pi=K)$

(i) Vortices, $+k$ at $z=0, \pm 2a, \pm 4a, \ldots$

(ii) Vortices, $-k$ at $z=\pm a, \pm 3a, \pm 5a, \ldots$



Hence the complex potential shall be

$$w=iK \log \frac{z\,(z^2-4a^2)\,(z^2-16a^2)\,(z^2-36a^2)}{(z^2-a^2)\,(z^2-9a^2)\,(z^2-25a^2)} \cdots$$

Thus, $$w=iK \log \frac{z\,\Pi\,[z^2-(2na)^2]}{\Pi\,[z^2-(2n+1)^2a^2]}=iK \log \frac{(z/2a)\,\Pi\,\{1-(z/2na)^2\}}{\Pi\,\{1-[z/(2n+1)a]^2\}}+\text{const.}$$

$$=iK \log \frac{\sin (\pi z/2a)}{\cos (\pi z/2a)}=iK \log \tan\left(\frac{\pi z}{2a}\right). \tag{1}$$

This is the required complex potential function that determines the velocity potential and stream function. Since

$$2i\psi=w(z)-\overline{w}(\bar{z})=iK\,[\log \tan (Az)+\log \tan (A\bar{z})], \qquad (A=\pi/2a).$$

$$\therefore \quad (2\psi/K)=\log \frac{\sin A\,(x+iy)\,\sin A\,(x-iy)}{\cos A\,(x+iy)\,\cos A\,(x-iy)}=\log\frac{\cosh 2Ay-\cos 2Ax}{\cosh 2Ay+\cos 2Ax} \tag{2}$$

whence the stream lines are given by

$$\cosh 2Ay=B\cos 2Ax, \qquad (A=\pi/2a,\ B=\text{const.})$$

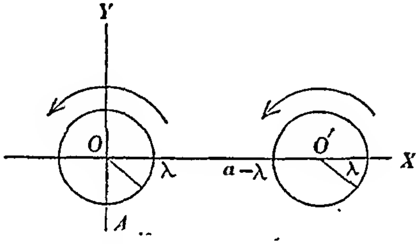
If the vortex K at $z=0$ has the velocity (u_0, v_0), then

$$u_0-iv_0=-\{D(iK\log\tan Az-iK\log z)\}_{z=0}, \ [D=d/dz]$$
$$=-iK\{A\,(\sec^2 Az/\tan Az)-z^{-1}\}\to 0 \text{ as } z\to 0$$

whence it follows that the vortices remain at rest.

To find the amount of flow between any two vertices A and B (say) we use the relation

$$\text{flow across } AB=\psi_B-\psi_A.$$



Now $y=0$ for ψ_A or ψ_B (on x-axis), therefore, by (2)

$$\psi=(K/2)\log[(1-\cos 2Ax)/(1+\cos 2A)]=K\log\tan Ax$$

$$\therefore \quad \psi_B-\psi_A=K\,[\log\tan A\,(a-\lambda)-\log\tan A\lambda]$$
$$=K[\log\tan[(\pi/2)-A\lambda]+\log\cot A\lambda].$$

Now, putting $K=k/2\pi$, $A=\pi/2a$ we get $\psi_B-\psi_A=(k/\pi)\log\cot(\pi\lambda/2a)$.

Ex. An infinite row of equidistant rectilinear vortices of equal numerical strength K, but alternately of opposite signs, are spaced at distances a apart in infinite fluid. Show that the complex potential is

$$w=iK\log\tan(\pi z/2a),$$

the origin of co-ordinates being at one of the vortices of positive sign, and hence show that the row remains at rest in this configuration.

Show further that if the very small radius of cross-section of each vortex filament is εa, then the amount of flow between two consecutive vortices is approximately $2K\log 2/\pi\varepsilon$. [*Del 1946 ; Osm 62 ; Pna 59 ; Pb 64*]

[The result follows from the proceeding problem because

$$\lambda=\varepsilon a,\ \cot(\pi\lambda/2a)=\{\cos(\pi\varepsilon/2)\}/\{\sin(\pi\varepsilon/2)\}=1/(\pi\varepsilon/2)=2/\pi\varepsilon]$$

Ex. 1. An infinite row of parallel rectilinear vortices, each of circulation $2\pi k$, intersects the z-plane at right angles at the points $z=na+\frac{1}{2}ib$, where $n=0, \pm 1, \pm 2, \ldots$ Another parallel row, in which each vortex has circulation $-2\pi k$, meets the z-plane at the points $z=(n+\frac{1}{2})\,a-\frac{1}{2}ib$, the two rows together forming a *Karman Street.* Show that the complex velocity potential in the z-plane is

$$ik\log\sin[\pi(z-\tfrac{1}{2}ib)/a]-ik\log\sin[\pi(z-\tfrac{1}{2}a+\tfrac{1}{2}ib)/a]$$

and prove that the velocity of advance of the rows is $(\pi k/a)\tanh(\pi b/a)$.

Find the velocity of the fluid at a general point in the plane which lies midway between the two vortex rows. [*Pna 1958 ; Pb 62*]

Ex. 2. Show that the stream function for a row of an infinite number of rectilinear vortices of equal strength k, evenly placed at intervals along the x-axis in an infinite fluid is

$$\psi=\tfrac{1}{2}k\,[\cosh(2\pi y/a)-\cos(2\pi x/a)].$$

all the vortices being parallel to the z-axis.

The position of a second row of such vortices of strength $-k$ would be obtained by a rigid body displacement of the first set defined by $x=\lambda a$, $y=-\mu a$. Show that such a double row or vortex *street* advances with speed

$$\frac{\pi k}{a}\left[\frac{\cosh 2\pi\mu+\cos 2\pi\lambda}{\cosh 2\pi\mu-\cos 2\pi\lambda}\right]^{\frac{1}{2}}$$

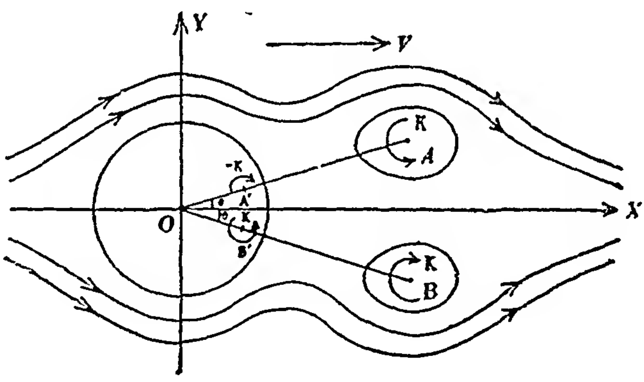
in a direction θ, with the street, given by

$$\tan\theta=\sin 2\pi\lambda/\sinh 2\pi\mu.$$

Ex. 3. A system consists of two infinite rows of equal parallel line vortices. One row has vortices of strength $+k$ at $z=na$, $(n=0, \pm 1, \pm 2, \ldots)$. The other has vortices of strength $-k$ at $z=(n+\lambda-i\mu)a$; $(n=0, \pm 1, \pm 2, \ldots)$ where λ, μ are real constants. Show that the vortices maintain their positions relative to one another but that the whole system advances in a direction θ with either row, where

$$\tan\theta=\sin 2\pi\lambda\div\sinh 2\pi\mu.$$

3.38. Stationary vortex pair in the wake of a cylinder in a stream. Suppose that there is a vortex pair placed symmetrically about OX



past which there is a stream V in the positive direction of x-axis. The complex potential of the arrangement is

$$f(z)=-Vz+iK\log\{(z-z_1)/z-\bar{z}_1)\}$$

where the vortex K is at A $(z=z_1)$, and the vortex $-k$ at $B(z=\bar{z}_1)$.

Now insert the circular cylinder $|z|=a$, so that the vortices lie in its wake ; application of *Circle theorem* then gives the complex potential

$$w=f(z)-V(a^2/z)-iK\log\{[(a^2/z)-\bar{z}_1]/[(a^2/z)-z_1]\}$$

$$=-V\left(z+\frac{a^2}{z}\right)+iK\log\frac{z-z_1}{z-\bar{z}_1}+iK\log\frac{z-(a^2/z_1)}{z-(a^2/\bar{z}_1)}+\text{const.}$$

$$=iK\log(z-z_1)+w_z+\text{const.}$$

where $$w_z=-V\left(z+\frac{a^2}{z}\right)+iK\log\frac{z-(a^2/z_1)}{(z-\bar{z}_1)[z-(a^2/\bar{z}_1)]}.$$

The velocity of the vortex K at A is given by $-(dw_z/dz)$ *when* $z=z_1$. *This is the velocity induced by other vortices on the vortex A.* Thus the vortex A will be stationary if $(dw_z/dz)=0$; and this is true of every other vortex, so that all the vortices will encounter a similar fate. Accordingly we get

$$V\left(1-\frac{a^2}{z_1^{\,2}}\right)+iK\left[\frac{1}{z_1-\bar{z}_1}+\frac{1}{z_1-(a^2/\bar{z}_1)}-\frac{1}{z_1-(a^2/z_1)}\right]=0$$

or $$V(z_1^{\,2}-a^2)+iK\,z_1^{\,2}\left[\frac{1}{z-\bar{z}_1}+\frac{\bar{z}_1}{z_1\bar{z}_1-a^2}-\frac{z_1}{z_1^{\,2}-a^2}\right]=0$$

or $$V(z_1^{\,2}-a^2)=-iK\,z_1^{\,2}\left[\frac{(z_1\bar{z}_1-a^2)\,(z_1^{\,2}-a^2)+(z_1-\bar{z}_1)^2a^2}{(z-\bar{z}_1)\,(z_1\bar{z}_1-a^2)\,(z_1^{\,2}-a^2)}\right] \tag{1}$$

Changing i to $-i$ on either side we get

$$V(\bar{z}_1^{\,2}-a^2)=iK\,\bar{z}_1^{\,2}\left[\frac{(z_1\bar{z}_1-a^2)\,(\bar{z}_1^{\,2}-a^2)+(\bar{z}_1-z_1)^2a^2}{(\bar{z}_1-z_1)\,(\bar{z}_1z_1-a^2)\,(\bar{z}_1^{\,2}-\bar{a}^2)}\right] \tag{2}$$

Dividing (1) by (2) we get

$$\frac{(z_1^{\,2}-a^2)}{(\bar{z}_1^{\,2}-a^2)}=\frac{z_1^{\,2}}{\bar{z}_1^{\,2}}\,\frac{(z_1\bar{z}_1-a^2)\,(z_1^{\,2}-a^2)+a^2(z_1-\bar{z}_1)^2}{(z_1\bar{z}_1-a^2)\,(\bar{z}_1^{\,2}-a^2)+a^2(z_1-\bar{z}_1)^2}\cdot\frac{\bar{z}_1^{\,2}-a^2}{z_1^{\,2}-a^2}.$$

or $$\frac{\bar{z}_1^{\,2}(z_1^{\,2}-a^2)^2}{z_1^{\,2}(\bar{z}_1^{\,2}-a^2)^2}=\frac{(z_1\bar{z}_1-a^2)\,(z_1^{\,2}-a^2)+a^2(z_1-\bar{z}_1)^2}{(\bar{z}_1z_1-a^2)\,(\bar{z}_1^{\,2}-a^2)+a^2(\bar{z}_1-z_1)^2}$$

By cross-multiplication, we get

$$(z_1\bar{z}_1-a^2)(z_1^{\,2}-a^2)(\bar{z}_1^{\,2}-a^2)\,[(z_1^{\,2}-a^2)\bar{z}_1^{\,2}-z_1^{\,2}(\bar{z}_1^{\,2}-a^2)]=$$
$$a^2(z_1-\bar{z}_1)^2[z_1(\bar{z}_1^{\,2}-a^2)+\bar{z}_1(z_1^{\,2}-a^2)][z_1(\bar{z}_1^{\,2}-a^2)-\bar{z}_1(z_1^{\,2}-a^2)]$$

or $$a^2(z_1\bar{z}_1-a^2)\,(z_1^{\,2}-a^2)\,(\bar{z}_1^{\,2}-a^2)\,(z_1^{\,2}-\bar{z}_1^{\,2})=$$
$$a^2(z-\bar{z}_1)^2(z_1\bar{z}_1-a^2)\,(z_1+\bar{z}_1)\,(\bar{z}_1-z_1)\,(\bar{z}_1z_1+a^2)$$

Cancelling $a^2(z_1\bar{z}_1-a^2)\,(z_1^{\,2}-\bar{z}_1^{\,2})$ from either side we get

$$(z_1^{\,2}-a^2)\,(\bar{z}_1^{\,2}-a^2)+(z_1-\bar{z}_1)^2(z_1\bar{z}_1+a^2)=0.$$

This we can rewrite this as

$$(z_1\bar{z}_1-a^2)^2+z_1\bar{z}_1(z_1-\bar{z}_1)^2=0. \tag{3}$$

Putting $z_1=be^{i\theta}$ in (3), we get $(b^2-a^2)^2=4b^4\sin^2\theta$ giving

$$(b^2-a^2)=2b^2\sin\theta,\text{ i.e. } a^2=b^2(1-2\sin\theta) \tag{4}$$

Simplifying (1) with the help of (3) and then eliminating θ with the help of (4), we get

$$K=-V(b^2-a^2)^2(b^2+a^2)/b^5. \tag{5}$$

Hence for a pair of vortices at a given distance b from the centre of a fixed cylinder in a given stream V in the positive direction of the real axis, the equations

$$a^2=b^2(1-2\sin\theta),\quad K=-V(b^2-a^2)^2\,(b^2+a^2)/b^3$$

determine the positions and strengths of the vortex pair so that they may remain at rest.

Exp. *Two point vortices each of strength K are situated at $(\pm a, o)$ and a point vortex of strength $-\frac{1}{2}K$ is situated at the origin. Show that the fluid motion is stationary and find the equations of stream lines. Show that the stream line which passes through the stagnation points meets the x-axis at $(\pm b, o)$ where*

$$3\sqrt{3}\,(b^2-a^2)^2=16a^3b.$$

Sol. The complex potential is given by

$$w=iK\log(z-a)+iK\log(z+a)-\tfrac{1}{2}iK\log z$$

or
$$2w=2iK\log(z^2-a^2)-iK\log z. \tag{1}$$

$$\therefore\quad 2\frac{dw}{dz}=2iK\frac{2z}{z^2-a^2}-\frac{iK}{z}=iK\frac{3z^2+a^2}{z(z^2-a^2)}.$$

The fluid motion will be stationary if the induced velocity at any of the vortex is zero. Thus the induced velocity of A (on account of O and B) is

$$-\frac{dw'}{dz}=-\left[\frac{dw}{dz}-\frac{iK}{z-a}\right]_{z=a}=-\left[\frac{d}{dz}\left(iK\log(z+a)-\frac{iK}{2}\log z\right)\right]_{z=a}=0$$

which implies stationary motion. The stagnation points, given by $dw/dz=0$, are

$$z=\pm ia/\sqrt{3},\quad \text{i.e.}\quad (0,\pm a/\sqrt{3}).$$

The stream lines are given by

$$2\psi=2K\,\tfrac{1}{2}\log[(x^2-y^2-a^2)^2+4x^2y^2]-\tfrac{1}{2}K\log(x^2+y^2)$$

or
$$[(x^2+y^2)^2-2a^2(x^2-y^2)+a^4]^2=A^2(x^2+y^2)$$

or
$$(x^2+y^2)^2-2a^2(x^2-y^2)+a^4=A\sqrt{(x^2+y^2)} \tag{2}$$

The stream lines will pass through $(o, \pm a/\sqrt{3})$ provided

$$(a^4/9)-2a^2(-a^2/3)+a^4=\pm Aa/\sqrt{3}$$

This leads to $A=\pm(16\sqrt{3}/9)a^3$.

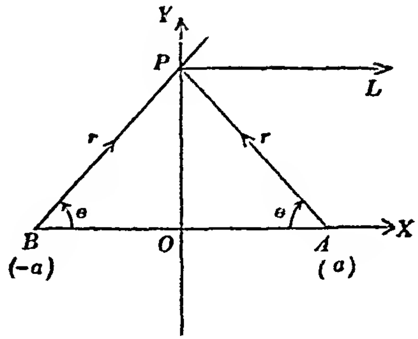
These stream lines will pass through $(\pm b, o)$ if

$$b^4-2a^2b^2+a^4=(16\sqrt{3}/9)\,a^3b,\ \text{or}\ 3\sqrt{3}\,(b^2-a^2)^2=16a^3b.$$

3.40. *Images in two dimensions. If the motion of the fluid in the x-y plane is due to a system of sources, sinks, doublets, vortices, etc. and if there is a curve C in the plane such that there is no flow across it, the system of sources, sinks, doublets, vortices, etc. on one side of C is said, to be the image of the sources, sinks, doublets, vortices, etc. on the other side of C.* Obviously the curve C must be a stream line for there is no flow across it and so if we introduce a rigid barrier coincident with C, the fluid motion will not be affected. Clearly, the velocity of the fluid at any point, normal to the rigid barrier must be zero.

In the following we shall obtain the image system for ane straight li and a circle for a *source*, a *doublet*, and a *vortex*.

(1) *Image of a source in a straight line.* Let there be two sources each of strength m at A and B on opposite sides of and equidistant from the line OP.



Since $\phi=-m \log r$

$\therefore \quad \partial\phi/\partial r=m/r.$

Thus velocity at P due to m at A

$=(m/r)$ along AP

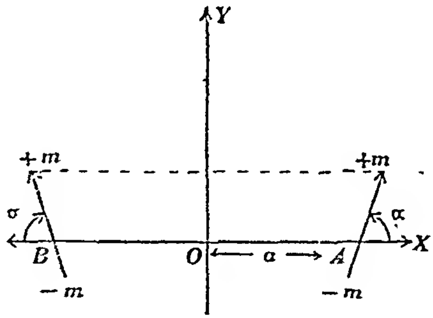
and the velocity at P due to m at B

$=(m/r)$ along BP

Hence the normal velocity at $P=-(m/r)\cos\theta+(m/r)\cos\theta=0.$

Thus, there is no flow across the straight line OP. This implies that *the image of a simple source in a straight line is an equal source equidistant from the line.*

(2) *Image of a doublet in a straight line.* A doublet is a combination of a source and a sink of equal strength which have their respective images in the straight line OY (say) as an equal source and sink of the same strength and placed equidistant from the straight line. It follows that *the image of a doublet of strength μ at $(a, 0)$ with its axis inclined at an angle α with OX, is an equal doublet at $B(-a, 0)$ inclined at $(\pi-\alpha)$ to OX,* i.e. *an equal anti-parallel doublet.*



Exp. 1. If fluid fill the region of space on the positive side of the x-axis, which is a rigid boundary, and if there be a source m at the point (0, a) and an equal sink at (0, b), and if the pressure on the negative side of the boundary be the same as the pressure of the fluid at infinity, show that the resultant pressure on the boundary is

$$\pi\rho m^2(a-b)^2/ab(a+b),$$

where ρ is the density of the fluid. [Bom 1962; Del 65, 36, 32]

Sol. The image of source m at $z=ia$ is an equal source m at $z=-ia$, and the image of sink, $-m$ at $z=ib$ is an equal sink at $z=-ib$, and this image system does away with the boundary $y=0$ (i.e. x-axis). Thus, the complex potential of the *entire* system is

$$w=-m\log(z-ai)+m\log(z-bi)-m\log(z+ai)+m\log(z+bi)$$

i.e. $\quad w=-m\log(z^2+a^2)+m\log(z^2+b^2) \qquad \text{(1)}$

The velocity at any point z is given by

$$q=\left|\frac{dw}{dz}\right|=2m\left|\frac{z}{z^2+b^2}-\frac{z}{z^2+a^2}\right|=2m(a^2-b^2)\left|\frac{z}{(z^2+a^2)(z^2+b^2)}\right|.$$

The velocity at any point on x-axis, distant x from the origin is $(z=x)$ is consequently

$$q=2m\ (a^2-b^2)x/(x^2+a^2)(x^2+b^2). \tag{2}$$

To determine the pressure, we need use Bernoulli-Cauchy pressure equation, viz. $(p/\rho)+\frac{1}{2}q^2=\text{const}$. Thus, since $q=0$ when $r\to\infty$, and $p=\Pi$ at ∞, we get

$$(p/\rho)=(\Pi/\rho)-\tfrac{1}{2}q^2, \text{ or } (\Pi-p)/\rho=\tfrac{1}{2}q^2.$$

Thus, the pressure gradient (or fall in pressure) is $(\Pi-p)$; hence the resultant pressure P on the boundary is

$$P=\int_0^\infty(\Pi-p)dx=\tfrac{1}{2}\rho\int_0^\infty q^2dx-2m^2\rho\int_0^\infty\frac{x^2(a^2-b^2)^2\,dx}{(x^2+a^2)^2(x^2+b^2)^2}.$$

Since, $$\frac{x^2}{(x^2+a^2)^2(x^2+b^2)^2}=\frac{a^2+b^2}{(b^2-a^2)^3}\frac{1}{x^2+a^2}-\frac{a^2}{(b^2-a^2)^2}\frac{1}{(x^2+a^2)^2}-\frac{a^2+b^2}{(b^2-a^2)^3}\frac{1}{x^2+b^2}-\frac{b^2}{(b^2-a^2)^2}\frac{1}{(x^2+b^2)^2}$$

$$\therefore\quad P=2m^2\rho\int_0^\infty\left\{\frac{a^2+b^2}{b^2-a^2}\left(\frac{1}{x^2+a^2}-\frac{1}{x^2+b^2}\right)-\frac{a^2}{(x^2+a^2)^2}-\frac{b^2}{(x^2+b^2)^2}\right\}dx$$

$$=2m^2\rho\left\{\frac{a^2+b^2}{b^2-a^2}\left(\frac{\pi}{2a}-\frac{\pi}{2b}\right)-\frac{\pi}{4a}-\frac{\pi}{4b}\right\}$$

$$=\pi\rho m^2\,\frac{2(a^2+b^2)-(a+b)^2}{2ab\,(a+b)}=\frac{\pi\rho m^2(a-b)^2}{2ab\,(a+b)}.$$

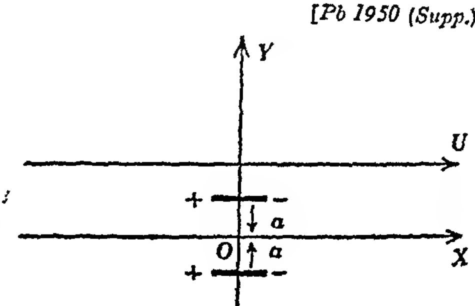
Exp. 2. The space on one side of an infinite plane wall, $y=0$ is filled with inviscid, incompressible liquid moving at infinity with velocity U in the direction of the axis of x. The motion of the liquid is wholly two-dimensional in the (x, y) plane. A doublet of strength μ is at a distance a from the wall, and points in the negative direction of the axis of x. Show that if $\mu<4a^2U$, the pressure of the fluid on the wall is a maximum at points distance $\sqrt{3}\,a$ from O, the foot of the perpendicular from the doublet on the wall, and is minimum at O.

If, $\mu=4a^2U$, find the points where the velocity of the liquid is zero; and show that the stream lines include the circle

$$x^2+(y-a)^2=4a^2$$

where the origin is taken at O. *[Pb 1950 (Supp.)]*

Sol. The complex potential for a doublet of strength μ at $z=z_0$ inclined at an angle α to the real axis is $\mu e^{i\alpha}/(z-z_0)$. [§3·21(3)p. 146.]



Since the image of the doublet in question, is an equal doublet, similarly oriented, the complex potential for the system, consisting of original doublet, image doublet and the stream U is

$$w=\frac{\mu e^{i\pi}}{z-ia}+\frac{\mu e^{i\pi}}{z+ia}-Uz=-\frac{2\mu z}{z^2+a^2}-Uz. \tag{1}$$

Thus, $$dw/dz=-U-2\mu\,(a^2-z^2)/(a^2+z^2)^2. \tag{2}$$

Now, by Bernoulli-Cauchy integral

$$(p/\rho)+\tfrac{1}{2}q^2=\text{const.}=(\Pi/\rho)+\tfrac{1}{2}U^2$$

i.e.
$$(\Pi-p)/\rho=\tfrac{1}{2}\,(q^2-U^2). \tag{3}$$

Since at any point on the wall, $z=x$ (the real axis), we get from (1)

$$|-dw/dz\,|^2=q^2=[U+2\mu\,(a^2-x^2)/(a^2+x^2)^2]^2 \tag{4}$$

Thus,
$$(\Pi-p)/\rho=[2\mu^2\,(a^2-x^2)/(a^2+x^2)^4]+[2\mu U\,(a^2-x^2)/(a^2+x^2)^2]. \tag{5}$$

For maximum or minimum value of the pressure, $dp/dx=0$, so that

$$2\mu^2\left[\frac{4x(a^2-x^2)}{(a^2+x^2)^4}+\frac{8x(a^2-x^2)^2}{(a^2+x^2)^5}\right]+2\mu U\left[\frac{2x}{(a^2+x^2)^2}+\frac{4x(a^2-x^2)}{(a^2+x^2)^3}\right]=0$$

or
$$4\mu x\,[2\mu\,(a^2-x^2)+U\,(a^2+x^2)^2]\,(3a^2-x^2)/(a^2+x^2)^5=0.$$

Thus, either $x=0$ or $x=a\sqrt{3}$. Now on the wall $(y=0)$, at $x=a\sqrt{3}$,

$$(dw/dz)=-U+(\mu/4a^2) \qquad \text{by (2).}$$

If $\mu<4a^2U$, then the value of (d^2p/dx^2) at $x=a\sqrt{3}$ is negative; so that the pressure of the fluid at the wall is a maximum. However, (d^2p/dx^2) at $x=0$ is positive, so that the pressure is a minimum. If $\mu=4a^2U$, then

$$U+\{8a^2U\,(a^2-z^2)/(a^2+z^2)^2\}=0 \qquad \text{by (2),}$$

or
$$z^4-6a^2z^2+9a^4=0,\ \Rightarrow\ (z^2-3a^2)^2=0 \quad \text{or } z=\pm a\sqrt{3}.$$

Thus, the stagnation-points are $(\pm\,a\sqrt{3},\,0)$.

From (1), by equating imaginary parts and using $\mu=4a^2U$, we get

$$\psi=-Uy+\frac{8a^2\,Uy(x^2+y^2-a^2)}{(x^2+y^2)^2+2a^2(x^2-y^2)+a^4}$$

Putting $\psi=0$, we get

$$(x^2+y^2)^2-6a^2x^2-10a^2y^2+9a^4=0,$$

or
$$(x^2+y^2-2ay-3a^2)\,(x^2+y^2+2ay-3a^2)=0.$$

This obviously includes the circle : $x^2+(y-a)^2=4a^2$.

Ex. 1. Find the velocity potential and the stream function due to a two-dimensional source of liquid. Find also the effect of the pressure of an infinite straight boundary in the plane of the motion of this liquid.

In the case of the motion of liquid in a part of plane bounded by a straight line due to a source, prove that, if $m\rho$ is the mass of liquid (of density ρ) generated at the source per unit of time, the pressure on the length $2l$ of the boundary immediately opposite to the source is less than that on an equal length at a great distance by

$$\frac{1}{2}\,\frac{m^2\rho}{\pi^2}\left(\frac{1}{c}\tan^{-1}\frac{l}{c}-\frac{l}{l^2+c^2}\right),$$

where c is the distance of the source from the boundary.

[*Ag 1965, 55* ; *Bom 52* ; *Jab 59*]

Ex. 2. Explain the terms source, sink and doublet as used in Hydrodynamics.

Two straight lines which are at right angles are the boundaries of a quarter plane occupied by a homogeneous liquid. There is a source and a sink of equal strength on bounding straight lines at a distance a from the point of intersection. Find an expression for the liquid motion in the plane. [*Del I.* ...

Ex. 3. A source is placed midway between two planes whose distance one another is $2a$. Find the equation of the stream lines when the two dimensions and show those particles, which at an infinite di... tant $\tfrac{1}{2}a$ from one of the boundaries, issued from the source in a... an angle $\pi/4$ with it.

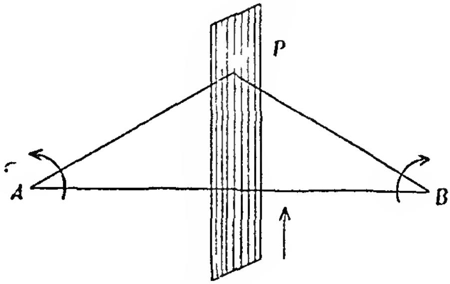
(3) *Image of a vortex filament in a plane.* Let A, B the vortex filaments of strength k and $-k$ be situated at z_1 and z_2 respectively. The complex potential of the vortex pair is

$$w = iK \log(z-z_1) - ik \log(z-z_2)$$

so that the current function ψ is given by

$$\psi = K \log(r_1/r_2)$$

where $|z-z_1| = r_1$ and $|z-z_2| = r_2$.



It is evident that there will be no flow across a plane bisecting AB at right angles, for a point P on it $r_1 = r_2$ so that $\psi = 0$. The motion would, therefore, be unaffected if the plane were made a rigid barrier. Thus the *image of a vortex filament in a plane to which it is parallel is an equal and opposite vortex filament at its optical image in the plane.*

Thus, if $AB = 2a$, the vortex filament A shall move (under the sole effect of B) parallel to the plane with uniform velocity

$$|dw/dz| = |iK/(z_1-z_2)| = K/AB = K/2a. \tag{1}$$

Also, the velocity midway between A and B due to both the vortices is $2K/a$, so the vortex moves parallel to the plane with one-fourth of the velocity of the liquid at the boundary.

Exp. 1. *Find the motion of a straight vortex filament in an infinite region bounded by an infinite plane wall to which the filament is parallel, and prove that the pressure defect at any point of the wall due to the filament is proportional to $\cos^2\theta \cos 2\theta$, where θ is the inclination of the plane through the filament and the point to the plane through the filament perpendicular to the wall.*

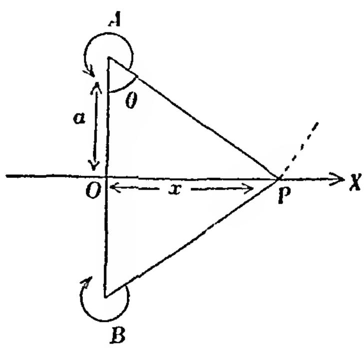
[*Ald 1955 ; Pb 50 (Sup)*]

Sol The vortex filament k at A gives rise to a vortex filament $-k$ at B, its optical image in the plane. Let the join of AB be y-axis, so that the co-ordinates of A and B are ia and $-ia$ respectively. Hence the complex potential shall be

$$w = ik \log[(z-ai)/(z+ai)] \tag{1}$$

which at time t shall become

$$w = ik \log\left(\frac{z-ia-Vt}{z+ia-Vt}\right) \text{ where } V = \frac{k}{2a}$$



This gives,

$$\frac{\partial w}{\partial t} = -iVk\left[\frac{1}{z-ia-Vt} - \frac{1}{z+ia-Vt}\right].$$

The value of $[(\partial\phi/\partial t) + i(\partial\psi/\partial t)]$ at a point P on OX ($y=0$), (Fig.) at $t=0$, is

$$\left(\frac{\partial\phi}{\partial t} + i\frac{\partial\psi}{\partial t}\right)_0 = \frac{2Vka}{x^2+a^2} = \frac{k^2\cos^2\theta}{a^2} \quad (\because x = \tan\theta).$$

Thus $$(\partial\phi/\partial t)_0=k^2\cos^2\theta/a^2.$$

Also at $t=0$, the velocity at P on Ox is the resultant of k/PA, k/PB perpendicular to PA and PB respectively and is obtained more easily from (1), i.e.

$$q=\left|-\frac{dw}{dz}\right|=\frac{2ka}{a^2+}=\frac{2k}{a}\cos^2\theta. \quad (y=0,\ x=a\tan\theta).$$

Now the pressure is given by the Cauchy's pressure equation :

$$\frac{p}{\rho}+\frac{1}{2}\ q^2-\frac{\partial\phi}{\partial t}=\text{const}=f(t),\ (\text{say}). \tag{2}$$

Thus the pressure at the point P at time $t=0$ is given by

$$\frac{p}{\rho}+\frac{2k^2\cos^4\theta}{a^2}-\frac{k^2\cos^2\theta}{a^2}=f(0).$$

Now as the point $P\to\infty$, $p\to\Pi$ and $\theta\to\frac{1}{2}\pi$; so that $f(t)=\Pi/\rho)$ for all t.

Hence, $$\frac{p-\Pi}{\rho}=\frac{k^2}{a^2}\cos^2\theta\ (1-2\cos^2\theta)=-\frac{k^2}{a^2}\cos^2\theta.\ \cos 2\theta.$$

Thus the *pressure defect* at any point P of the wall OX is as stated.

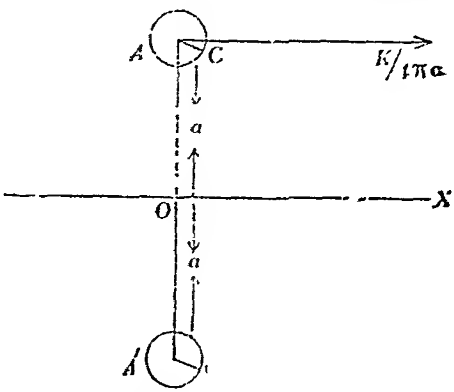
NOTE. The force on the plane due to the motion is

$$\frac{k^2\rho}{a^2}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\cos^2\theta\cos 2\theta.\ a\sec^2\theta\ d\theta=0.$$

Exp. 2. Prove that a thin cylindrical vortex of strength k, running parallel to a plane boundary at distance a will travel with velocity $(k/4\pi a)$; and show that a stream of fluid will flow past between the travelling vortex and the boundary, of total amount

$$(k/2\pi)\ [\log\,(2a/c)-\tfrac{1}{2}]$$

per unit length along the vortex, when c is the (small) radius of the cross-section of the vortex. [*Del 1938*]



Sol. The cylindrical vortex k at A $(z=ia)$ will give rise to an image $-k$ at A' $(z=-ia)$ and consequently, the complex potential will be

$$w=\frac{ik}{2\pi}\log\left(\frac{z-ia}{z+ia}\right). \tag{1}$$

The velocity at A shall be *solely due to its image*, and is given by

$$\frac{dw'}{dz}=\frac{d}{dz}\left(-\frac{ik}{2\pi}\log\,(z+ia)\right)_{z=ia},$$

$$\therefore\ \left|\frac{dw'}{dz}\right|=q=\frac{k}{2\pi}\left|\frac{1}{z+ia}\right|_{z=ia}=\frac{k}{4\pi a}.$$

Now let us reduce the system to relative equilibrium by superimposing an equal and opposite velocity to that of the vortex velocity, viz. $k/4\pi a$; this evidently adds an additional term $(ky/4\pi a)$, $[(\partial\psi/\partial y)=(k/4\pi a)]$ to the stream function derivable from (1).

Thus $$\psi=\frac{k}{2\pi}\log\left|\frac{z-ia}{z+ia}\right|+\frac{k}{4\pi a}y=\frac{k}{4\pi}\log\frac{x^2+(y-a)^2}{x^2+(y+a)^2}+\frac{k}{4\pi a}y.$$

We need total flow Q between the travelling vortex and the plane given by $Q=\psi_{(0,\,0)}-\psi_{(0,\,a-c)}$. Thus

$$Q=-(k/4\pi)\log\{c^2/(2a-c)^2\}-(k/4\pi a)(a-c)$$
$$=(k/2\pi)[\log\{(2a/c)-1\}-(a-c)/2a]$$
$$=(k/2\pi)[\log(2a/c)+\log\{1-(c/2a)\}-\tfrac{1}{2}+\tfrac{1}{2}(c/a)]$$
$$\doteq (k/2\pi)[\log(2a/c)-\tfrac{1}{2}]\text{ ; neglecting } c^2 \text{ in expansion of log series.}$$

Ex. 1. If a rectilinear vortex moves (in two dimensions) in fluid bounded by a fixed plane, prove that a stream line can never coincide with a line of constant pressure.

Ex. 2. An infinitely long line vortex of strength m, parallel to the axis of z, is situated in infinite liquid bounded by a rigid wall in the plane $y=0$. Prove that, if there be no field of force, the surfaces of equal pressure are given by

$$[(x-a)^2+(y-b)^2][(x-a)^2+(y+b)^2]=c[y^2+b^2-(x-a)^2]$$

where (a, b) are the co-ordinates of the vortex and c is a parametric constant.
[*Bom 1957 ; Lkn 59 ; Osm 59*]

(4) *Image of a vortex in a quadrant.* If the vortex k be at (x, y), then the image system consists of

(i) $-k$ at $(x, -y)$,
(ii) $-k$ at $(-x, y)$,
(iii) $+k$ at $(-x, -y)$.

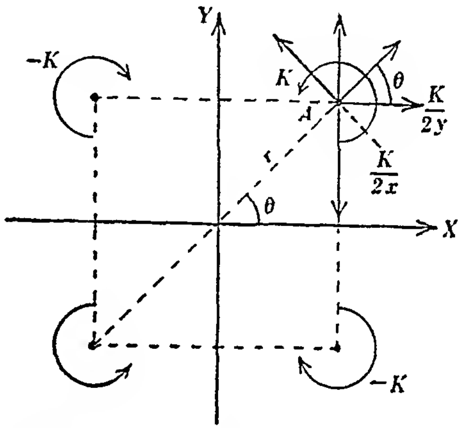


Figure shows the arrangement of images. The velocity of the vortex at A (x, y) is *due solely to its images* and since vortex moves parallel to the plane with velocity $(k/2a)$, [§3·40(3), p. 180], hence its components are as shown in the figure. The radial and cross-radial components are given by

$$\frac{dr}{dt}=\frac{k}{2y}\cos\theta-\frac{k}{2x}\sin\theta=\frac{k}{2r}\,\frac{\cos\theta}{\sin\theta}-\frac{k}{2r}\,\frac{\sin\theta}{\cos\theta}=\frac{k}{r}\,\frac{\cos 2\theta}{\sin 2\theta}.$$

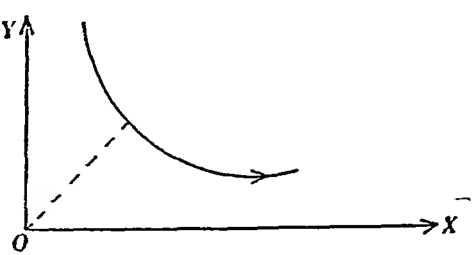
$$r\frac{d\theta}{dt}=\frac{k}{2r}-\frac{k}{2y}\sin\theta-\frac{k}{2x}\cos\theta=\frac{k}{2r}-\frac{k\sin\theta}{2r\sin\theta}-\frac{k}{2r}\,\frac{\cos\theta}{\cos\theta}=\frac{-k}{2r}.$$

Therefore by division,

$$\frac{1}{r}\,\frac{dr}{d\theta}=-2\,\frac{\cos 2\theta}{\sin 2\theta}$$

whence integration provides

$$\log r+\log\sin 2\theta=\log a$$
or $$r\sin 2\theta=a \qquad (1)$$
where a is a const. of integration.



This is *Cote's spiral* and its trace is on p. 182, botton.

To obtain its Cartesian equivalent, we rewrite (1) as

$$\tfrac{1}{2}ar = r\sin\theta \cdot r\cos\theta$$

whence: $\tfrac{1}{4}(x^2+y^2)\,a^2 = x^2y^2$ or $\dfrac{1}{x^2}+\dfrac{1}{y^2}=\dfrac{4}{a^2}$.

Ex. Exemplify the theory of images in two-dimensional liquid motions by finding the velocity potential of infinite liquid bounded by two plane walls enclosing a right angle,

(i) when there is a single line source of strength m,

(ii) when there is a single line vortex of strength m.

Prove that, in the latter case, the path of the vortex is a curve of the type

$$\frac{1}{x^2}+\frac{1}{y^2}=\frac{1}{a^2}$$

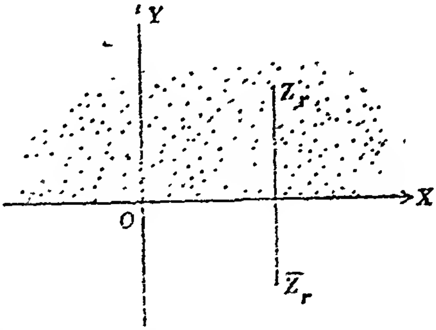
and the rate of description of the vectorial area about the origin is constant. [*London 1932, 10*]

[Since $x\dot{y}-\dot{x}y=-\tfrac{1}{2}k$, the vortices describe the *Cote's spiral* in the same way as a particle under the central repulsive force varying as the inverse of the cube of the distance. Obviously the rate of description of vectorial area about the origin is constant.]

(5) *Images in a plane : a general method.* Suppose all te sources eli in the half plane $y>0$ and that their complex potential is given by

$$f(z) = -\Sigma m_r \log(z-z_r).$$

Now, when the plane barrier $Y=0$ is inserted, the complex potential reduces to



$$w = f(z)+\bar{f}(z)$$
$$= -\Sigma m_r \log(z-z_r) - \Sigma m_r \log(z-\bar{z}_r).$$

Since *on* $y=0$, $z=\bar{z}$, it follows that w is purely real and *the plane* $y=0$ is the *stream line* $\psi=0$. Also when z_r lies in the region $y>0$, then $\bar{z}_r$ lies in the region $y<0$, so that no new singularities are introduced into the region $y>0$.

Suppose now that all the sources and sinks lie in the half plane $x>0$ and the usual complex potential is $f(z)=-\Sigma m_r \log(z-z_r)$. Now when the plane barrier $X=0$ is inserted, the complex potential reduces to

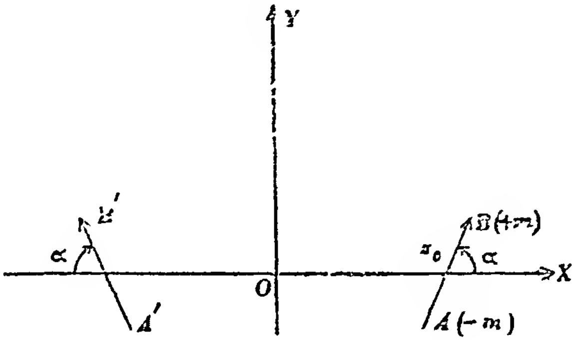
$$w = f(z)+\bar{f}(-z) = -\Sigma m_r \log(z-z_r) - \Sigma m_r \log(-z-\bar{z}_r)$$

for *on* $X=0$, $-z=\bar{z}$, so that w is purely real and the *plane* $x=0$ is *the stream line* $\psi=0$.

Two-dimensional doublet. Suppose now that there is a two-dimensional doublet of strength μ inclined at α to x-axis. Then the image

is an equal anti-parallel doublet. The complex potential for the isolated doublet at z_0 is (vide §3·21(3), p. 146)

$$w_1=f_1(z)=\mu\, e^{i\alpha}/(z-z_0).$$



When the plane $x=0$ is inserted, it reduces to

$$w=f(z)=\frac{\mu\, e^{i\alpha}}{z-z_0}+\frac{\mu\, e^{i(\pi-\alpha)}}{-z-\bar{z}_0}=\frac{\mu\, e^{i\alpha}}{z-z_0}-\frac{\mu\, e^{-i\alpha}}{z+\bar{z}_0}$$

by an argument similar to what has been said above.

(6) *Image of a source in front of a circle.* When a source m at $z=f$ is present *alone* in the fluid, the complex potential is $-m\log(z-f)$. But when we insert a cylinder, the complex potential, by circle theorem, becomes

$$w=-m\log(z-f)-m\log[(a^2/z)-f] \tag{1}$$

We re-write (1), by adding an *immaterial constant* term $-\log(-f)$. Thus

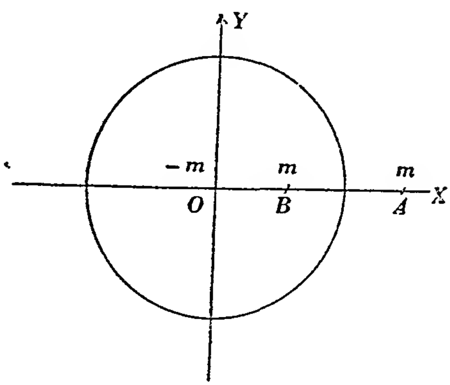
$$w=-m\log(z-f)-m\log[z-(a^2/f)]+m\log z. \tag{2}$$

This is the complex potential of

(i) a source m at $A(z=f)$

(ii) a source m at B $[z=a^2/f]$

(iii) a sink $-m$ at O, $(z=0)$



Since $OB=a^2/f$, it follows that A and B are inverse points with respect to the circular section of the cylinder and B is inside the section when A is outside.

From (2), by equating the imaginary parts after putting $z=ae^{i\theta}$ on the circle, and setting $(a^2/f)=f'$, we get

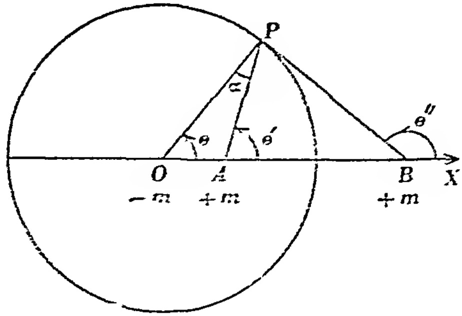
$$\psi=-m\tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta-f}\right)-m\tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta-f'}\right)+m\theta$$

$$=-m\tan^{-1}\left[\frac{a\sin\theta\,(2a\cos\theta-f'-f)}{a\cos\theta\,(2a\cos\theta-f'-f)}\right]+m\theta$$

$$=-m\tan^{-1}(\tan\theta)+m\theta=0.$$

Hence for *this complex potential, circle is a stream line*. Thus *the image system for a source outside a circle* (i.e. *circular cylinder*) *consists of an equal source at the inverse point and an equal sink at the centre of the circle.*

SECOND METHOD. Let us consider three points O, A and B on a straight line OX. Place a sink $-m$ at O, a source m each at B and A.



Since $\psi=-m\theta$, for a *single* two-dimensional source, therefore the stream function at any point P will be

$$\psi=+m\theta-m\theta'-m\theta''$$
$$=-m(\alpha+\theta'');$$

since $\theta'-\theta=\alpha$.

Now draw a circle with centre O and radius $\sqrt{OA.OB}$, so that A and B are inverse points with respect to the circle. If P be any point on this circle, then

$$OA.OB=OP^2 \qquad \text{or} \qquad [OA/OP]=[OP/OB]$$

so that triangles OAP and OPB are similar. Hence

$$\alpha=\pi-\theta'' \qquad \text{or} \qquad \alpha+\theta''=\pi$$

Thus $\qquad \psi=-m\pi$ (a constant).

It follows that on the circle, ψ is constant, so that the *circle is a stream line.*

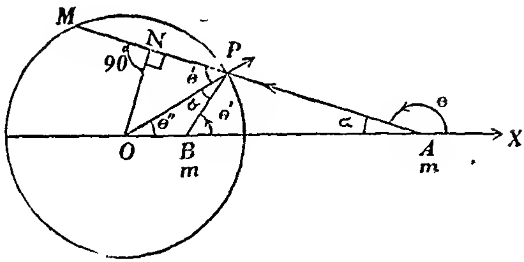
Hence *the image of a source at B with regard to a circle is an equal source at the inverse point A with an equal sink at the centre O of the circle.*

THIRD METHOD. The image system can also be found by direct calculation of *radial velocity* on a circular cylinder.

Let us place a source m at A outside the circle and an equal source m at B, the inverse point of A with respect to the circle (Fig. p. 186).

The velocity at P due to m at $A=(m/AP)$ along AP.

The velocity at P due to m at $B=(m/BP)$ along BP.



Hence the the velocity at P normal to the circle, due to m at A and m at B is

$$=(m/AP)\cos\angle OPA+(m/BP)\cos\angle OPB.$$

Now triangles OAP and OBP are similar, so that

$$\angle OPB=\angle OAP=\alpha \text{ (say) and } \angle OPN=\angle PBA=\theta'$$

Now, $\cos\alpha=\dfrac{AN}{OA}=\dfrac{AP+PN}{OA}=\dfrac{AP+OP\cos\theta'}{OA}=\dfrac{AP}{OA}+\dfrac{OP}{OA}\cos\theta'$

$$=\frac{BP}{OP}+\frac{BP}{AP}\cos\theta' \qquad \left[\text{for } \frac{OB}{OP}=\frac{OP}{OA}=\frac{BP}{AP}\right].$$

Hence *normal velocity* $=\dfrac{m}{AP}\cos(\pi-\theta')+\dfrac{m}{BP}\left(\dfrac{BP}{OP}+\dfrac{BP}{AP}\cos\theta'\right)=\dfrac{m}{OP}$.

Clearly if we place a sink $-m$ at O, the normal velocity at P shall be zero.

Thus the image system consists of an equal source at B, the inverse point, and an equal sink at O, the centre of the circle.

Cor. The image of a source at B and an equal sink at O is an equal source at A, the inverse point of B where B lies *inside* the circle.

Exp. 1. Two-dimensional source is parallel to the axis of a rigid cylinder of circular section. In a plane perpendicular to the axis, the source and axis of the cylinder are represented by the points A and O respectively. P is a point in this plane and is on the cylinder. Show that the speed of the liquid at P is proportional to

$$\triangle OPA/AP^2.$$

Sol. Let A' be the point inverse to the point A with regard to the circle, centre O and radius a. Then the image of source m at A $(OA=f>a)$ is a source m at A' and sink $-m$ at O. Consequently the complex potential is given by

$$w=-m\log(z-f)-m\log(z-a^2/f)+m\log z$$

$$\text{Thus, } -\frac{dw}{dz}=\frac{m}{z-f}+\frac{m}{z-a^2/f}-\frac{m}{z}=\frac{m(a^2-z^2)}{z(z-f)(z-f')} \tag{1}$$

where $f'=a^2/f$. If the point P lies on the circle, then $z=ae^{i\theta}$ and (1) yields

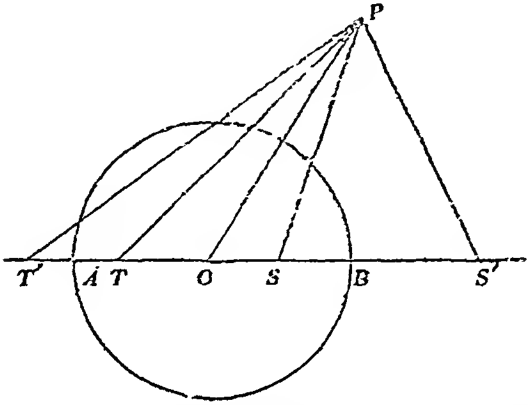
$$\left|-\frac{dw}{dz}\right|=\frac{a^2\,|\,(1-e^{2i\theta})\,|}{OP.AP.A'P}=\frac{2a\sin\theta}{AP.[OP.(AP/f)]}=\frac{4\triangle OPA}{a\,AP^2}.$$

Since $A'P/AP=OP/OA$. The final conclusion is $q\sim\triangle OPA/AP^2$.

Exp. 2. A source S and a sink T of equal strengths m are situated within the space bounded by a circle whose centre is O. If S and T are at equal distances from O on opposite sides of it and on the same diameter AOB, show that the velocity of the liquid at any point P is

$$2m\,.\,\frac{OS^2+OA^2}{OS}\;\frac{PA\,.\,PB}{PS\,.\,PS'\,.\,PT\,.\,PT'}$$

where S′ and T′ are the inverse points of S and T with respect to the circle.
[*Ban 1964 ; Bom 58, 56 ; Del 37 ; Osm 59 ; Raj 66, 60 ; Ut 62*]



Sol. Let $OS=OT=f$, whence $OS'=a^2/f=OT'$. Now, to do away with the circular boundary, we obtain its equivalent image system. Let us introduce a source and a sink of equal strengths m at O, the centre. Their introduction does not alter the dynamical configuration.

Now, image of source m at S $(z=f)$ and sink $-m$ at O $(z=0)$ is source m at $S'(z=a^2/f=f')$. And image of sink $(-m)$ at T $(z=-f)$ and source m at $O(z=0)$ is a sink $-m$ at T' $(z=-a^2/f=-f')$. Thus, the required complex potential w is

$$w=-m\log(z-f)-m\log(z-f')+m\log(z+f)+m\log(z+f')$$

the contributions of $+m$ and $-m$ at O cancelling each other. Thus,

$$\begin{aligned}(dw/dz)&=-m[1/(z-f)+1/(z-f')-1/(z+f)-1/(z+f')]\\&=-2m[f/(z^2-f^2)+f'/(z^2-f'^2)]\\&=-2m(a^2+f^2)(z^2-a^2)/f(z^2-f^2)(z^2-f'^2)\end{aligned}$$

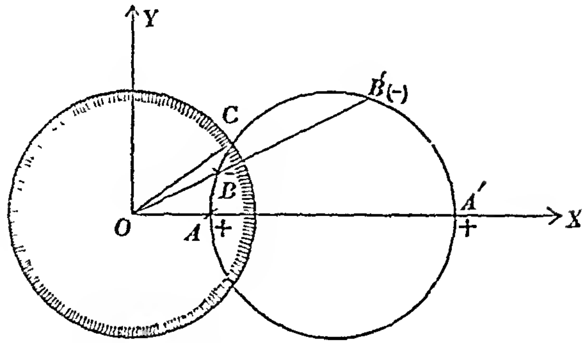
$$\therefore\quad q=\left|\frac{dw}{dz}\right|=\frac{2m}{f}\;\frac{(a^2+f^2)\;|(z-a)(z+a)|}{|(z-f)(z+f)|\;|(z+f')(z-f')|}$$

$$=\frac{2m(OS^2+OA^2)}{OS}\cdot\frac{PA\,.\,PB}{PS\,.\,PS'\,.\,PT\,.\,PT'}$$

Exp. 3. In the case of a source and an equal sink inside a circular cavity show that one of the stream lines is an arc of the circle which passes, through the source and sink and cuts orthogonally the boundary of the cavity.
[*Ban 1953 ; Bom 63 (old) ; Del 59 ; Osm 59*]

Sol. Let the source m and the sink $-m$ be situated at the points A and B within the circular cavity with centre at the origin. We introduce a source m and equal sink $-m$ at O ; this introduction does not alter the given configuration. Now $+m$ at $A(z=f)$ and $-m$ at O produce image source m at A' $(z=a^2/f=f'$ say). And $-m$ at B $(z=be^{i\alpha})$ and $+m$ at O yield the image sink $-m$ at $(z=a^2e^{i\alpha}/b=b'$ say). Thus, the *entire* system consists of

(i) *a source of strength* m *at* A $(z=f)$
(ii) *a source of strength* m *at* A' $(z=f')$
(iii) *a sink of strength* $-m$ *at* B $(z=be^{i\alpha})$
(iv) *a sink of strength* $-m$ *at* B' $(z=b'e^{i\alpha})$.



Hence the complex potential function is

$$w=-m\log(z-f)-m\log(z-f')+m\log(z-be^{i\alpha})+m\log(z-b'e^{i\alpha}).$$

From this equation we can obtain velocity potential and stream function by equating the real and imaginary parts on either side.

Since $OA \cdot OA'=OB \cdot OB'=a^2$; the points A, A', B, B' are concyclic. Let the circle through these four points meet the original circle in C and C'.

Since $$OA \cdot OA'=a^2=OC^2$$

$\therefore$ OC is a tangent at C to the circle through C, C'. Therefore the two circles cut orthogonally. Also the circle $A'B'CC'$ passes through A and B, the source and the sink, hence it must be a stream line.

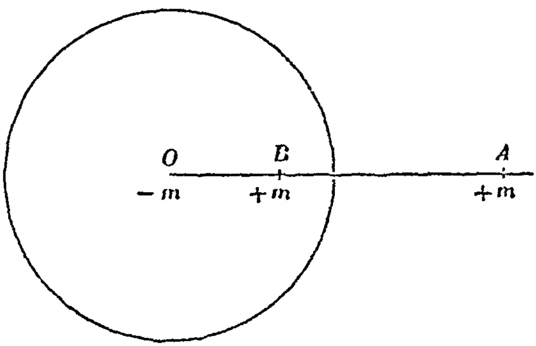
This is otherwise also clear ; for the stream lines for a source and an equal sink are circles, the circle through the above four points is a stream line. C and C' are *stagnation-points.*

Exp. 4. Show that the force per unit length exerted on a circular cylinder, radius a, *due to a source of strength* m, *at a distance* c *from the axis is*

$$2\pi\rho m^2 a^2/c(c^2-a^2).$$

[*Bom 1955 ; Jad 58 ; Kr 60 ; Pb 62, 58*]

Sol. In Fig. $OA=c$; $OB=a^2/c=c'$ (say).



The complex potential due to source m at $A(z=c)$ and its image system, m at B and $-m$ at O is

$$w=m \log z-m \log (z-c)-m \log (r-c').$$

Thus, $$dw/dz=(m/z)-\{m/(z-c)\}-\{m/(z-c')\}.$$

Since components of force are given by Cauchy-Blasius theorem, viz.

$$X-iY=-\pi\rho \text{ [sum of the residues of } (dw/dz)^2 \text{ within cylinder]} \quad (1)$$

we proceed to find the residues of $(dw/dz)^2$. Now

$$(dw/dz)^2=(m^2/z^2)+\{m^2/(z-c)^2\}+\{m^2/(z-c')^2\}+\{2m^2/(z-c)(z-c')\}-\{2m^2/z(z-c')\}-\{2m^2/z(z-c)\}.$$

Putting the expression on the right into partial fractions we get

$$(dw/dz)^2=(m^2/z^2)+\{m^2/(z-c)^2\}+\{m^2/z-c')^2\}+\{2m^2/(z-c)(c-c')\}+\{2m^2/(z-c')(c'-c)\}+(2m^2/(c'z)-\{2m^2/(z-c')c'\}+\{2m^2/(z-c)c\}+(2m^2/cz).$$

Obviously the poles *inside* the cylindrical contour are $z=0$ and $z=c'$. The sum of the residues, i.e. the sum of the co-efficients of z^{-1} and $(z-c')^{-1}$ is

$$(2m^2/c')+(2m^2/c)+2m^2/(c'-c)-(2m^2/c')=\{2m^2c'/c(c'-c)\}=\{(2m^2a^2/c(a^2-c^2)\}. \quad (2)$$

From (1) and (2)

$$X-iY=-\pi\rho \,.\, 2m^2a^2/c(a^2-c^2).$$

Hence $$X=2\pi\rho m^2a^2/c(c^2-a^2),\ Y=0.$$

Thus the cylinder is attracted towards the source, and a sketch of stream lines reveals that the pressure is greater on the opposite side of the cylinder to that of the source.

Ex. A source of fluid situated in space of two dimensions, is of such strength that $2\pi\rho\mu$ represents the mass of the fluid of density ρ emitted per unit of time. Show that the force necessary to hold a circular disc at rest in the plane of the source is

$$2\pi\rho a^2\mu^2/r(r^2-a^2),$$

where a is the radius of the disc and r the distance of the source from its centre. In what direction is the disc urged by the pressure?

[*Ag 1959, 56 ; Ald 61 ; Gor 61 ; Raj 64*]

[Here the strength of the source is, by definition, μ, because mass of fluid emitted is $2\pi\rho\mu$ per unit of time.]

Exp. 5. *Within a circular boundary of radius a there is a two-dimensional liquid motion due to a source producing liquid at the rate m, at a distance f from the centre, and an equal sink at the centre. Find the velocity potential, and show that the resultant pressure on the boundary is*

$$\rho m^2f^3/2a^2\pi(a^2-f^2)$$

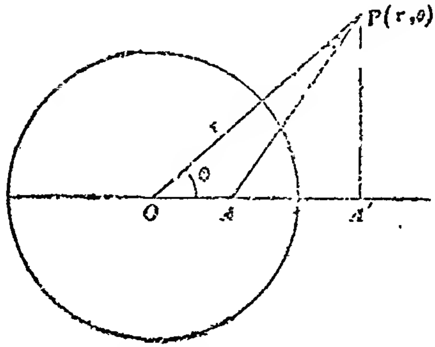
where ρ is the density of the liquid.

Deduce as a limit the velocity potential due to a doublet at the centre.

[*Ag 1961, 59, 51 ; Alg 61 ; Bon 47 ; Cal 54 ; Del 57, 51 ; Mad 59, 57 ; Pna 65 ; Pb 56 (S) ; Sag 54; Ut 64, 61*]

Sol. Here, the strength of the source is $m/2\pi=m'$ (say). Now, to do away with the cylindrical boundary, we consider its effect in the form of images. The image of source m' at $A(z=f)$ and sink $-m'$ at $O(z=0)$ is a source m' at $A'(z=a^2/f=f'$, say) Consequently, the required complex potential is

$$w=-m' \log(z-f)-m' \log(z-f')+m' \log z \quad (1)$$



To find the velocity potential, we equate the real parts in (1), and remembering that real part of $\log \zeta=\log|\zeta|$ we get

$$\phi=-m' \log AP-m' \log A'P+m' \log OP=-m' \log (AP \,.\, A'P/OP).$$

Now, $(dw/dz)=-m'[1/(z-f)+1/(z-f')-1/z]$.

Since the components of the fluid thrust are given by Cauchy-Blasius theorem, viz.

$$X-iY=-\pi\rho\ [\text{Sum of the residues of } (dw/dz)^2 \text{ within the cylinder}] \qquad (2)$$

we proceed to find the residues of $(dw/dz)^2$. Now

$$(1/m'^2)(dw/dz)^2=[1/(z-f)^2]+[1/(z-f')^2]+(1/z^2)+[2/(z-f)(z-f')]-[2/z(z-f')]-[2/z(z-f)].$$

Putting the expression on the right into *partial fractions*, we get

$$(dw/dz)^2=m'^2\{(1/z^2)+[1/(z-f)^2]+[1/(z-f')^2]+[2/(z-f)(f-f')]+[2/(z-f')(f'-f)]+(2/zf')-[2/(z-f')f']-[2/f(z-f)]+(2/fz)\}.$$

Obviously the poles *inside* the circular boundary are $z=0$ and $z=f$. The sum of the residues, i.e. the co-efficients of z^{-1} and $(z-f)^{-1}$ is

$$2m'^2\{(1/f')+(1/f)+1/(f-f')-(1/f)\}=[2m'^2f/f'(f-f')]=[m^2f^3/2a^2\pi^2(f^2-a^2)].$$

Substitutions in (2) lead to

$$X-iY=-\pi\rho.[m^2f^3/2a^2\pi^2(f^2-a^2)].$$

Thus $$X=\rho m^2f^3/2\pi a^2(a^2-f^2)\ ;\ Y=0.$$

To obtain the doublet at the centre, we have to make $f\to 0$, $m'f\to\mu$ [or $a^2/f\to\infty$]

Now from (1)

$$w/m'=-\log(1-f/z)-\log(1-z/f')+\text{const.}$$
$$=(f/z)+(z/f') \qquad \text{expanding logarithms and neglecting const.}$$

or $$w\doteq(\mu/z)+(\mu z/a^2) \qquad [m'f\to\mu \text{ and } f'=a^2f]$$

Equating real parts we get

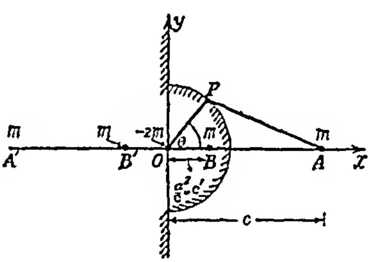
$$\phi=\mu\cos\theta\,(r^{-1}+a^{-2}r).$$

Exp. 6. A line source is in the presence of an infinite plane on which is placed a semi-circular cylindrical boss ; the direction of the source is parallel to the axis of the boss, the source is at distance c from the plane and the axis of the boss, whose radius is a. Show that the radius to the point on the boss at which the velocity is a maximum makes an angle θ with the radius to the source where

$$\theta=\cos^{-1}\{(a^2+c^2)/\sqrt{2(a^4+c^4)}\}.$$

[*Del 1962 ; Osm 60*]

Sol. The image of source m at A $(z=c)$ in the cylindrical boundary is a source m at B $(z=a^2/c=c')$ and a sink, $-m$ at O $(z=0)$. This does away with the cylindrical boundary. And the image of source m at A $(z=c)$ in the line $x=0$ is source m at A' $(z=-c)$, that of m and $-m$ at B and O is m at $B'(z=-c')$ and $-m$ at O. Hence the complex potential shall be



$$w=2m\log z-m\log(z^2-c^2)-m\log(z^2-c'^2)$$

$$\therefore \quad \frac{dw}{dz}=-2m\left\{\frac{z}{z^2-c^2}+\frac{z}{z^2-c'^2}-\frac{1}{z}\right\}=\frac{-2m(z^4-c^2c'^2)}{z(z^2-c^2)(z^2-c'^2)}$$

The velocity at *any point* P $(z=ae^{i\theta})$ is given by

$$q=\left|\frac{dw}{dz}\right|=\frac{2ma^4\,|(e^{4i\theta}-1)|}{a\,|(a^2e^{2i\theta}-c^2)|\,.\,|\{a^2e^{2i\theta}-(a^4/c^2)\}|}$$
$$=\frac{4mac^2\sin 2\theta}{(a^4+c^4-2a^2c^2\cos 2\theta)}. \qquad (1)$$

Thus, $\qquad (4mac^2/q)=(a^4+c^4-2a^2c^2\cos 2\theta)/\sin 2\theta.$

If q is maximum, then $(1/q)$ must be minimum. Now, put $(4mac^2/q)=K$, so that

$$K=(a^4+c^4)\operatorname{cosec} 2\theta-2a^2c^2\cot 2\theta$$

$$(dK/d\theta)=-2(a^4+c^4)\operatorname{cosec} 2\theta\cot 2\theta+4a^2c^2\operatorname{cosec}^2 2\theta \tag{2}$$

$$(d^2K/d\theta^2)=4(a^4+c^4)\operatorname{cosec}\theta\,(\operatorname{cosec}^2 2\theta+\cot^2 2\theta)-8a^2c^2\operatorname{cosec}^2 2\theta\cot 2\theta$$

$$=4\operatorname{cosec}\theta\{(a^2\operatorname{cosec}\theta-c^2\cot\theta)^2+a^4\cot^2\theta+c^4\operatorname{cosec}^2\theta\} \tag{3}$$

Since $\theta\leqslant\frac{1}{2}\pi$, it is clear that $(d^2K/d\theta^2)>0$, so that K must be minimum and hence q must be maximum. From (2), putting $(dK/d\theta)=0$, we obtain

$$(a^4+c^4)\operatorname{cosec} 2\theta\cot 2\theta=4a^2c^2\operatorname{cosec}^2 2\theta \text{ or } \cos 2\theta=2a^2c^2/(a^4+c^4)$$

Thus $2\cos^2\theta=(a^4+c^4+2a^2c^2)/(a^4+c^4)$, or $\cos^2\theta=(a^2+c^2)^2/2(a^4+c^4)$,

i.e. $$\cos\theta=(a^2+c^2)/\sqrt{2(a^4+c^4)}. \tag{4}$$

Ex. 1. In the two-dimensional motion of an infinite liquid there is a rigid boundary consisting of that part of the circle $x^2+y^2=a^2$, which lies in the first and fourth quadrants and the parts of the axis of y which lie outside the circle. A simple source of strength m is placed at the point $(f, 0)$ where $f>a$. Prove that the speed of the fluid at the point $(a\cos\theta,\ a\sin\theta)$ of the semi-circular boundary is

$$4maf^2\sin 2\theta/(a^4+f^4-2a^2f^2\cos 2\theta).$$

Find at what points of the boundary the pressure is least. *[Osm 1961]*

[By Bernoulli-Cauchy pressure equation, $p+\frac{1}{2}\rho q^2=\text{const.}$, it follows that p is least when q is maximum. Thus, at a point $P\ (z=ae^{i\theta})$ where θ is given by (4), the pressure is least. At every other point, p is greater than that at P.]

Ex 2. (a) In liquid bounded by the axes of x and y in the first quadrant there is a source of strength m at distance a from the origin on the bisector of the angle xoy. Prove that the complex potential is $-m\log(a^4+z^4)$.

(b) OX, OY are fixed rigid boundaries and there is a source at $(a,\ b)$. Find the form of the stream lines and show that the dividing line is

$$xy\,(x^2-y^2-a^2+b^2)=0.$$

(b′) The two-dimensional motion due to a source whose polar coordinates are $r=a$, $\theta=\beta$, $(\beta<\pi/4)$ is bounded by the lines $\theta=0$ and $\theta=\pi/2$. Show that the equations of the stream lines are

$$r^4\sin(4\theta-\alpha)-2r^2a^2\cos 2\beta\sin(2\theta-\alpha)-a^4\sin\alpha=0,$$

where α is a parameter. In particular, find the equation of the stream line from the source to the stagnation point on the line $\theta=0$. *[Pb 1963]*

Ex. 3. A source is situated at the point $(b,\ b)$ in the region bounded by the axis of x and the circle $x^2+y^2=a^2$, the source being outside the circle. Show that the fluid velocity vanishes at the points $(\pm a, 0)$ and that it will vanish at one other point on the circle provided that $2b<(2+\sqrt{2})a$.

Ex. 4. (i) Prove that if there are any number of sources at points on a circle, the circle is a stream line provided that there is no boundary and that the algebraic sum of the strengths of the sources is zero *(Del 1949 ; I.A.S. 59)*

(ii) Show that the same is true if the region of flow is bounded by a circle which cuts orthogonally the circle in question.

Ex. 5. Show that he motion of a liquid within a circular boundary of radius a due to a source m at a distance $a/2$ from the centre and an equal sink at the centre is given by

$$w=-m\log[(z-\tfrac{1}{2}a)(z-2a)/z].$$

Provo that the velocity of a fluid particle at the boundary is

$$[4m \sin \theta/a\,(5-4 \cos \theta)],$$

θ being the angle between the radius through the source and the radius to the particle. Show how to calculate the pressure on the boundary. [Bom 1957]

Ex. 6. If a circle be cut in half by the y-axis, forming a rigid boundary and a source of strength m, be on the x-axis at a distance a, eqnal to half the radius, from the centre, prove that the stream lines are given by

$$(16a^4+r^4) \cos 2\theta-17a^2r^2=(16a^4-r^4) \sin 2\theta . \cot (\psi/m)$$

ψ being a suitably adjusted value of the stream fnnction.

Show further that the stream line $\psi=\frac{1}{2}\, m\pi$ leaves the source in a direction perpendicular to OX and enters the sink at an angle $\pi/4$ with OX, and sketch the stream lines. [Pb 1960]

Ex. 7. In the caso of two-dimensional fluid motion produced by a source of strength m placed at a point P ontside a rigid circular disc of radius a whose centre is O, show that the velocity of slip of the fluid in contact with the disc is greatest at the points where the line joining P to the cnds of the diameter at right angles to OP cut the circle. Prove that its magnitude at these points is

$$2m . OP/(OP^2-a^2). \qquad \text{[Bom 1964 (old) ; Gti 55 ; Pb 50(S)]}$$

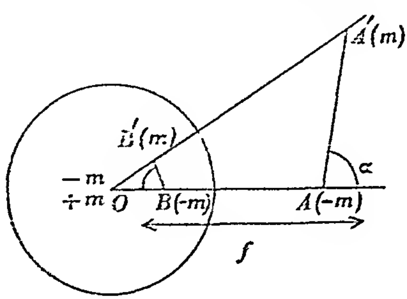
Ex. 8. Provo *Legally's Theorm** on the resultant force and couple on a cylinder in a uniform (two-dimensional) current of liquid in which sources and doublets are present.

(7) *Image of a doublet in front of a circle.* The complex potential for a doublet of strength μ, inclined at an angle α to OX, when present *alone* at $z=f$ is $\mu\, e^{i\alpha}/(z-f)$. If the circular cylinder $|z|=a$, $(a<f)$ is inserted into the field, the complex potential, by Circle theorem, becomes

$$W=\frac{\mu e^{i\alpha}}{z-f}+\frac{\mu e^{-i\alpha}}{(a^2/z)-f}=\frac{\mu e^{i\alpha}}{z-f}+\frac{\mu z\, e^{i(\pi-\alpha)}}{(z-f')f} \quad \left(\frac{a^2}{f}=f'\right)$$

$$=\frac{\mu e^{i\alpha}}{z-f}+\left(\frac{\mu a^2}{f^2}\right)\frac{e^{i(\pi-\alpha)}}{(z-f')}+\frac{\mu e^{i(\pi-\alpha)}}{f}.$$

Omitting the constant term (which is immaterial for flow patterns), this equations gives the complex potential of doublet of strength μ at $z=f$, inclination α and a doublet of strength $\mu'=(\mu a^2/f^2)$ at $z=a^2/f$ which the image doublet inclined at $(\pi-\alpha)$. Thus it follows that the *image is anti-parallel doublet of strength* $(\mu a^2/f^2)$ *situated at its inverse point.*



SECOND METHOD. Let AA' be the doublet outside the circle, then the image of m at A' is a source m at B', the inverse point of A' with respect to the circle and sink $-m$ at O ; that of $-m$ at A is $-m$ at B, the inverse point of A with respect to the circle and $+m$ at O. Clearly the source m and sink $-m$ at O cancel each other.

*See §18 p. 41, 'Hydrodynamics' by D.H. Wilson.

Since, $\qquad OB.OA = OB'\,OA' = a^2$

and $\qquad \angle BOB' = \angle AOA'$

triangles OAA' and OBB' are similar, whence

$$\frac{BB'}{AA'} = \frac{OB'}{OA'} = \frac{OB'}{OA}\cdot\frac{OA'}{OA'} = \frac{a^2}{OA.OA'}.$$

Now let $A' \to A$, so that AA' makes a constant angle α with OA, then $B' \to B$ and BB' makes $(\pi - \alpha)$ with OB. Therefore, the strength of the image doublet is

$$\mu' = \lim_{B' \to B} m.BB' = \lim_{A \to A'} m\frac{AA'.a^2}{OA.OA'} = \mu\frac{a^2}{f^2} \qquad [m.AA' \to \mu]$$

And ultimately the given doublet is at A and the image doublet at B, the inverse point of A with respect to the circle.

Hence the image of a doublet of strength μ at a distance $f > a$ from the centre of a circle of radius a is a doublet at the inverse point, of strength $\mu a^2/f^2$, and the axes of the doublet and its image are anti-parallel.

Exp. Within a rigid boundary in the form of the circle

$$(x+a)^2+(y-4a)^2=8a^2,$$

there is a liquid motion due to a doublet of strength μ at the point $A(0, 3a)$ with its axis along the axis of y.

Show that the velocity potential is

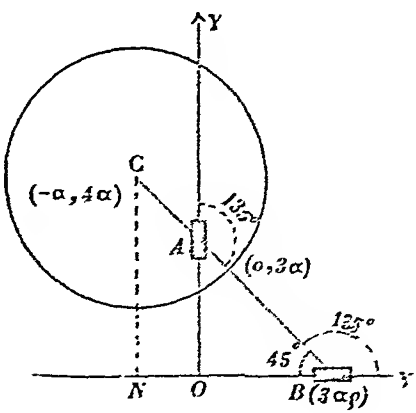
$$\mu\left[4\frac{x-3a}{(x-3a)^2+y^2}+\frac{y-3a}{x^2+(y-3a)^2}\right]$$

[Ban 1965, 54; Del 63, 52, 45, 35; Cal 56; Raj 65; Sag 55; Ut 65]

Sol. The inverse point of $A(0, 3a)$ is $B(3a, 0)$ with regard to the circle of centre C. The distance $CB = 4\sqrt{2}a$, by distance formula as well as from $CA\,CB = 8a^2$: the definition of inverse points. To find the situation of B, we see that line CA is

$$(x+a)/\cos 135 = (y-4a)/\sin 135 = r.$$

Where this line meets x-axis, $y=0$, so that $r=-4\sqrt{2}a$. Thus, the image doublet is situated on x-axis. Since the image of the doublet at A with regard to a circle is another doublet at the inverse point B, the axes of the doublets making supplementary angles with CAB, (i.e. anti-parallel), it follows that the axis of the image-doublet at B is directed along OX.



If the moment of the image-doublet at B be μ', then by [§ 3.40(7)].

$$\mu' = \mu\,(\text{radius})^2/(\text{doublet distance})^2 = \mu\,8a^2/(CA)^2 = 4\mu.$$

Hence the complex potential function is

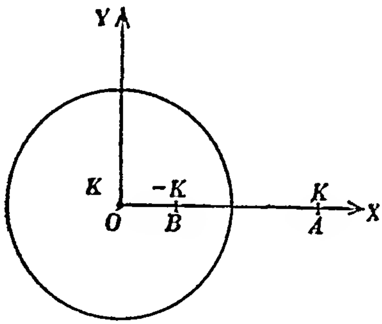
$$w = \frac{\mu' e^{i0}}{z-3a} + \frac{\mu e^{\frac{1}{2}i\pi}}{z-3ia}$$

or
$$\phi+i\psi=\mu\left[\frac{4}{(x-3\alpha)+iy}+\frac{i}{x+i(y-3\alpha)}\right]$$
$$=\mu\left[4\,\frac{(x-3\alpha)-iy}{(x-3a)^2+y^2}+i\,\frac{x-i(y-3\alpha)}{x^2+(y-3\alpha)^2}\right].$$

To obtain the velocity potential, we need *equate the real parts only*, which give
$$\phi=\mu\left[4\,\frac{x-3\alpha}{(x-3\alpha)^2+y^2}+\frac{y-3\alpha}{x^2+(y-3\alpha)^2}\right].$$

(8) *Image of a vortex outside a circular cylinder.* When a vortex filament k at $z=f\,(>a)$ is present *alone* in the fluid, the complex potential is $ik\log(z-f)$. But when we insert the circular cylinder $|z|=a$, the complex potential becomes
$$w=ik\log(z-f)-ik\log[(a^2/z)-f] \tag{1}$$
by virtue of the *Circle theorem*. We rewrite (1) by adding the constant term $ik\log(-f)$
$$w=ik\log(z-f)-ik\log(z-a^2/f)+ik\log z.$$

This is the complex potential of
(i) a vortex k at A, $z=f$;
(ii) a vortex $-k$ at B, $z=a^2/f$;
(iii) a vortex k at O, $z=0$ (centre).



Since $OB=a^2/f$, it follows that A and B are inverse points with regard to the circular section of the cylinder and B is inside the section when A is outside. By putting $z=a\,e^{i\theta}$, we notice that $\psi=0$ so that *the circle is a stream line.*

Thus the image system for a vortex k outside the circular cylinder consists of a vortex of strength $(-k)$ at the inverse point and a vortex of strength k at the centre.

We may notice that the vortex k at $A(z=f)$ describes a circle round the cylinder with velocity (counter-clockwise)
$$\left|-\frac{dW}{dz}\right|_{z=f}=\left|\frac{ik}{z-(a^2/f)}-\frac{ik}{z}\right|_{z=f}=\left|\frac{ik}{f}\cdot\frac{a^2}{f^2-a^2}\right|=\frac{k}{f}\;\frac{a^2}{(f^2-a^2)}$$
where $W=w-w_f=w-ik\log(z-f)$.

Cors. 1. The image of a vortex $-k$ at B and a vortex k at the centre is a vortex k at A, the inverse point of B, where B lies inside the cylinder.

2. The term $ik\log z$ also represents the circulation round the cylinder so that the image system can also be stated as under:

The image system for a vortex k outside the circular cylinder consists of a vortex of strength $(-k)$ at the inverse point and a circulation of strength k round the cylinder.

Ex. A vortex of strength k is placed at the point (f, o) outside a circular cylinder, centre (o, o) of radius a. By calculating the forces exerted on the image system, prove that the cylinder is acted on by a force of magnitude $2\pi\rho k^2a^2/f(f^2-a^2)$. In what direction is the cylinder urged by this force?

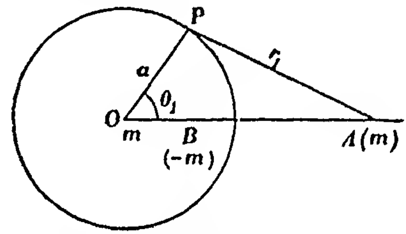
[*Mad 1953*]

[*Replace m by k and c by f in Exp. 4* p. *188*].

Exp. A long fixed cylinder of radius a is surrounded by infinite frictionless incompressible liquid, and there is in the liquid a vortex filament of strength k, which is parallel to the axis of the cylinder at a distance c $(c>a)$ from this axis. Given that there is no circulation round any circuit enclosing the cylinder but not the filament, show that the speed q of the fluid at the surface of the cylinder is

$$\frac{k}{a}\left(1-\frac{c^2-a^2}{r^2}\right)$$

r being the distance of the point considered from the filament.



Sol. The image system consists of a vortex of strength $-k$ at the inverse point B $[z=a^2/c=c'(\text{say})]$ and a vortex of strength k at the centre $(z=0)$. Therefore, the complex potential of the *entire* system is

$$w=ik\log(z-c)-ik\log(z-c')+ik\log z$$

Equating the imaginary parts, we get

$$\psi=\tfrac{1}{2}k\{\log[(r\cos\theta-c)^2+r^2\sin^2\theta]-\log[(r\cos\theta-c')^2+r^2\sin^2\theta]+\log r^2\}$$

$$=\tfrac{1}{2}k\,[\log(r^2+c^2-2cr\cos\theta)-\log(r^2-2r\,c'\cos\theta+c'^2)+\log r^2]$$

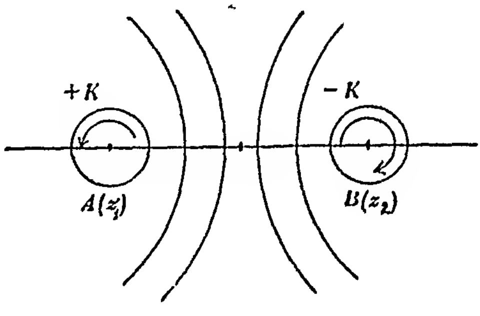
$$\therefore\qquad \frac{\partial\psi}{\partial r}=k\left[\frac{r-c\cos\theta}{r^2+c^2-2rc\cos\theta}-\frac{r-c'\cos\theta}{r^2+c'^2-2rc'\cos\theta}+\frac{1}{r}\right]$$

$$\text{At}\quad r=a,\left(\frac{\partial\psi}{\partial r}\right)=k\left[\frac{a-c\cos\theta}{r_1^2}-\frac{c}{a}\,\frac{c-a\cos\theta}{r_1^2}+\frac{1}{a}\right]$$

$$=\frac{k}{a}\left[1-\frac{c^2-a^2}{r_1^2}\right]$$

Replacing r_1 by $r=AP$, we get $q=\dfrac{k}{a}\left(1-\dfrac{c^2-a^2}{r^2}\right)$.

(9) ***Image of a vortex inside a circular cylinder.*** Consider a vortex pair, where the vortices are situated at $z=z_1$ and $z=z_2$. The complex potential is



$w=ik\log[(z-z_1)/(z-z_2)]$ and the stream function is

$$\psi=k\log\left\{\frac{|z-z_1|}{|z-z_2|}\right\}$$

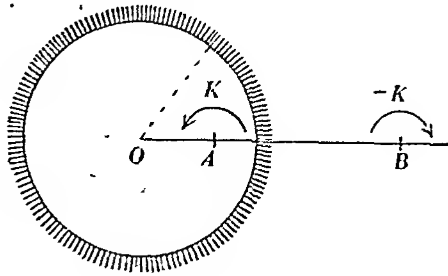
so that the stream lines are

$$r_1=c\,r_2 \qquad (1)$$

where $r_1=|z-z_1|$: $r_2=|z-z_2|$; and c is any constant.

The equation (1) represents a system of co-axial circles with A and B as limiting points. Further the motion is *not steady* and the stream lines are not fixed but follow closely the vortices moving through the liquid.

If now, some circle of the co-axial system is replaced by an indentical rigid boundary and *kept fixed*, then the *image system of a vortex inside a circular cylinder would be an equal and opposite vortex at the inverse point. The vortex A will move round a circle of radius a, centre O, with velocity*



$$q=k/AB=k/(OB-r)=k/[(a^2/r)-r].$$

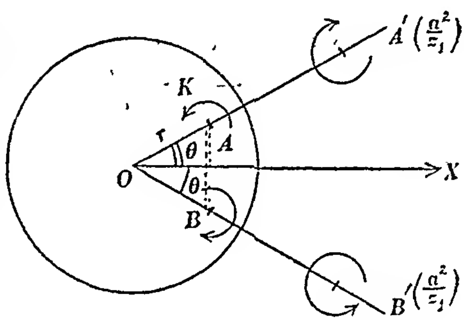
Since $q=\omega \times \mathbf{r}$, $\Rightarrow$ $\omega=q/r$, it follows that the angular velocity of A is $k/(a^2-r^2)$.

Exp. 1. If a pair of equal and opposite vortex filaments are situated inside, or outside, a circular cylinder of radius a at equal distances from its axis, prove that the equation of the cylinder described by each vortex is

$$(r^2-a^2)^2.(r^2 \sin^2 \theta-b^2)=4a^2b^2r^2 \sin^2 \theta$$

where b is a constant and θ is measured from the line through the centre perpendicular to the join of the vortices. [Bom 1958; Ban 47; Raj 66]

Sol. Case (i) *Let the vortex pair be inside the cylinder.* The image of vortex k at $A(z=z_1)$ is a vortex $-k$ at A' $(z=a^2/\bar{z}_1$, and the image of $-k$ at $B(z=\bar{z}_1)$ is an equal and opposite vortex k at B' $(z=a^2/z_1)$.



Hence the complex potential of the system of four vortices is
$w=ik \log(z-z_1)-ik \log(z-\bar{z}_1)-ik \log[z-(a^2/\bar{z}_1)]+ik \log[z-(a^2/z_1)]$
$=ik \log(z-z_1)+w'.$

where, $w'=-ik \log(z-\bar{z}_1)-ik \log(z-a^2/\bar{z}_1)+ik \log(z-a^2/z_1)$ and the complex velocity of the vortex k at A is the value of $(-dw'/dz)$ when $z=z_1$. Now if $w'=\phi'+i\psi'$, then

$$\psi'=k \log(AB'/AA'.AB).$$

Thus the stream lines are given by

$$AB'/AA'.AB=\text{const}.$$

If A is the point (x_1, y_1), or (r, θ), then

$$\frac{r^2+(a^4/r^2)-2r(a^2/r)\cos 2\theta}{[(a^2/r)-r]^2.(2r \sin \theta)^2}=\text{const}.$$

or
$$\frac{r^4+a^4-2a^2r^2\cos 2\theta}{(a^2-r^2)^2 . r^2 \sin^2\theta}=\text{const.}=\frac{1}{b^2} \qquad \text{(say)}$$

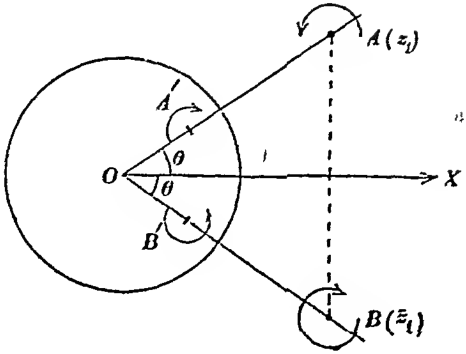
or
$$b^2(r^4+a^4-2a^2r^2\cos 2\theta)=r^2(a^2-r^2)^2\sin^2\theta$$

or
$$b^2[(r^2-a^2)^2+2a^2r^2(1-\cos 2\theta)]=r^2(a^2-r^2)^2\sin^2\theta$$

or
$$(r^2-a^2)^2\,[b^2-r^2\sin^2\theta]=-4a^2b^2r^2\sin^2\theta$$

or
$$(r^2-a^2)^2\,(r^2\sin^2\theta-b^2)=4a^2b^2r^2\sin^2\theta.$$

CASE (ii). *Now let the vortices be outside the cylinder. Let vortex k be at $A(z=z_1)$, then $-k$ will be at B $(z=\bar{z}_1)$.*



The complex potential, when the cylinder $|z|=a$ is not present in the field, is given by

$$ik\log[(z-z_1)/(z-\bar{z}_1)]$$

We now insert the cylinder and the application of Circle theorem provides the complex potential

$$w=ik\log[(z-z_1)/(z-\bar{z}_1)]-ik\log\{[(a^2/z)-\bar{z}_1]/[(a^2/z)-z_1]\}$$
$$=ik\log(z-z_1)+w'$$

where $w'=-ik\log\{(z-\bar{z}_1)[z-(a^2/\bar{z}_1)]/[z-(a^2/z_1)]\}$

and we have ignored an *irrelevant* constant.

Then as before, if $w'=\phi'+i\psi'$, then $\psi'=\log(AB'/AA'.AB)$.

This is what has been obtained earlier and consequently the same result will be obtained again.

Exp. 2. *Three vortex filaments, each of strength K, are symmetrically placed inside a circular cylinder of radius a, and pass through the corners of an equilateral triangle of side $\sqrt{3}b$. If there is no circulation in the fluid other than that due to the vortices, show that they will revolve about the axis of the cylinder with angular velocity*

$$K(a^6+2b^6)/b^2(a^6-b^6).$$

[*Del 1957 ; Jad 59 ; Osm 61 ; Pb 61*]

Sol. Vortex filaments, each of strength k are placed at A, B, C ; A', B', C' are the inverse points of A, B, C with regard to the circular boundary.

Since each side $=\sqrt{3}b$, therefore, $OB=OC=OA=b$.

Further $OA.OA'=a^2$, therefore, $OA'=(a^2/b)=b'=OB'=OC'$.

Thus $\qquad BB'=CC'=AA'=(b'-b)$.

The image system for the vortices, each of strength K at A, B, C consists of the vortices, each of strength $-K$, at A', B', C'.

Of any vortex pair, it is known that each moves perpendicular to their join (AA' say) with a velocity K/r, where r is the distance between them.

The vortex at A moves *solely due to the other* vortices at B, C, A', B' and C' and describes a circle with a certain angular velocity about O. It will be convenient, therefore, to resolve all velocities in a direction perpendicular to OA, i.e. along AP (vide Fig. p. 198)

Let $\qquad AB'=AC'=r$ and $\angle OAB'=\angle OAC'=\theta$.

Then $\cos\theta=\dfrac{OA^2+AB'^2-OB'^2}{2OA.AB'}=\dfrac{b^2+r^2-b'^2}{2br}$

and $\quad AB'^2=OA^2-OB'^2-2OA.OB'\cos 120,$

i.e. $\quad r^2=b^2+b'^2+b.b'=a^2+b^2+b'^2.$

Now we calculate the velocities as under:

(i) Velocity due to A' along

$AP=K.1/(A'A)=K\{1/[(a^2/b)-b]\}$

(ii) Velocity due to B' is along AQ and its magnitude is $=K/r$.

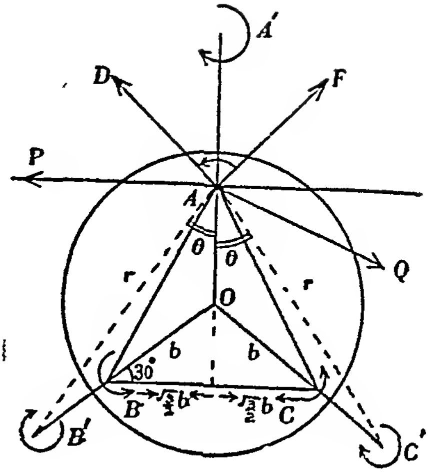
(iii) Velocity due to B is along AD and its magnitude is

$$(K/AB)=K/\sqrt{3}b$$

(iv) Velocity due to C is along AE and its magnitude is

$$(K/AC)=K/\sqrt{3}b$$

(v) Velocity due to C' is along AF and its magnitude is $=K/r$.



Thus the velocities perpendicular to OA are

$$=K\left[\frac{b}{a^2-b^2}-\frac{1}{r}\cos\theta+\frac{1}{\sqrt{3}b}\cos 30^\circ+\frac{1}{\sqrt{3}b}\cos 30^\circ-\frac{1}{r}\cos\theta\right]$$

$$=K\left[\frac{b}{a^2-b^2}-\frac{2r\cos\theta}{r^2}+\frac{1}{b}\right]=K\left[\frac{b}{a^2-b^2}+\frac{1}{b}-\frac{1}{r^2}\left(b^2+r^2-\frac{a^4}{b^2}\right)\frac{1}{b}\right]$$

$$=K\left[\frac{b}{a^2-b^2}+\frac{a^4-b^4}{b^3r^2}\right]=K\left[\frac{b}{a^2-b^2}+\frac{a^4-b^4}{b[a^4+b^4+a^2b^2]}\right]$$

$$=K\left[\frac{b^2(a^4+b^4+a^2b^2)+(a^2-b^2)(a^4-b^4)}{b(a^6-b^6)}\right]=K\frac{a^6+2b^6}{b(a^6-b^6)}.$$

Thus, $b\dot{\theta}=K(a^6+2b^6)/b(a^6-b^6)$; $\Rightarrow\dot{\theta}=K(a^6+2b^6)/b^2(a^6-b^6)$.

3.50. Conformal invariance of hydrodynamical singularities. An essential feature of conformal mapping is that the vanishing of the Laplacian ($\nabla^2=0$) remains invariant under transformation. Thus a harmonic function remains harmonic after conformal transformation to a new cordinate plane. This may be ascertained for transformation $\zeta=f(z)$. Thus if $\phi(x, y)$ which in terms of ξ, η may be written $\Phi(\xi, \eta)$, then

$$\frac{\partial^2\Phi}{\partial\xi^2}+\frac{\partial^2\Phi}{\partial\eta^2}=\left|\frac{dz}{d\zeta}\right|^2\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)$$

so that if dz/dt is not infinite then

$$(\partial^2\phi/\partial x^2)+(\partial^2\phi/\partial y^2)=0;\Rightarrow(\partial^2\Phi/\partial\xi^2)+(\partial^2\Phi/\partial\eta^2=0.$$

Thus, if $\phi(x, y)$ is a general potential function in the z-plane, then $\Phi(\xi, \eta)$ is necessarily a general potential function in the ζ-plane. In other words, (velocity) potential functions transform into (velocity) potential functions. Furthermore, any curve or boundary (i.e. stream line) in the z-plane along which ϕ is constant is mapped into a

new curve or boundary (i.e. stream line) in the ζ-plane along which Φ is constant.

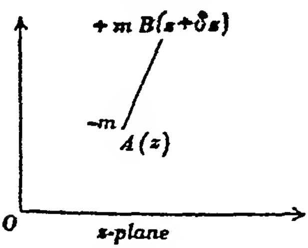
We now discuss the retention of some important hydrodynamic singularities on conformal transformation.

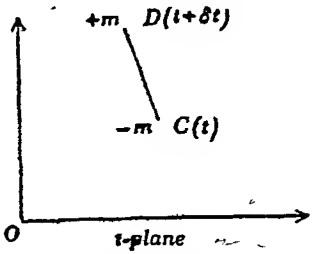
(1) Source. Let there be a source of strength m at the point $P(z_0)$ in the z-plane and enclose it by a small curve C. Let $P'(t_0)$ be the corresponding point in t-plane and let C' be the corresponding curve in the t-plane, then C' *must enclose* P'; because the domain D in the z-plane is mapped in one-to-one fashion onto D' in the t-plane under analytic function $t=f(z)$ transformation.

The flow across C by the definition of a source is $2\pi m\rho$. The flow across C is also given in terms of the stream function by $-\rho\int_C d\psi$, and since each point on C' corresponds to one and only one point on C, we must have, $-\rho\int_C d\psi=-\rho\int_{C'} d\psi$ taken in the *same* sense. This means that the flow across C' is $2\pi m\rho$, and this will be the same for any small closed curve surrounding $P'(t_0)$. Thus, *there must be a source* m at $P'(t_0)$. Hence, we can say that in *a conformal transformation a source is transformed into an equal source.* If C' encircles P' only once, the source will be of strength m. If C' encircles P' *n times* when C encircles P once, the strength of the source at P will be (m/n).

NOTE. Other hydrodynamic singularities must transform into singularities of the same character as is evident from the fact that they are all derivable from a source. Thus, a sink transforms conformally into a sink, a doublet into a doublet, a vortex into a vortex, a vortex source into a vortex source, and so on. However, we prove each case separately.

(2) *Doublet.* Let the doublet of strength μ at the point A (z) in the z-plane be obtained by the combination of source $-m$ at A and $+m$ at B, so that $(\text{Lim } m \cdot AB)=\mu$; as $\delta z\to 0$, $m\to\infty$.





On transforming, we get a source m at D and a sink $-m$ at C in the t-plane. If AB is small enough, magnification gives

$$CD/AB=|(dt/dz)| \text{ so that } m.CD=m.AB\,|(dt/dz)|.$$

Proceeding to the limits, the result is $\mu'=\mu\,|(dt/dz)|$; which is the strength of the doublet in t-plane.

If the doublet in the z-plane is inclined at α with the real axis, the doublet in t-plane will be inclined with the real axis at an angle β, where

$$\beta=\arg \delta t=\arg \{(dt/dz)\, \delta z\} \qquad [\because \delta t=(dt/dz).\delta z]$$
$$=\arg (dt/dz)+\arg \delta z=\arg (dt/dz)+\alpha.$$

Thus, *a doublet of strength μ and inclined at α with real axis in the z-plane transforms conformally into a doublet of strength $\mu'\{=\mu|(dt/dz)|\}$ and inclination β $\{=\alpha+\arg (dt/dz)\}$ with the real axis in the t-plane.*

(3) *Vortex filament.* Let there be a vortex at $P(z_0)$ of strength K in the z-plane and let $P'(t_0)$ be the corresponding point in the t-plane; the connecting mapping being $t=f(z)$. Let C be any small closed curve surrounding $P(z_0)$ and C' the corresponding small closed curve surrounding P' (t_0).

The circulation round C, by definition, is : $-\int_C d\phi=-\Big[\phi\Big]_C=K.$

Since each point on C' corresponds to one and only one point on C, we must have $-\int_C d\phi=-\int_{C'} d\phi$ and thus, circulation around any small closed curve C' surrounding $P'(t_0)$ is K. Therefore, there must be a vortex of strength K at $P'(t_0)$ Hence, *a vortex is transformed into an equal vortex at the corresponding point.*

These vortices however do not necessarily continue to move so as to occupy corresponding points ; if however we know the motion of one, the motion of the other is usually deduced by a device due to *Routh* (vide §3.62 p 213)

(4) *Kinetic energy.* Let $P(z_0)$ be a point inside a small $\triangle ABC$ in the z-plane. Let $P'(t_0)$ be the corresponding point to P inside the corresponding small $\angle A'B'C'$ in the t-plane. The kinetic energy of the fluid in the two triangles is respectively

$$\left.\begin{aligned} T_1&=\tfrac{1}{2}\rho q^2\, \triangle ABC=\tfrac{1}{2}\rho|(dw/dz)|^2 \triangle ABC \\ T_2&=\tfrac{1}{2}\rho q'^2 \triangle A'B'C'=\tfrac{1}{2}\rho|(dw/dt)|^2 \triangle A'B'C' \end{aligned}\right\} \qquad (1)$$

where w is the complex potential for the fluid motion. Now

$$\triangle'=|(dt/dz)|^2\triangle \text{ and } |(dw/dz)|=|(dw/dt)(dt/dz)|,$$

by conformality, we see from (1) that $T_1=T_2$.

NOTES. (1) *The complex potential, $w=\phi+i\psi$, of a flow is invariant under a conformal mapping*; because ϕ and ψ are both harmonic and hence conformally invariant.

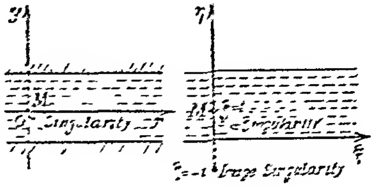
The complex potential, $w=\phi+i\psi$, performs a conformal mapping onto the w-plane, where ψ-lines and ϕ-lines are respectively horizontal and vertical lines.

(2) The retention of the character of the hydrodynamic singularities during transformations is of considerable importance in solving certain problems. In some transformations *mathematical singularities*

appear from the transformation. Physically, these correspond to stagnation points and are not termed hydrodynamic singularities.

3.51. Source or vortex between parallel walls. Liquid fills the doubly infinite strip between barriers along the lines $y=\pm\frac{1}{2}c$, and there is a source of strength m (or vortex of strength K) at the origin. Determine the motion and the paths of the particles.

Here, the image system is a row of an infinite number of sources (or an infinite number of vortices), all along y-axis, and to avoid the longer computations, we transform the region $y=\pm\frac{1}{2}c$ in z-plane, to the upper half of ζ-plane by the transformation $\zeta=i\,e^{\pi z/c}$; $z=0$ corresponding to $\zeta=i$.



Let the strength of the singularity be $M(=m$ if the singularity is source, and K if the singularity is vortex). Then the complex potential of m at $\zeta=i$ and its image m at $\zeta=-i$ is

$$w=-m\log(\zeta-i)-m\log(\zeta+i)=-m\log(-e^{2\pi z/c}+1) \qquad (1)$$

or $\phi+i\psi=-m\log(-2e^{\pi z/c})\{(e^{\pi z/c}-e^{-\pi z/c})/2\}$

$$=-(m\pi z/c)-m\log\sinh(\pi z/c) \qquad (2)$$

neglecting an added constant.

The paths of the particles are given by $\psi=$constant, which from (2) or (1) are given by

$$\text{const.}=\tan^{-1}\{e^{2\pi x/c}.\sin(2\pi y/c)/[1-e^{2\pi x/c}\cos(2\pi y/c)]\}$$

or $\qquad e^{2\pi x/c}[\cos(2\pi y/c)+\lambda\sin(2\pi y/c)]=1 \qquad (\lambda$ is some const.)

For the case of vortex K, the complex potential of K at $\zeta=i$ and its image $-K$ at $\zeta=-i$ is

$$w=iK\log(\zeta-i)-iK\log(\zeta+i)=iK\log\{(e^{\pi z/c}-1)(e^{\pi z/c}+1)\}$$

or $$\phi+i\psi=iK\log\tanh(\pi z/2c).$$

Now, $\qquad 2i\psi=w(z)-\bar{w}(\bar{z})=iK\log\tanh(\pi z/2c)\tanh(\pi\bar{z}/2c)$

$$\therefore \qquad \psi=\tfrac{1}{2}K\log\left[\tanh\frac{\pi(x+iy)}{2c}\tanh\frac{\pi(x-iy)}{2c}\right]$$

$$=\tfrac{1}{2}K\log\left(\frac{\cosh(\pi x/c)-\cosh i(\pi y/c)}{\cosh(\pi x/c)+\cosh i(\pi y/c)}\right)$$

$$=\tfrac{1}{2}K\log\left(\frac{\cosh(\pi x/c)-\cos(\pi y/c)}{\cosh(\pi x/c)+\cos(\pi y/c)}\right)$$

Hence, the stream lines are given by

$$\cosh(\pi x/c)=\lambda\cos(\pi y/c)$$

which are also the paths of the particles.

The velocity of the vortex at O, i.e. (u_0, v_0) is given by $(D=d/dz)$

$$u_0 - iv_0 = -\{D[iK \log \tanh(\pi z/2c) - iK \log(\pi z/2c)]\}_{z=0}$$
$$= iK.(\pi/2c)\{\operatorname{cosech}(\pi z/2c) - (\pi z/2c)^{-1}\} \to 0 \text{ as } z \to 0.$$

Thus, $u_0=0$, $v_0=0$ and the vortex at O and thereby all its images remain at *rest*.

Ex. 1. (a) Determine the path of a single vortex in a corner between planes meeting at right angles. What is the ultimate direction in which the vortex moves? *(Lkn 1956)*

(b) Define a *vortex pair,* and obtain its complex potential. Deduce that a single vortex in the presence of a plane will move parallel to the plane, and find the velocity of this motion, as well as the pressure on the plane area due to this motion.

Use the method of conformal transformation to show that two equal but opposite vortices of proper strength can remain at rest behind a circular cylinder in a uniform stream, if they are suitably placed. *(Pna 1960)*

Ex, 2. The irrotational motion in two-dimensions of a fluid bounded by the lines $y=\pm b$ is due to a doublet μ at the origin, the axis of the doublet being in the positive direction of the axis of x. Prove that the motion is given by

$$w=(\mu\pi/2b)\coth[\pi(x+iy)/2b].$$

Show also that the points where the fluid is moving parallel to the axis of y lie on the curve

$$\cosh(\pi x/b)=\sec(\pi y/b). \qquad (Kr\ 1961)$$

3.52. *Applications of conformal invariance*

Exp. 1. Between the fixed boundaries $\theta=\pi/4$ and $\theta=-\pi/4$, there is a two-dimensional liquid motion due to a source of strength m at the point $r=a$, $\theta=0$, and an equal sink at the point $r=b$, $\theta=0$. Use the method of images to show that the stream function is

$$-m\tan^{-1}\{r^4(a^4-b^4)\sin 4\theta/[r^8-r^4(a^4+b^4)\cos 4\theta+a^4b^4]\}.$$

Show also that the velocity at (r, θ) is

$$4m(a^4-b^4)r^3/\sqrt{(r^8-2a^4r^4\cos 4\theta+a^8)(r^8-2b^4r^4\cos 4\theta+b^8)}.$$

[Ag 1945 ; Alg 60, 57 ; Gti 56 ; Jab 61, 60 ; Sag 56 ; Ut 63 ; I.A.S. 53]

Sol. The boundaries $\theta=\pm\pi/4$ in the x-y plane can be transformed to the imaginary axis $\varphi=\pm\pi/2$ in the ζ-plane by the mapping $\zeta=z^2$, i.e. $Re^{i\varphi}=(re^{i\theta})^2$. This yields $R=r^2$ and $\theta=\varphi/2$. The points $(a, 0)$ and $(b, 0)$ in the z-plane go over to the points $(a^2, 0)$, $(b^2, 0)$ in the ζ-plane. Now image of $+m$ at $(a^2, 0)$ in η-axis ($\xi=0$) is $+m$ at $(-a^2, 0)$; image of $-m$ at $(b^2, 0)$ in η-axis ($\xi=0$) is $-m$ at $(-b^2, 0)$. We can now do away with the boundary $\xi=0$, and the complex potential due to m at $(a^2, 0)$ and $(-a^2, 0)$; due to $-m$ at $(b^2, 0)$ and $(-b^2, 0)$ is simply

$$w=-m\log(\zeta-a^2)-m\log(\zeta+a^2)+m\log(\zeta-b^2)+m\log(\zeta+b^2)$$
$$=-m\log[(\zeta^2-a^4)/(\zeta^2-b^4)]=-m\log[(z^4-a^4)/(z^4-b^4)] \quad (\because \zeta=z^2) \qquad (1)$$

Putting $z=re^{i\theta}$; equating the imaginary parts we get

$$\psi=-m[\tan^{-1}\{r^4\sin 4\theta/(r^4\cos 4\theta-a^4)\}-\tan^{-1}\{r^4\sin 4\theta/(r^4\cos 4\theta-b^4)\}]$$
$$=-m\tan^{-1}\{r^4(a^4-b^4)\sin 4\theta/[r^8-r^4(a^4+b^4)\cos 4\theta+a^4b^4]\}.$$

Also, $$\frac{dw}{dz}=-m\left[\frac{4z^3}{z^4-a^4}-\frac{4z^3}{z^4-b^4}\right]=-\frac{4m(a^4-b^4)z^3}{(z^4-a^4)(z^4-b^4)}$$

Thus, $$q=\left|\frac{dw}{dz}\right|=\frac{4m(a^4-b^4)\,|z^3|}{|z^4-a^4|\;|z^4-b^4|}=\frac{4m(a^4-b^4)r^3}{|r^4e^{4i\theta}-a^4|\;|r^4e^{4i\theta}-b^4|}$$

$$=4m(a^4-b^4)r^3/\sqrt{(r^8-2a^4r^4\cos 4\theta+a^8)}\cdot$$

Ex. 1. Between the fixed boundaries $\theta=\pm\pi/4$, there is a two-dimensional liquid motion due to a source of strength m at the point $r=a$, $\theta=0$, and an equal sink at the point $r=b$, $\theta=0$. Show that the stream function ψ is given by $\psi=-m\alpha+m\beta$, where

$\tan\alpha=r^4\sin 4\theta/(r^4\cos 4\theta-a^4)$ and $\tan\beta=r^4\sin 4\theta/(r^4\cos 4\theta-b^4)$. [*Mar 1960*]

Exp. 2. Between the fixed boundaries $\theta=\pi/6$ *and* $\theta=-\pi/6$, *there is a two-dimensional liquid motion due to a source at the point* $(r=c,\ \theta=\alpha)$, *and a sink at the origin, absorbing water at the same rate as the source produces it. Find the stream function, and show that one of the stream lines is a part of the curve*

$$r^3\sin 3\alpha=c^3\sin 3\theta.$$

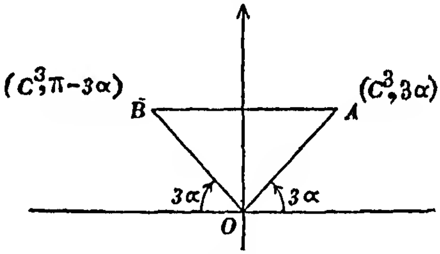
[*Ag 1960, 57 ; Alg 64, 58 ; Ald 62 ; Del 54, 50 ; Raj 62*]

Sol. Let us transform the z-plane to the ζ-plane by the transformation

$$z^3=\zeta \qquad \text{where } z=re^{i\theta}, \text{ and } \zeta=R\,e^{i\beta},$$

i.e. $$r^3\,e^{3i\theta}=R\,e^{i\beta}, \text{ so that } r^3=R,\ 3\theta=\beta.$$

Hence the boundaries $\theta=\pm\pi/6$ transform to boundary (vertical line) $\beta=\pm\pi/2$. The point $(c,\ \alpha)$ transforms to $(c^3,\ 3\alpha)$ and the origin $(0,\ 0)$ in z-plane corresponds to the origin $(0,\ 0)$ in the ζ-plane.



Let the strength of the source be m. Since sink absorbs water at the same rate as is created by source m, the strength of the sink is $-m$.

The image of $+m$ at $(c^3,\ 3\alpha)$ is $+m$ at $[c^3,\ (\pi-3\alpha)]$; and the image of $-m$ at $(0,\ 0)$ is $-m$ at $(0,\ 0)$.

We can now do away with the boundary $\beta=\pm\pi/2$ (the vertical line) and the required complex potentia w is given by

$$W=2m\log\zeta-m\log(\zeta-c^3e^{3i\alpha})-m\log(\zeta-c^3e^{i(\pi-3\alpha)})$$
$$=6m\log z-m\log(z^3-c^3e^{3i\alpha})(z^3-c^3e^{-i3\alpha})$$
$$=6m\log z-m\log(z^6-c^6-2ic^3z^3\sin 3\alpha)$$

or $\phi+i\psi=6m(\log r+i\theta)-m\log[r^6\cos 6\theta+2c^3r^3\sin 3\theta\sin 3\alpha-c^6)+i(r^6\sin 6\theta-2c^3r^3\sin 3\alpha\cos 3\theta)]$

Equating the imaginary parts, we get

$$\psi=6m\theta-m\tan^{-1}\left[\frac{r^6\sin 6\theta-2c^3r^3\sin 3\alpha\cos 3\theta}{r^6\cos 6\theta+2c^3r^3\sin 3\theta\sin 3\alpha-c^6}\right].$$

One of the stream lines is when $\psi=0$, which gives

$$\tan 6\theta=\frac{r^6\sin 6\theta-2c^3r^3\sin 3\alpha\cos 3\theta}{r^6\cos 6\theta+2c^3r^3\sin 3\alpha\sin 3\theta-c^6}.$$

Setting $\tan 6\theta=\sin 6\theta/\cos 6\theta$, cross-multiplying and cancelling $r^6\sin 6\theta\cos 6\theta$, we get

$$c^6\sin 6\theta=2r^3c^3\sin 3\alpha\,[\cos 3\theta\cos 6\theta+\sin 3\theta\sin 6\theta]$$

or $$c^3\sin 3\theta\cos 3\theta=r^3\sin 3\alpha\cos 3\theta$$

or $$\cos 3\theta\,[r^3\sin 3\alpha-c^3\sin 3\theta]=0$$

either $\cos 3\theta=0$ which gives $3\theta=\pm\pi/2$, i.e. $\theta=\pm\pi/6$ which are the given boundaries or

$$r^3 \sin 3\alpha = c^3 \sin 3\theta$$

which is the other result stated.

Exp. 3. In the part of an infinite plane bounded by a circular quadrant AB and the productions of the radii OA, OB, there is a two-dimensional motion due to the production of liquid at A, and its absorption at B, at the uniform rate m. Find the Velocity Potential of the motion; and show that the liquid which issues from A in the direction making an angle μ with OA follows the path whose polar equation is

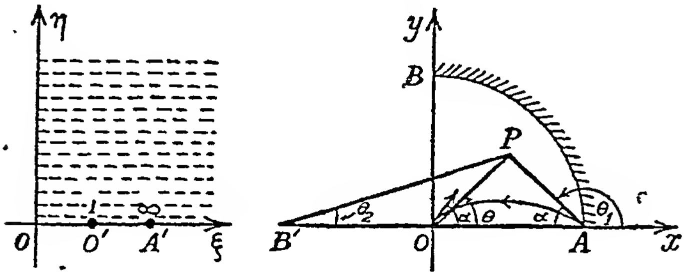
Exp. 4. In a region bounded by a fixed quadrantal arc and its radii, deduce the motion due to a source and an equal sink situated at the ends of one of the bounding radii. Show that the stream line leaving either end at an angle α *with the radius is*

$$r^2 \sin(\alpha+\theta)=a^2 \sin(\alpha-\theta)$$

where a is the radius of the circle [*Alg 1956 ; Ald 64 ; Bom 55 : Gu 59, 53*]

Sol. The *interior* of the quadrant of the circle can be transferred to the upper half of the ζ-plane by the transformation

$$\zeta=\{(a^2+z^2)/(a^2-z^2)\}^2$$



The source m at A ($z=a$) goes to source m at $\zeta=\infty$ and source $(-m)$ i.e. sink at 0 goes over to sink $-m$ at $\zeta=1$. The point B ($z=ia$) goes to $\zeta=0$.

The image of $-m$ at $\zeta=1$ in the line $\eta=0$ is an equal sink at $\zeta=1$, and the image of source m at ∞ is a source m at ∞ in the line $\eta=0$. Hence the complex potential shall be

$$\begin{aligned} w&=2m \log(\zeta-1)-2m \log(\zeta-\zeta_0) \qquad [\zeta_0\to\infty]\\ &=2m \log(\zeta-1)-2m \log(-\zeta_0)(1-\zeta/\zeta_0)\\ &=2m \log(\zeta-1)+\text{const (expanding log series and taking limits)}\\ &=2m \log\{[(a^2+z^2)^2/(a^2-z^2)^2]-1\}+\text{const.}\\ &=2m \log\{4a^2z^2/(a^2-z^2)^2\}+\text{const.}\\ &=-4m \log\{(z^2-a^2)/z\}\ ; \quad \text{omitting constant.} \end{aligned} \tag{1}$$

Thus, $\phi+i\psi=-4m \log\{r^2 e^{2i\theta}-a^2)/r\, e^{i\theta}\}$

$$=-4m \log\{[(r^2-a^2)\cos\theta+i\,(r^2+a^2)\sin\theta]/r\}$$

hence $\psi=-4m \tan^{-1}[\{(r^2+a^2)/(r^2-a^2)\}\tan\theta]$

or $$\tan(-\psi/4m)=\{(r^2+a^2)/(r^2-a^2)\}\tan\theta. \tag{2}$$

Thus, $r=a$ or $\theta=\pi/2$ corresponds to $(\psi/4m)=-\pi/2$ and $\theta=0$ (line OA) corresponds to $\psi=0$. Thus, (1) really determines the motion within the fluid.

From (1), setting $4m=m'$,

NOTE. In some transformations mathematical singularities appear from the transformation. Hydrodynamically, these correspond to stagnation points and are not termed hydrodynamic singularities (vide source, vortex, etc).

Exp. 5. Prove that in two-dimensional irrotational fluid motion, if the stream lines are confocal ellipses

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1\text{ ; then }\psi=A\log\left[\sqrt{a^2+\lambda}+\sqrt{b^2+\lambda}\right]+B,$$

and the velocity at any point is inversely proportional to the square root of the rectangle under the focal radii of the point.

[*Ald 1963 ; Ban 56 ; Del 55, 39 ; Gti 64, 63 ; Jab 62 ; Lkn 56 ; Mod 58 ; Osm 59 ; Pb 53*]

Sol. The conformal transformation $z=c\cos w$ is known to yield the given type of confocal ellipses and hence eliminating ϕ we get

$$(x^2/c^2\cosh^2\psi)+(y^2/c^2\sinh^2\psi)=1. \tag{1}$$

Thus, if $\psi=$const., the stream lines are confocal ellipses. And comparing (1), with the given set of ellipses, we get

$$c^2\cosh^2\psi=a^2+\lambda,\ c^2\sinh^2\psi=b^2+\lambda\ ;\Rightarrow a^2-b^2=c^2,\text{ i.e. } ae=c.$$

Also $$\sqrt{a^2+\lambda}+\sqrt{b^2+\lambda}=c\,(\cosh\psi+\sinh\psi)=ce^{\psi}$$

or $$\psi=\log\left[\sqrt{a^2+\lambda}+\sqrt{b^2+\lambda}\right]-\log c \tag{2}$$

If $w=\phi+i\psi$ is the complex potential of some fluid motion, then so is Aw. Hence (2) gives

$$\psi=A\log\left[\sqrt{a^2+\lambda}+\sqrt{b^2+\lambda}\right]+B$$

Velocity : To find the velocity at any point, we have to calculate $|dw/dz|$.

Now $dz/dw=-c\sin w=-c\sqrt{1-\cos^2 w}=-\sqrt{c^2-z^2}$. Thus,

$$q=|-dw/dz|=|\sqrt{(c-z)}\ \sqrt{(c+z)}|^{-1}=|\sqrt{(z-ae)}\ \sqrt{(z+ae)}|^{-1} \tag{3}$$

Now, if the focii are $(\pm ae, 0)$, denoted by S and H and P is the point z, then $SP=\sqrt{z-ae}$, $HP=\sqrt{z+ae}$, then (3) yields $q=1/SP.HP$ which is the result to be established.

Exp. 6. A columnar vortex filament moves in an infinite liquid bounded by two plane walls intersecting at right angles and parallel to the vortex. Prove that the path of any point of the vortex has for equation

$$r\sin 2\theta=\text{const.}$$

Prove that when the vortex is at the same distance a from each wall, the equation of the stream lines is

$$\{r^2+(4a^4/r^2)\}\operatorname{cosec}2\theta=\text{constant.}$$

Prove also that

$$r^2=2a^2+(k^2t^2/8\pi^2a^2)$$

where K is the strength of the vortex and $t=0$ when the vortex is at the point $r=\sqrt{2}a$, $\theta=\pi/4$. (London 1922, 20)

Sol. We have proved in § 3.40(4), p. 182 that $r\sin 2\theta=$const.

Now initially $r=\sqrt{2a}$, $\theta=\pi/4$, therefore const.$=\sqrt{2a}$,

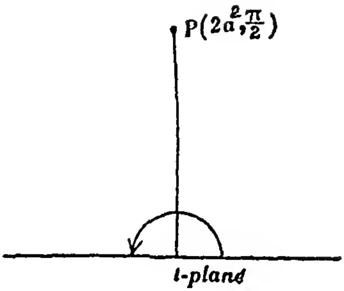
whence $$r\sin 2\theta=\sqrt{2a} \tag{1}$$

As to the second part, let us effect the transformation

$t=z^2$, i.e. $Re^{i\chi}=r^2e^{2i\theta}$; $\Rightarrow R=r^2$; $\chi=2\theta$.

When $\theta=0$; $\chi=0$, and when $\theta=\pi/4$, $\chi=\pi/2$.

The rigid boundary in the z-plane now transforms to a line in t-plane extending from $\chi=0$ to $\chi=\pi$ and the point $(\sqrt{2}, \frac{1}{4}\pi)$ transforms to $(2a^2, \frac{1}{2}\pi)$ [See Fig.]



Thus, $w=iK\log\dfrac{t-2a^2e^{i\pi/2}}{t-2a^2e^{-i\pi/2}}$

$$= iK\log\left(\frac{z^2-2a^2i}{z^2+2a^2i}\right)$$

or $$\phi+i\psi=iK\log\frac{r^2\cos 2\theta+i(r^2\sin 2\theta-2a^2)}{r^2\cos 2\theta+i(r^2\sin 2\theta+2a^2)}.$$

Equating the imaginary parts we obtain

$$\psi=(K/2)\log\{(r^4-4a^2r^2\sin 2\theta+4a^4)/(r^4+4a^2r^2\sin 2\theta+4a^4)\}$$

The stream lines are given by $\psi=$const., i.e.

$$r^4-4a^2r^2\sin 2\theta+4a^4=A(r^4+4a^2r^2\sin 2\theta+4a^4)$$

or $$\{r^2+4a^4/r^2)\}\operatorname{cosec} 2\theta=B \text{ (constant)}.$$

Now from (1)

$$\operatorname{cosec}^2 2\theta=(r^2/2a^2) \text{ or } \cot 2\theta=\sqrt{r^2-2a^2}/a\sqrt{2}$$

Also $$(K/2r)\cot 2\theta=dr/dt \qquad \text{[§ 3.40(4) p. 182]}$$

Therefore, $$\frac{dr}{dt}=\frac{K}{\sqrt{2a}}\,\frac{\sqrt{r^2-2a^2}}{r}$$

Integrating we get,

$$\int_{a\sqrt{2}}^{r}\frac{r\,dr}{\sqrt{r^2-2a^2}}=\frac{Kt}{a\sqrt{2}} \qquad \text{giving } r^2=2a^2+\frac{K^2t^2}{2a^2}.$$

Since $K=k/2\pi$, the result follows:

$$r^2=2a^2+(k^2t^2/8\pi^2a^2).$$

Exp. 7. The space enclosed between the planes $x=0$, $x=a$, $y=0$ on the positive side of $y=0$ is filled with uniform incompressible liquid. A rectilinear vortex parallel to the axis of z has co-ordinates (x', y'). Determine the velocity at any point of the liquid and show that the path of the vortex is given by

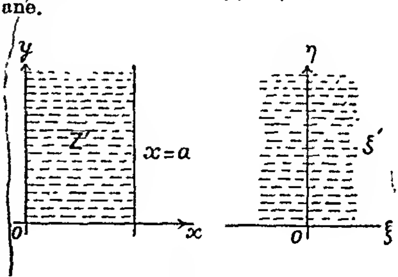
$$\cot^2(\pi x/a)+\coth^2(\pi y/a)=\text{const.}$$

[*Del 1966, 49 ; Krk 65 ; Pb 56, 52, 48*]

Sol. The mapping $\zeta=-\cos(\pi z/a)$ transforms the region $x=0$, $y=0$, $x=a$ of the z-plane into the region $\eta \geqslant 0$ of the ζ-plane.

The point $z'=x'+iy'$ corresponds to $\zeta'=-\cos(\pi z'/a)$. Since the image of the vortex K at ζ' is the vortex $-K$ at $\overline{\zeta}'$ so far as the boundary $\eta=0$ is concerned, the complex potential shall be

$$w=iK\log(\zeta-\zeta')-iK\log(\zeta-\overline{\zeta'})$$
$$=iK\,[\log\{-\cos(\pi z/a)+\cos(\pi z'/a)\}-\log\{-\cos(\pi z/a)+\cos(\pi\bar{z}'/a)\}]$$



$$=iK \log \frac{\sin \lambda (z+z') \sin \lambda (z-z')}{\sin \lambda (z+\bar{z}') \sin \lambda (z-\bar{z}')} \quad (\lambda=\pi/2a). \tag{1}$$

The velocity at any point of the liquid not occupied by the vortex is given by $|dw/dz|$.

The velocity of the vortex at $z=z'$ is due *solely* to its images, and the complex potential of the images at $z=z'$ is

$$w'=[w-iK \log (z-z')] \text{ at } z=z'$$

Now $$\frac{dw'}{dz}=iK \lambda\{\cot \lambda(z+z')-\cot \lambda (z+\bar{z}')-\cot \lambda (z-\bar{z}')\}_{z=z'} \tag{2}$$

where we have used $[\cot \lambda\{(z-z')-\{1/\lambda (z-z')\}] \to 0$ as $z \to z'$.

Thus, (2) gives

$$-\dot{x}'+i\dot{y}'=iK\lambda \{\cot 2\lambda (x'+iy')-\cot 2\lambda x'+i \coth 2\lambda y'\}.$$

Since $\cot 2\lambda (x'+iy')=(\sin 4\lambda x'-i \sinh 4\lambda y')/(\cosh 4\lambda y'-\cos 4\lambda x')$, we get on equating the real and imaginary parts

$$\dot{x}'=K\lambda \{2 \cosh 2\lambda y' \sin^2 2\lambda x'/\sinh 2\lambda y' (\cosh 4\lambda y'-\cos 4\lambda x')\}$$

$$\dot{y}'=K\lambda \{-2 \cos 2\lambda x' \sinh^2 2\lambda y'/\sin 2\lambda x' (\cosh 4\lambda y'-\cos 4\lambda x')\}$$

Division of the last two equations gives

$$\frac{dx'}{dy'}=-\frac{\cosh 2\lambda y' \sin^3 2\lambda x'}{\cos 2\lambda x' \sinh^3 2\lambda y'}, \Rightarrow \frac{\cos 2\lambda x'}{\sin^3 2\lambda x'}\, dx'+\frac{\cosh 2\lambda y'}{\sinh^3 2\lambda y'}\, dy'=0$$

To obtain the path, we integrate the preceding result and replace x', y' by the current coordinates (x, y). Thus

$$\operatorname{cosec}^2 2\lambda x+\operatorname{cosech}^2 2\lambda y=\text{const.}$$

or $$\cot^2 (\pi x/a)+\coth^2 (\pi y/a)=\text{const.} \quad [\lambda=\pi/2a]$$

Ex. 1. Use the *method of images* to prove that if there be a source m at the point z_0 in a fluid bounded by the lines $\theta=0$ and $\theta=\pi/3$, the solution is given by

$$w=-m \log \{(z^3-z_0^3)(z^3-\bar{z}_0^3)\}$$

where $$z=x+iy,\ z_0=x_0+iy_0, \text{ and } \bar{z}_0=x_0-iy_0.$$

[*Ag 1953, 46 ; Ban 61 ; Del 48 ; Lkn 62 ; Pna 59 ; Sag 57*]

Ex. 2. The motion of a liquid is in two dimensions, and there is a constant source at one point A in the liquid and an equal sink at another point B. Find the form of the stream lines and prove that the velocity at any point P varies $(AP.BP)^{-1}$, the plane of motion being unlimited.

If the liquid is bounded by the planes $x=0$; $x=a$; $y=0$; $y=a$, and if the source is at the point $(0, a)$ and the sink at $(a, 0)$, find an expression for the velocity potential. [*Ag 1952*]

Ex. 3. An area A is bounded by that part of the x-axis for which $x > a$ and by that branch of $x^2-y^2=a^2$ which is in the positive quadrant. There is a two-dimensional unit source at $(a, 0)$ which sends out liquid uniformly in all directions. Show by means of the transformation $w=\log (z^2-a^2)$ that in steady motion the stream lines of the liquid within the area A are portions of rectangular hyperbolas. Draw the stream lines corresponding to $\psi=0, \frac{1}{4}\pi, \frac{1}{2}\pi$.

If r_1, r_2 are the distances of a point P within the fluid from the points $(\pm a, 0)$, show that the velocity of the fluid at P $(OP=r)$ is measured by $2r/r_1r_2$, O being the origin. [*Alg 1952*]

Ex. 4. The internal boundary of a liquid is composed of the two orthogonal circles

$$x^2+y^2+2y=1 \; ; \; x^2+y^2-2y=1.$$

A source producing liquid at the rate m is placed at one of the points of intersection $(z=1)$. Show that the complex potential of the fluid motion is $\frac{m}{2\pi}\log[z(z^2+1)/(z-1)^4]$, and that the two circles are the only stream lines possessing *double points.*

Ex. 5. Liquid flows steadily and irrotationally in two dimensions in a space with fixed boundaries the cross section of which consists of the two lines $\theta=\pm\pi/10$ and the curve $r^5\cos 5\theta=K^5$.

Prove that, if V is the velocity of the liquid in contact with one of the plane boundaries at unit distance from their intersection, the volume of liquid which passes per unit time through a circular ring in the plane $\theta=0$ is

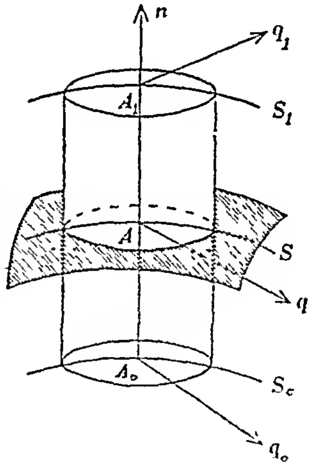
$$\tfrac{1}{8}\pi\, Va^2\,(a^4+12a^2b^2+8b^4),$$

where a is the radius of the ring, and b the distance of its centre from the intersection of the plane boundaries.

3.60. *Vortex sheet.* Consider the surface in a fluid and a point A of the same occupying the position of centroid of an element dS of the said surface. Let the points A_0, A_1 be taken on the normal at A such that

$$AA_0=-\tfrac{1}{2}\,\varepsilon\,\mathbf{n}, \quad AA_1=\tfrac{1}{2}\,\varepsilon\,\mathbf{n},$$

where $\mathbf{n}$ is the unit normal at A and ε is an infinitesimal positive scalar. Thus when A describes S, A_0 and A_1 describe S_0 and S_1 parallel and on opposite sides of and equidistant from S. The normals at the boundary of dS together with dS_1 and dS_2 enclose a cylindrical element of volume $dv=\varepsilon\, dS$.



Now suppose that the fluid moves irrotationally everywhere except in the part bounded by S_0 and S_1. Let ζ be the vorticity vector at A, then

$$\zeta\, dv=\zeta\,\varepsilon dS=\omega\, dS \quad \text{(say)},$$

where $\zeta\varepsilon=\omega$. Let $\varepsilon\to 0$ in such a way that ω remains constant, (i.e. $\zeta\to\infty$). Then *the surface S is called a* vortex sheet *of vorticity ω per unit area. We may note that the normal component of velocity is continuous across the vortex sheet.*

With the notations of Fig.

$$\mathbf{q}_1=\mathbf{q}+\tfrac{1}{2}\,\varepsilon\,(\mathbf{n}\,.\,\nabla)\,\mathbf{q}\,; \quad \mathbf{q}_0=\mathbf{q}-\tfrac{1}{2}\,\varepsilon\,(\mathbf{n}\,.\,\nabla)\,\mathbf{q}$$

so that $\mathbf{q}=\frac{1}{2}(\mathbf{q}_0+\mathbf{q}_1)$ which is independent of ε. The result suggests that *the velocity of a point A of a vortex sheet is the arithmetic mean of the velocities just above and just below A on the normal at A.*

An extension of Gauss's theorem is

$$\int_V \operatorname{curl}\mathbf{q}\, dv=\int_S d\mathbf{S}\times\mathbf{q} \quad \text{or} \quad \int_V \zeta\, dv=\int_S \mathbf{n}\times\mathbf{q}\, dS \qquad (1)$$

Therefore, if we apply (1) to the elementary cylinder of volume dv we get

$$\zeta\varepsilon dS \doteq \mathbf{n}\times(\mathbf{q}_1-\mathbf{q}_0)\, dS \quad \text{to this order.}$$

Dividing by dS and letting $\varepsilon\to 0$, we obtain

$$\boldsymbol{\omega}=\mathbf{n}\times(\mathbf{q}_1-\mathbf{q}_0).$$

Obviously a non-zero volue of $\boldsymbol{\omega}$ is associated with discontinuity of the components of $\mathbf{q}_0$, $\mathbf{q}_1$ perpendicular to $\mathbf{n}$, so that the *tangential velocity changes abruptly across a vortex sheet.*

As $\boldsymbol{\omega}.\mathbf{n}=0$, it follows that $\boldsymbol{\omega}$ is perpendicular to $\mathbf{n}$ and is therefore tangential to the vortex sheet, *i.e.* a vortex sheet may be thought of as covered by a system of vortex filaments.

Cor. A two-dimensional vortex sheet is represented by a curve AB in the plane of motion, such that the tangential velocity changes abruptly but there is no change in the normal velocity as we cross the curve AB.

Exp. *The motion of liquid at rest at infinity is due to a circular cylindrical vortex sheet of radius a and of constant strength k. Determine the complex potential inside and outside the sheet.*

Sol. Suppose that throughout the interior of the circle $|z|=a+\delta a$, the spin is constant and equal to ω. The complex potential at external points is obviously $i\omega(a+\delta a)^2 \log z$.

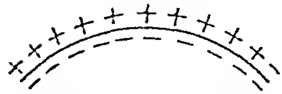
Let us remove the liquid from within the circle $|z|=a$, so that we are left with an annular vortex of internal radius a which becomes the given vortex sheet if $\delta a\to 0$, $2\omega\delta a\to k$.

From what has been said, it follows that the complex potential at external points is

$$w=i\omega(a+\delta a)^2\log z-i\omega a^2\log z$$
$$=i\omega\,.\,2a\,.\,\delta a\log z=ika\log z.$$

The circle $|z|=a$ is a fixed boundary, and space enclosed by it is clearly simply-connected. It follows that the liquid within it must be at rest and so w must be constant.

3.61. Potential field due to a given vortex. Let us consider an arbitrary surface covered uniformly with sources $(++\ldots++)$ on one side and sinks $(--\ldots--)$ on the other so that the surface may be thought of as a double layer of small but finite thickness h rather than the arbitrary surface (Fig.). We can now do away with the difficulty of the discontinuity of potential. By far the largest part of the stream coming from the sources flows directly to the sinks and produces a large difference of potential in the small distance h. Let us choose the strengths of the uniformly distributed doublets such that the potential difference is everywhere equal to k. The rest of the field is built by comparatively a small quantity of source stream.



If M be the mass of liquid flowing past a source of strength m, and q_r is the velocity at a distance r from it, then

$$M=4\pi r^2 q_r, \text{ as also } M=4\pi m,$$

so that
$$q_r=(M/4\pi r^2)=-\partial\phi/\partial r.$$

Thus the potential of the source is

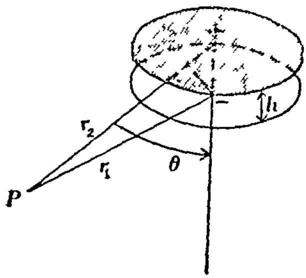
$$\phi=\int_o^r \frac{M}{4\pi r^2}\,dr=\text{const.}-\frac{M}{4\pi r}.$$

Similarly the potential of the sink is

$$\phi=\text{const.}+(M/4\pi r).$$

The potential at a point P due to the element dS of the doublet sheet is (Fig.)

$$d\phi=-\frac{mdS}{4\pi r_2}+\frac{mdS}{4\pi r_1}=\frac{mdS}{4\pi}\left(\frac{1}{r_1}-\frac{1}{r_2}\right) \quad (1)$$



where m is the source intensity per unit surface and r_2, r_1 the distance from P to the source and sink elements respectively. Clearly $r_2=r_1+h\cos\theta$ (Fig.).

Since h is very small,

$$\frac{1}{r_2}=\frac{1}{r_1}\left(1-\frac{h}{r_1}\cos\theta\right)$$

or

$$\frac{1}{r_1}-\frac{1}{r_2}=\frac{h}{r_1^2}\cos\theta \qquad (2)$$

From (1) and (2) we get

$$d\phi=(mhdS\cos\theta/4\pi r_1^2).$$

The velocity inside the double layer equals the source intensity per unit area, i.e. m ; *since almost all the flow is confined to the double layer itself*. Thus the potential increases by mh which must be equal to k if $h\to 0$. Hence

$$d\phi=(k/4\pi)(\cos\theta\, dS/r_1^2).$$

Integration then provides

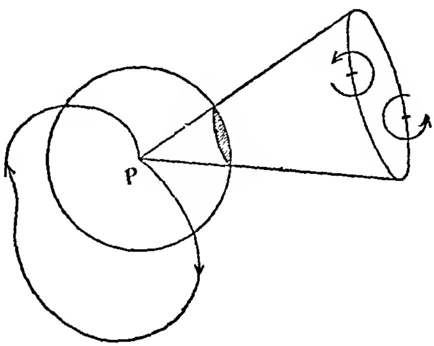
$$\phi=\frac{k}{4\pi}\iint\frac{\cos\theta\, dS}{r_1^2}=\frac{k}{4\pi}\iint d\omega$$

where $d\omega$ is the solid angle subtended by the surface dS at P. Thus finally

$$\phi=k\omega/4\pi \qquad (3)$$

where ω is the solid angle subtended at P (x, y, z) by the surface having the closed vortex filament for edge.

Cor. If P moves round a closed curve interlinking the vortex ring (Fig.) the solid angle increases by 4π, so that this *potential function is a cyclic quantity increasing by the cyclic constant k every time when the circuit is completed.*



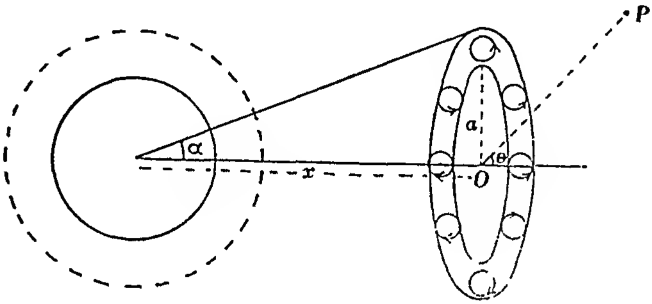
Exp. Prove that in regions remote from a single thin vortex ring the stream lines approximate to the curves

$$r = c \sin^2 \theta$$

where r denotes the distance of a point P from the centre O of the ring, and θ the angle which the line OP makes with the axis of the ring and c is a constant.

[*Ald 1958 ; Osm 59*]

Sol. The velocity potential ϕ due to a vortex ring is given by



$$\phi = k\omega/4\pi \qquad [\S 3.61(3) \text{ p. } 211] \qquad (1)$$

Also the solid angle of a right circular cone of semi-vertical angle α is

$$\omega = 2\pi\,(1-\cos\alpha) \qquad (2)$$

From (1) and (2) we get

$$\phi = \tfrac{1}{2}k\,(1-\cos\alpha) = \tfrac{1}{2}k[1 - x/\sqrt{(a^2+x^2)}]$$
$$= \tfrac{1}{2}k[1 - \{1+(a/x)^2\}^{-1/2}]$$
$$= \tfrac{1}{2}k\left[\frac{1}{2}\,\frac{a^2}{x^2} - \frac{1\cdot 3}{2\cdot 4}\,\frac{a^4}{x^4} + \cdots\right] \quad \text{if } \frac{a}{x} < 1.$$

Thus, to this degree of approximation, the value of ϕ at a distance r from the centre of the vortex ring is

$$\phi = \frac{k}{4}\,\frac{a^2}{r^2}\cos\theta \qquad (1)$$

The stream lines are given by

$$\frac{dr}{-(\partial\phi/\partial r)} = \frac{r\,d\theta}{-r^{-1}(\partial\phi/\partial\theta)}$$

whence we get, from (1) and (2)

$$(dr/r) = (2\cos\theta\,d\theta/\sin\theta).$$

Integrating we get

$$\log r = \log \sin^2\theta + \text{const. or } r = c\sin^2\theta$$

which is the required result.

Ex. 1. (*a*) Prove that for a single thin vortex ring of radius a, the stream function at a point near the ring and distant x from its plane is approximately equal to

$$-kx^2a^2/4(a^2+x^2)^{3/2},$$

where k is the circulation through the ring.

(*b*) Obtain the approximate formula

$$(k/4\pi b)[\log(8b/a) - \tfrac{1}{4}]$$

Also, $|d\zeta/dz| = nc\,|z^{n-1}| = ncr^{n-1}$

$\therefore \log|d\zeta/dz|_1 = \log(ncr_1^{\,n-1})$

Also $\chi(\xi_1, \eta_1) = -\frac{1}{2}K \log \eta_1$

$$= -\tfrac{1}{2}K \log(cr_1^{\,n} \sin n\theta_1)$$

because χ is obtained from

$$-(\partial\chi/\partial\eta_1) = (-\partial\psi'/\partial\eta)_{\zeta_1};$$

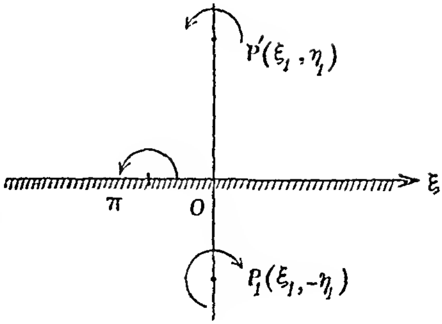
$$(\partial\chi/\partial\xi_1) = (\partial\psi'/\partial\xi_1)_{\zeta_1}$$

[ξ_1 means at (ξ_1, η_1)]

and the value of ψ' [i e. the current function due to K at ζ_1 alone] is

$$\psi' = -K \log|\zeta - \bar{\zeta}_1|$$

$$= -\tfrac{1}{2}K \log[(\xi-\xi_1)^2 + (\eta+\eta_1)^2].$$



The path is determined by the constant value of χ' given by

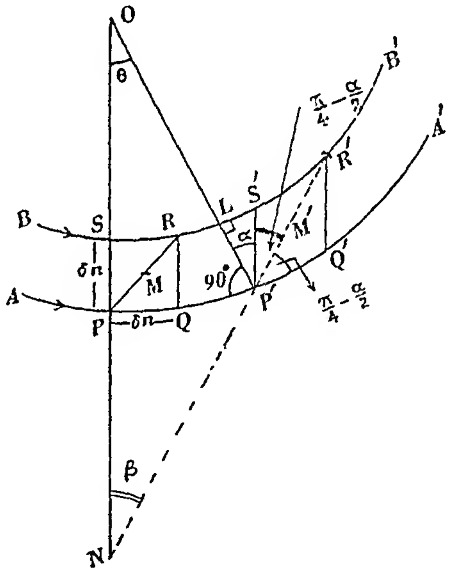
$$\chi'(x_1, y_1) = \chi(\xi_1, \eta_1) + \tfrac{1}{2}K \log|d\zeta/dz|_1.$$

This gives: $\quad -\frac{1}{2}K \log(cr_1^{\,n} \sin n\theta_1) + \frac{1}{2}K \log(ncr_1^{\,n-1}) = \text{const.}$

or $\quad r_1 \sin n\theta_1 = \text{const}$, the Cotes Spiral.

3.70. *Analysis of two-dimensional steady motion of a liquid drop.*

Let us consider two neighbouring stream lines AA' and BB': these being *coincident* with the paths of the fluid particles for the motion is steady.



Here P, P' (vide Fig) are the positions of any particle at time t and $t+\delta t$. Let the normals to AA' at these points meet in O, the centre of curvature. Let $PS = \delta n$, the element of the normal be regarded positive towards the centre of curvature. Along AA' and BB' we mark lengths PQ, SR each equal to δn and consider the square drop of fluid $PQRS$ at any time t, which at time $t+\delta t$, occupies the prism represented by the rhombus $P'Q'R'S'$; the distortion being the result of different velocities at P and S which are q and $q+(\partial q/\partial n)\delta n$

Let $\angle POP' = \theta$, $\angle R'NO = \beta$, $\angle S'P'O = \alpha$.

From the triangle NOP', $\theta + \beta = \frac{1}{4}\pi + \frac{1}{2}\alpha$.

The motion under investigation is necessarily restricted to an infinitesimal element, during an infinitesimal time δt, so that the quantities involved, i e. θ and α, are necessarily infinitesimal.

small,† it may be found by expanding the functions g in (161) in powers of η. The outer and inner solutions are joined at the points of inflexion, and values there are denoted by a subscript j.

The inner solution is found by approximating to the equation for u^2 in terms of ϕ and ψ. Since $z = u_1^2-u^2$,

$$\nu\frac{\partial^2}{\partial\psi^2}(u^2) = -\frac{2u_1^2 u_1'}{u}\left[1-\frac{1}{2u_1 u_1'}\frac{\partial}{\partial\phi}(u^2)\right], \tag{166}$$

where it is to be specially noted that the dash denotes differentiation with respect to ϕ. If the term $(2u_1 u_1')^{-1}\partial(u^2)/\partial\phi$ is replaced by a function of ϕ or u^2 alone, the equation (166) may be treated, for each value of ϕ, as an ordinary differential equation for u^2 in terms of ψ. At the wall $\partial(u^2)/\partial\phi = 0$; but if we simply neglect it in (166), the resulting solution gives values of $\partial^2(u^2)/\partial\psi^2$ which are of one sign for all values of ψ, and hence, since

$$\frac{\partial^2 u}{\partial y^2} = \tfrac{1}{2}u\frac{\partial^2}{\partial\psi^2}(u^2), \tag{167}$$

there are no points of inflexion. Since the inner solution is to be joined to the outer solution at a point of inflexion, it is essential that $\partial^2(u^2)/\partial\psi^2$ should vanish when $u = u_j$, where u_j is the value of u at the point of inflexion as determined from the outer solution. Moreover, the function by which we replace $(2u_1 u_1')^{-1}\partial(u^2)/\partial\phi$ in (166) should vanish at the wall. Now u/u_j satisfies both these requirements, and leads to a solution in elementary functions.‡ Hence (166) is replaced by

$$\nu\frac{\partial^2}{\partial\psi^2}(u^2) = -\frac{2u_1^2 u_1'}{u}\left(1-\frac{u}{u_j}\right), \tag{168}$$

or, with η as defined in (159),

$$\frac{d^2}{d\eta^2}\left(\frac{u^2}{u_1^2}\right) = -\frac{8\phi u_1'}{u}\left(1-\frac{u}{u_j}\right). \tag{169}$$

† Since
$$y = \int_0^\psi \frac{d\psi}{u} = 2(\nu\phi)^{\frac{1}{2}}\int_0^\eta \frac{d\eta}{u}$$
(see equation (20)), and $u = 0$ at $\eta = 0$, a small value of η corresponds to a comparatively large value of $y/\nu^{\frac{1}{2}}$.

‡ $(u/u_j)^2$ also satisfies the above requirements, and has a double zero at the wall, but leads to the appearance of elliptic integrals in the solution. The results in the cases where both procedures—i.e. the substitution of u/u_j and of $(u/u_j)^2$—have been tested were found to be very similar, and so the procedure which leads to the simpler integral is adopted.

This equation integrates to

$$\frac{d}{d\eta}\left(\frac{u^2}{u_1^2}\right) = \frac{4}{\mu_1}(-u_1' u_j \phi)^{\frac{1}{2}}\left[C^2+2\frac{u}{u_j}-\left(\frac{u}{u_j}\right)^2\right]^{\frac{1}{2}}, \tag{170}$$

where C^2 is a constant of integration whose value is

$$\begin{aligned} C^2 &= \left[\frac{d}{d\eta}(u^2)\right]^2_{\eta=0}\Big/(-16u_1^2 u_1' u_j \phi) \\ &= \nu\left[\frac{\partial}{\partial\psi}(u^2)\right]^2_{\psi=0}\Big/(-4u_1^2 u_1' u_j) \\ &= \nu\left(\frac{\partial u}{\partial y}\right)^2_{y=0}\Big/(-u_1^2 u_1' u_j). \end{aligned} \tag{171}$$

The equation (170) may be integrated again, with the condition $u = 0$ when $\eta = 0$, to give

$$\eta = \frac{u_j^{\frac{3}{2}}}{2u_1(-u_1'\phi)^{\frac{1}{2}}}\left\{C-\left[C^2+2\frac{u}{u_j}-\left(\frac{u}{u_j}\right)^2\right]^{\frac{1}{2}} +\sin^{-1}\frac{1}{(1+C^2)^{\frac{1}{2}}}-\sin^{-1}\frac{1-u/u_j}{(1+C^2)^{\frac{1}{2}}}\right\}. \tag{172}$$

but, as we shall see presently, this formula is not used in the final solution.

By putting $u = u_j$ in (170) we find

$$\left[\frac{d}{d\eta}(u^2)\right]_j = 4u_1(-u_1' u_j \phi)^{\frac{1}{2}}(C^2+1)^{\frac{1}{2}}. \tag{173}$$

By putting $u = u_j$ in (172) we could also find the value of η_j from this solution. Now $d(u^2)/d\eta = -\partial z/\partial\eta$, and hence the values of $d(u^2)/d\eta$ and η at the point of inflexion ($u = u_j$) are known from the outer solution. We have already arranged that $d^2(u^2)/d\eta^2$ for the inner solution shall vanish at $u = u_j$; we now determine C so that $[d(u^2)/d\eta]_j$ for the inner solution, as given by (173), has the value determined from the outer solution. But then we cannot ensure that the value of η_j for the inner solution is equal to its value as determined from the outer solution. In fact (169) is a second-order differential equation: the conditions $u = 0$ at $\eta = 0$ and

$$d(u^2)/d\eta = [d(u^2)/d\eta]_j$$

at $u = u_j$ determine the solution completely. Thus of the four conditions we should like to impose at the join,

$$\eta = \eta_j, \qquad u = u_j, \qquad \frac{d}{d\eta}(u^2) = \left[\frac{d}{d\eta}(u^2)\right]_j, \qquad \frac{d^2}{d\eta^2}(u^2) = 0,$$

we can impose only the last three. Hence in the resulting graph of u^2 against η there is a discontinuity in η at $u = u_j$. When we come to find u in terms of y this difficulty is overcome, as we shall see, by a procedure which is effectively the same as displacing the outer solution parallel to the η-axis by an amount equal to the discontinuity.

To express u in terms of y for the inner solution ($u \leqslant u_j$), we have, from equation (20),

$$y = \int\limits_0^{\psi} \frac{d\psi}{u} = 2(\nu\phi)^{\frac{1}{2}} \int\limits_0^{\eta} \frac{d\eta}{u} \tag{174}$$

$$= \frac{1}{u_1}\left(\frac{\nu}{-u_1' u_j}\right)^{\frac{1}{2}} \int\limits_0^{u} \left[C^2+2\frac{u}{u_j}-\left(\frac{u}{u_j}\right)^2\right]^{-\frac{1}{2}} du, \tag{175}$$

the last expression being obtained from equation (170). Hence, for $u \leqslant u_j$, we find by performing the integration in (175) and inverting,

$$u/u_j = 1-(1+C^2)^{\frac{1}{2}}\sin\{\sin^{-1}(1+C^2)^{-\frac{1}{2}}-u_1(-u_1'/\nu u_j)^{\frac{1}{2}}y\}, \tag{176}$$

u_j being determined from the outer solution and C by making $d(u^2)/d\eta$ continuous at the join as previously explained.

For the outer solution ($u \geqslant u_j$) we have

$$u/u_1 = (1-z/u_1^2)^{\frac{1}{2}}, \tag{177}$$

where z is given by (158) or (163), for example. This gives u in terms of η, and to connect u and y we express y in terms of η by the formula

$$y = \frac{1}{u_1}\left(\frac{\nu u_j}{-u_1'}\right)^{\frac{1}{2}}\sin^{-1}(1+C^2)^{-\frac{1}{2}}+2(\nu\phi)^{\frac{1}{2}}\int\limits_{\eta_j}^{\eta}\frac{d\eta}{(u_1^2-z)^{\frac{1}{2}}}. \tag{178}$$

The first term is the value of y at the join as determined from (176). In the lower limit of the integral in the second term the value of η_j is to be taken from the outer solution. This is effectively the same as taking the value from the inner solution and supposing the curve of z against η displaced parallel to the η-axis through a distance equal to the difference of the two values.

From the final formulae it is apparent that the value of η for the inner solution is not required, except to check that the discontinuity in η is small. In the cases in which it has been calculated it is found that the discontinuity is a very small fraction of the value of η

which corresponds to the boundary layer thickness δ, as determined by taking $y = \delta$ where $u/u_1 = 0{\cdot}995$.

The solution is now complete. Although the procedure adopted is complicated, the results are comparatively easy to apply since most of the formulae are given explicitly.

It remains to determine the position of the point of separation From (171) we see that $C = 0$ at separation: hence from (173)

$$\left[\frac{d}{d\eta}(u^2)\right]_j = 4u_1(-u_1' u_j \phi)^{\frac{1}{2}} \tag{179}$$

at separation, and this is equal to $-(\partial z/\partial\eta)_j$, which, together with u_j, is determined from the outer solution. Hence (179) is an equation to determine the value, ϕ_s, of ϕ at separation.

The method was applied by Karman and Millikan (*loc. cit.*) to solve the problem in which

$$u_1 = \beta_0 - \beta_1 x. \qquad u_1^2 = \beta_0^2 - 2\beta_1\phi. \tag{180}$$

This corresponds to flow along a flat plate against a linearly decreasing adverse pressure gradient, with a constant value of u for all values of y at the leading edge. Separation was found to occur at $\beta_1 x/\beta_0 = 0{\cdot}102$, whereas the value given by the use of the momentum equation with a quartic expression for the velocity (equation (145)) is $0{\cdot}156$. At separation, the boundary layer thickness, δ, defined as the value of y for which $u/u_1 = 0{\cdot}995$, is given by $\delta(\beta_1/\nu)^{\frac{1}{2}} = 2{\cdot}43$, and the Reynolds number R_δ $(= u_1\delta/\nu)$ by $(\beta_1\nu)^{\frac{1}{2}}R_\delta/\beta_0 = 2{\cdot}18$. The corresponding numbers for the solution from the momentum equation are $3{\cdot}46$ and $2{\cdot}92$ respectively.

A solution of the same problem by a method of expansion in series, which we shall consider more fully in the following section, gave $\beta_1 x/\beta_0 = 0{\cdot}120$ at separation, compared with the value $0{\cdot}102$ from the method of outer and inner solutions, and $0{\cdot}156$ from the momentum equation. Of these three methods, that of expansion in series (in spite of the slowness of the convergence) is almost certainly the most accurate. If the result given by it be taken as correct, then the method of outer and inner solutions gives an answer 15 per cent. in error, and the use of the momentum equation an answer 30 per cent. in error. Whether the percentage errors in other problems (reckoned as percentages of the distance between the pressure minimum and the point of separation, for example) will be of the same order of magnitude, remains to be decided.

Millikan† and von Doenhoff‡ have applied the method of outer and inner solutions to the pressure distribution measured by Schubauer at an elliptic cylinder (p. 162). Millikan approximates to u_1^2 by two different cubics in ϕ for two ranges of values of ϕ, while von Doenhoff uses two quadratics. With the distance from the forward stagnation point expressed as a multiple x of the minor axis, Millikan finds

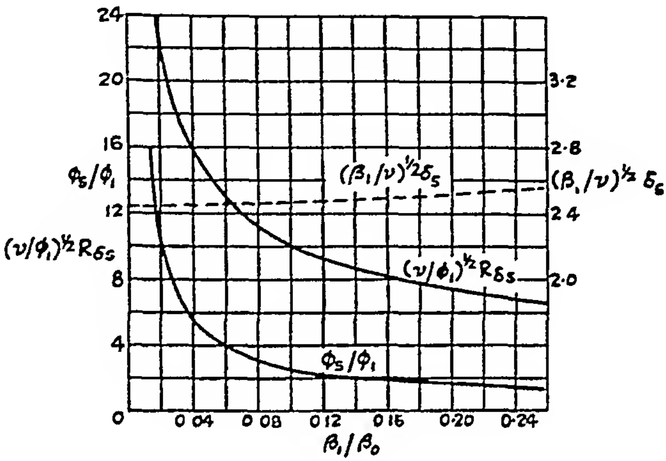


FIG. 46.

separation at $x = 1{\cdot}88$ and von Doenhoff at $x = 1{\cdot}92$, compared with an observed value 1·99 and no separation at all by Pohlhausen's method. The pressure minimum is at $x = 1{\cdot}30$.

Kármán and Millikan (*loc. cit.*) have also considered by their method the problem in which the graph of u_1 against x or u_1^2 against ϕ is composed of two straight lines:

$$\left.\begin{aligned}
u_1 &= \beta_0 x && (0 \leqslant x \leqslant x_1)\\
&= (\beta_0+\beta_1)x_1-\beta_1 x && (x \geqslant x_1),\\
\phi &= \tfrac{1}{2}\beta_0 x^2 && (0 \leqslant x \leqslant x_1)\\
&= -\tfrac{1}{2}(\beta_0+\beta_1)x_1^2+(\beta_0+\beta_1)xx_1-\tfrac{1}{2}\beta_1 x^2 && (x \geqslant x_1),\\
u_1^2 &= 2\beta_0\phi && (0 \leqslant \phi \leqslant \phi_1)\\
&= 2(\beta_0+\beta_1)\phi_1-2\beta_1\phi && (\phi \geqslant \phi_1),\\
\phi_1 &= \tfrac{1}{2}\beta_0 x_1^2.
\end{aligned}\right\} \quad (181)$$

This represents in certain circumstances a first, very rough, approximation to the velocity distribution outside the boundary layer at

† *Journ. Aero. Sciences*, 3 (1936), 91–94.
‡ *N.A.C.A. Technical Note* No. 544 (1935).

an aerofoil section. If the suffix s denotes values at separation, then ϕ_s/ϕ_1, $(\beta_1/\nu)^{\frac{1}{2}}\delta_s$ and $(\nu/\phi_1)^{\frac{1}{2}}R_{\delta s}$ are functions of β_1/β_0 only, where δ is the boundary layer thickness (defined as the value of y for which $u/u_1 = 0{\cdot}995$) and $R_\delta = u_1\,\delta/\nu$. The results obtained are reproduced in Fig. 46.

63. Approximate methods of calculating steady two-dimensional boundary layer flow. Application of the solution with a linear pressure gradient.

If
$$u_1 = \beta_0 - \beta_1 x, \tag{182}$$
with $u = u_1 = \beta_0$ at $x = 0$, then
$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = -\beta_1(\beta_0 - \beta_1 x), \tag{183}$$
and the equation of motion is
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\beta_1(\beta_0 - \beta_1 x) + \nu\frac{\partial^2 u}{\partial y^2}. \tag{184}$$
This equation may be solved in series. Write
$$\left.\begin{aligned} \xi = \beta_1 x/\beta_0, \qquad \eta = \tfrac{1}{2}(\beta_0/\nu x)^{\frac{1}{2}}y = \tfrac{1}{2}(\beta_1/\nu\xi)^{\frac{1}{2}}y, \\ \psi = (\nu\beta_0 x)^{\frac{1}{2}}f(\xi,\eta) = (\nu\beta_0 x)^{\frac{1}{2}}\{f_0(\eta) - 8\xi f_1(\eta) + (8\xi)^2 f_2(\eta) - \ldots\}, \end{aligned}\right\} \tag{185}$$
where ψ is the stream-function, so that
$$u = \frac{\partial\psi}{\partial y} = \tfrac{1}{2}\beta_0 f'_\eta(\xi,\eta) = \tfrac{1}{2}\beta_0\{f'_0(\eta) - 8\xi f'_1(\eta) + \ldots\}. \tag{186}$$

If we substitute in (184) and equate the coefficients of the various powers of ξ on the two sides of the equation, we obtain a series of ordinary differential equations for the f's. The boundary conditions are (since $u = v = 0$ at $y = 0$) $f_r(0) = f'_r(0) = 0$ and (since $u \to u_1$ as $\eta \to \infty$) $f'_0(\eta) \to 2$, $f'_1(\eta) \to \frac{1}{4}$, $f'_r(\eta) \to 0$ for $r \geqslant 2$ as $\eta \to \infty$. The function f_0 is the same as the function f which satisfies equation (45) in § 53. The equations are third-order equations, and all except the first are non-homogeneous linear equations. The first seven functions $(f_0, f_1, \ldots, f_6)$ have been tabulated by Howarth,† who also obtained rough values of f_7 and f_8. For values of ξ in the neighbourhood of $0{\cdot}1$ the first nine terms are no longer sufficient to give a sufficiently accurate representation; the value of ξ at separation is greater than $0{\cdot}1$ and cannot be determined from the series correct to three decimal places. Several approximate methods (specially devised for the particular problem) were applied to extend the solution to the position of separation, and checks were applied. The results

† *Proc. Roy. Soc. A*, 164 (1938), 547–564.

all agreed in giving separation at $\xi = 0{\cdot}120$. Now

$$\left.\begin{aligned} \frac{\nu^{\frac{1}{2}}(\partial u/\partial y)_{y=0}}{\beta_0\beta_1^{\frac{1}{2}}} &= \frac{1}{4\xi^{\frac{1}{2}}} f''_\eta(\xi, 0), \\ \frac{\beta_1^{\frac{1}{2}}\delta_1}{\nu^{\frac{1}{2}}} &= \xi^{\frac{1}{2}} \int\limits_0^\infty \left\{2 - \frac{f'_\eta(\eta,\xi)}{1-\xi}\right\} d\eta, \\ \frac{\beta_1^{\frac{1}{2}}\vartheta}{\nu^{\frac{1}{2}}} &= \xi^{\frac{1}{2}} \int\limits_0^\infty \left\{\frac{f'_\eta(\eta,\xi)}{1-\xi} - \frac{1}{2}\left[\frac{f'_\eta(\eta,\xi)}{1-\xi}\right]^2\right\} d\eta, \end{aligned}\right\} \tag{187}$$

where δ_1 and ϑ are the displacement and momentum thicknesses as defined in equations (36) and (37), while

$$u_1 = \beta_0(1-\xi), \qquad -u_1' = \beta_1. \tag{188}$$

Hence the quantities

$$\left.\begin{aligned} \frac{\nu^{\frac{1}{2}}(\partial u/\partial y)_{y=0}}{u_1(-u_1')^{\frac{1}{2}}} &= \frac{\nu^{\frac{1}{2}}(\partial u/\partial y)_{y=0}}{\beta_0\beta_1^{\frac{1}{2}}(1-\xi)}, \\ \frac{(-u_1')^{\frac{1}{2}}\delta_1}{\nu^{\frac{1}{2}}} &= \frac{\beta_1^{\frac{1}{2}}\delta_1}{\nu^{\frac{1}{2}}}, \\ \chi = \frac{(-u_1')^{\frac{1}{2}}\vartheta}{\nu^{\frac{1}{2}}} &= \frac{\beta_1^{\frac{1}{2}}\vartheta}{\nu^{\frac{1}{2}}}, \end{aligned}\right\} \tag{189}$$

are functions of ξ only. They are tabulated against ξ in Table 8, which also contains tables of $d\chi/d\xi$ and $\chi\Big/\dfrac{d\chi}{d\xi}$.

For small values of ξ, the first term in the expansion of χ is

$$\chi = \xi^{\frac{1}{2}} \int\limits_0^\infty f_0'(\eta)[1-\tfrac{1}{2}f_0'(\eta)]\, d\eta = 0{\cdot}664\xi^{\frac{1}{2}}. \tag{190}$$

TABLE 8

ξ	$\dfrac{\nu^{\frac{1}{2}}(\partial u/\partial y)_{y=0}}{u_1(-u_1')^{\frac{1}{2}}}$	$\dfrac{(-u_1')^{\frac{1}{2}}\delta_1}{\nu_1}$	χ	$\dfrac{d\chi}{d\xi}$	$\chi\Big/\dfrac{d\chi}{d\xi}$
0·0000	∞	0·000	0·000	∞	0·000
0·0125	2·773	0·199	0·076	3·17	0·024
0·0250	1·817	0·292	0·110	2·39	0·046
0·0375	1·360	0·371	0·137	2·08	0·066
0·0500	1·064	0·447	0·162	1·93	0·084
0·0625	0·843	0·523	0·186	1·85	0·100
0·0750	0·663	0·603	0·209	1·82	0·115
0·0875	0·503	0·691	0·231	1·81	0·128
0·1000	0·345	0·794	0·254	1·84	0·138
0·1125	0·184	0·931	0·276	1·88	0·147
0·120	0·000	1·110	0·290	1·92	0·151

Table 9 contains values of u/u_1 for several values of ξ and η.

TABLE 9

Values of u/u_1

η \ ξ	0·0125	0·025	0·0375	0·050	0·0625	0·075	0·0875	0·100	0·1125	0·120
0·0	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
0·2	0·125	0·117	0·108	0·099	0·089	0·078	0·066	0·052	0·034	0·010
0·4	0·251	0·237	0·222	0·205	0·188	0·168	0·146	0·120	0·085	0·038
0·6	0·377	0·358	0·338	0·317	0·293	0·267	0·237	0·202	0·152	0·085
0·8	0·498	0·477	0·455	0·430	0·403	0·372	0·337	0·294	0·234	0·149
1·0	0·611	0·590	0·567	0·541	0·513	0·480	0·442	0·394	0·325	0·227
1·2	0·711	0·692	0·670	0·645	0·617	0·585	0·546	0·498	0·426	0·318
1·4	0·796	0·779	0·760	0·738	0·712	0·682	0·646	0·598	0·527	0·416
1·6	0·864	0·850	0·834	0·815	0·794	0·769	0·736	0·692	0·625	0·517
1·8	0·914	0·904	0·891	0·877	0·860	0·839	0·812	0·776	0·716	0·616
2·0	0·949	0·942	0·934	0·923	0·910	0·894	0·872	0·844	0·794	0·708
2·2	0·972	0·967	0·962	0·954	0·946	0·934	0·918	0·897	0·858	0·787
2·4	0·985	0·983	0·979	0·975	0·969	0·961	0·951	0·936	0·908	0·853
2·6	0·993	0·991	0·990	0·987	0·984	0·979	0·972	0·962	0·943	0·903
2·8	0·997	0·996	0·995	0·994	0·992	0·989	0·985	0·978	0·967	0·940
3·0	0·998	0·998	0·998	0·997	0·996	0·995	0·992	0·989	0·982	0·965
3·2	0·999	0·999	0·999	0·999	0·999	0·998	0·997	0·994	0·991	0·981
3·4	1·000	1·000	1·000	1·000	1·000	0·999	0·999	0·998	0·995	0·990
3·6	..	..	..	..	..	1·000	1·000	0·999	0·998	0·995
3·8	..	..	..	..	..	..	..	1·000	0·999	0·998
4·0	..	..	..	..	..	..	..	..	1·000	0·999
4·2	..	..	..	..	..	..	..	..	..	1·000

The above solution has been made the basis of a general method for any distribution of u_1 in a retarded region (u_1' negative).‡ In the first place, the graph of u_1 against x may be replaced approximately by a polygon. The value of ϑ being known at the first vertex (which may be taken at the pressure minimum, the solution up to the pressure minimum being supposed found by expansion in series from the stagnation point, or from the momentum equation, or by any other suitable method), and u_1' being the slope of the first side of the polygon, χ is known at the first vertex. The value, ξ_0, of ξ corresponding to the first vertex is found from a graph or table of χ against ξ. The value of $-\beta_1$ for the first side of the polygon is the slope of the side, and β_0 is obtained by equating $\beta_0(1-\xi_0)$ to the value of u_1 at the first vertex. The above solution is then applied along the first side of the polygon, with

$$\xi = \xi_0 + \beta_1 x/\beta_0, \tag{191}$$

where x is measured from the first vertex of the polygon. At the second vertex we make ϑ continuous, and since there is a discon-

† Howarth, *op. cit.*, 565–578.

tinuity in u_1' there is a discontinuity in χ and therefore also in ξ. This discontinuity is found, and we proceed along the second side in the same way as along the first, and so on.

The discontinuity in ξ at each vertex of the polygon implies a discontinuity in the skin-friction; and since the skin-friction is one of the most important results of the calculation this is a grave objection. Conversely, if we made the skin-friction continuous at the vertices, we should introduce discontinuities in ϑ. There would be a violation of the momentum equation (38) (since $d\vartheta/dx$ would become infinite), corresponding to a series of impulses applied at the vertices of the polygon.

The discontinuities can be avoided by keeping ϑ continuous but taking the limit when the sides of the polygon tend to zero. In place of the relation (191) between ξ and x, together with a series of discontinuities in ξ, we then obtain a differential equation for ξ in terms of x. If two vertices are taken at a distance δx apart, then from (191) the variation in ξ corresponding to the side of the polygon joining them is $\beta_1 \delta x/\beta_0$, which, since $\beta_1 = -u_1'$ and $\beta_0(1-\xi) = u_1$, is equal to $-u_1'(1-\xi)\delta x/u_1$. To obtain the total variation in ξ corresponding to a variation δx in x we must add on the discontinuity at the second vertex. This is obtained by making ϑ or $\nu^{\frac{1}{2}}\chi/(-u_1')^{\frac{1}{2}}$ continuous, i.e.

$$\delta\left[\frac{\chi}{(-u_1')^{\frac{1}{2}}}\right] = 0 \quad \text{or} \quad \frac{\delta\chi}{\chi} = \tfrac{1}{2}\frac{u_1''}{u_1'}\delta x.$$

Since $\delta\chi = (d\chi/d\xi)\,\delta\xi$, this gives a variation

$$\delta\xi = \tfrac{1}{2}\frac{\chi}{d\chi/d\xi}\frac{u_1''}{u_1'}\delta x$$

in ξ. The total variation in ξ for a variation δx in x is therefore

$$\delta\xi = \left[\tfrac{1}{2}\frac{\chi}{d\chi/d\xi}\frac{u_1''}{u_1'} - \frac{u_1'(1-\xi)}{u_1}\right]\delta x, \tag{192}$$

and the differential equation required is

$$\frac{d\xi}{dx} = \tfrac{1}{2}\frac{\chi}{d\chi/d\xi}\frac{u_1''}{u_1'} - \frac{u_1'}{u_1}(1-\xi). \tag{193}$$

Since $\chi \div d\chi/d\xi$ is a known function of ξ, while u_1''/u_1', u_1'/u_1 are known functions of x, (193) gives ξ in terms of x if the initial value of ξ is known. This initial value of ξ is found as above from the initial value of χ. If, however, we start from the pressure minimum, where $u_1' = 0$, $\chi = 0$ and $\xi = 0$. Moreover, since $\chi = 0{\cdot}664\xi^{\frac{1}{2}}$ for

small ξ, the equation (193) has then a singular point at the origin, through which an infinite number of integral curves pass, so that it is necessary to determine also the initial value of $d\xi/dx$. This determination is effected by considering the first side of the polygon, with vertices δx apart, as having zero slope, while the slope of the second side is the value of u_1' at δx, i.e. $u_1''\delta x$. The value of χ at a distance δx from the pressure minimum is therefore $(-u_1''\delta x)^{\frac{1}{2}}\vartheta_0/\nu$, where ϑ_0 is the value of ϑ at the pressure minimum, the variation of ϑ along the first side of the polygon being ignored since it is $O(\delta x)$. Equating the value of χ thus found to $0{\cdot}664(\delta\xi)^{\frac{1}{2}}$, where $\delta\xi$ is the value of ξ at δx, we find for the initial value of $d\xi/dx$,

$$\left(\frac{d\xi}{dx}\right)_0 = -\frac{2{\cdot}269u_1''\vartheta_0^2}{\nu}. \tag{194}$$

The values of ξ corresponding to values of x may now be found from (193); the corresponding values of the skin-friction and the displacement thickness are found by graphical interpolation from Table 8, and the velocity distributions from Table 9.

In this method of procedure we must again suppose that the graphs of velocity against distance from the wall at various sections are members of a singly-infinite family of curves—namely, the velocity curves obtained from (186). At the separation point ξ will be $0{\cdot}120$, and for values of ξ between 0 and $0{\cdot}120$ the skin-friction for this singly-infinite system of curves takes all positive values.

We have seen in § 54 that if $u_1 = cx^m$ there is a solution for which $(\partial u/\partial y)_{y=0} = 0$ for all x if $m = -0{\cdot}0904$. The method described above was tested by using (193) to find the value of m for which the skin-friction vanishes everywhere—i.e. for which $\xi = 0{\cdot}120$ and $d\xi/dx = 0$. (193) then reduces to

$$\frac{0{\cdot}151}{2}(m-1)-0{\cdot}880m = 0,$$

whence $m = -0{\cdot}0938$. It may be noted that when the momentum equation with a quartic expression for the velocity is used to determine the corresponding value of m we require $\Lambda = -12$ in (145) and (146), whence it is found that $m = -0{\cdot}100$.

With values of u_1', u_1'' determined graphically from the experimental values of u_1 obtained by Schubauer at the surface of an elliptic cylinder (pp. 162, 172), and with the solution up to the pressure minimum found from the momentum equation with a quartic

expression for the velocity, Howarth found that, according to (193) and (194), separation occurs (i.e. $\xi = 0{\cdot}120$) at a distance from the forward stagnation point equal to 1·925 times the minor axis.

64. Approximate methods of calculating steady two-dimensional boundary layer flow. Expansion in powers of y; generalization of the solution with $u_1 = cx^m$; approximate solution in closed form for a nearly linear velocity distribution in an accelerated region; an iterative process.

Green† has attempted to find a solution by expanding the stream-function in a series of powers of y with coefficients which are functions of x. When the expressions

$$\left.\begin{aligned} \psi &= f_1\frac{y^2}{2!}+f_2\frac{y^3}{3!}+f_3\frac{y^4}{4!}+\ldots, \\ u &= \frac{\partial\psi}{\partial y} = f_1 y+f_2\frac{y^2}{2!}+f_3\frac{y^3}{3!}+\ldots, \\ v &= -\frac{\partial\psi}{\partial x} = -\left\{f_1'\frac{y^2}{2!}+f_2'\frac{y^3}{3!}+\ldots\right\} \end{aligned}\right\} \tag{195}$$

(where the f's are functions of x and dashes denote differentiation with respect to x) are substituted into the equation of steady motion, and the coefficients of the various powers of y on the two sides of the equation are equated, it is found that

$$\nu f_2 = -u_1 u_1', \qquad f_3 = 0, \qquad \nu f_4 = f_1 f_1', \qquad \nu f_5 = 2f_1 f_2', \ldots. \tag{196}$$

The function f_1, which is $(\partial u/\partial y)_{y=0}$, must be determined so as to make $u \to u_1$ at the outside of the boundary layer. Apart from inaccuracies which may arise in numerical work from the repeated differentiations required by (196), the main difficulty lies in the determination of f_1. Green applied this method to an experimental pressure distribution for flow past a circular cylinder, and developed a trial and error step-by-step method of determining f_1. (Basically, the method depends on making $u = u_1$ and $\partial u/\partial y = 0$ at $y = \delta$, and eliminating δ between the resulting equations.) The pressure distribution and the calculated skin-friction are shown in Chap. IX, Fig. 164.

Several of the methods described in previous sections give values of the skin-friction which are more reliable than the values of the velocity in the middle of the boundary layer. Expansion in powers of y may then be used to obtain improved values of the velocity.

A method has been suggested by Falkner and Skan‡ of generalizing the solution described in § 54 for the case $u_1 = cx^m$ (c and m constants) so as to derive approximate solutions for any distribution of u_1. If in the equation of steady motion we write

$$\eta = (u_1/\nu x)^{\frac{1}{2}} y, \qquad \psi = (u_1 \nu x)^{\frac{1}{2}} f(x, \eta), \tag{197}$$

† *Phil. Mag.* (7), 12 (1931), 2–30; *A.R.C. Reports and Memoranda*, No. 1313 (1930).

‡ *A.R.C. Reports and Memoranda*, No. 1314 (1930).

the equation becomes

$$M\left[\left(\frac{\partial f}{\partial \eta}\right)^2 - 1\right] - \tfrac{1}{2}(M+1)f\frac{\partial^2 f}{\partial \eta^2} - \frac{\partial^3 f}{\partial \eta^3} + x\left[\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \eta \partial x} - \frac{\partial^2 f}{\partial \eta^2}\frac{\partial f}{\partial x}\right] = 0, \tag{198}$$

where
$$M = u_1' x/u_1. \tag{199}$$

In the special case $u = cx^m$, $M = m$, f is a function of η only; the term in square brackets goes out and (198) reduces to an ordinary differential equation.

In the general case Falkner and Skan replace (198) by an ordinary differential equation whose coefficients are functions of x:

$$G_1(x)\left[\left(\frac{\partial f}{\partial \eta}\right)^2 - 1\right] - G_2(x)f\frac{\partial^2 f}{\partial \eta^2} - \frac{\partial^3 f}{\partial \eta^3} = 0, \tag{200}$$

and determine G_1 and G_2 so that (200) shall agree with (198) as closely as possible. The method employed by Falkner and Skan is to make (200) agree with (198) for small values of η. Now from (197)

$$u = \frac{\partial \psi}{\partial y} = u_1\frac{\partial f}{\partial \eta}, \qquad v = -\frac{\partial \psi}{\partial x} = -\tfrac{1}{2}(u_1 \nu/x)^{\frac{1}{2}}\left[(M+1)f + \eta(M-1)\frac{\partial f}{\partial \eta} + 2x\frac{\partial f}{\partial x}\right], \tag{201}$$

and since u and v must vanish at $\eta = 0$ for all values of x, $\partial f/\partial \eta$ and f must vanish at $\eta = 0$ for all values of x. If we put $\eta = 0$ in (198), we obtain simply

$$\left(\frac{\partial^3 f}{\partial \eta^3}\right)_{\eta=0} = -M$$

(cf. equation (16)), whilst with $\eta = 0$ equation (200) becomes

$$\left(\frac{\partial^3 f}{\partial \eta^3}\right)_{\eta=0} = -G_1(x).$$

In order that these should be identical, we must have

$$G_1(x) = M. \tag{202}$$

If we differentiate (198) and (200) with respect to η, and put $\eta = 0$, we obtain in both cases $\partial^4 f/\partial \eta^4 = 0$. If we differentiate twice and put $\eta = 0$, we obtain from (198)

$$\left(\frac{\partial^2 f}{\partial \eta^2}\right)^2_{\eta=0}[2M - \tfrac{1}{2}(M+1)] - \left(\frac{\partial^5 f}{\partial \eta^5}\right)_{\eta=0} + x\left(\frac{\partial^2 f}{\partial \eta^2}\frac{\partial^3 f}{\partial \eta^2 \partial x}\right)_{\eta=0} = 0,$$

and from (200) $$\left(\frac{\partial^2 f}{\partial \eta^2}\right)^2_{\eta=0}[2G_1(x) - G_2(x)] - \left(\frac{\partial^5 f}{\partial \eta^5}\right)_{\eta=0} = 0.$$

In order that these may be identical we require that

$$G_2(x) = \frac{M+1}{2} - \frac{x}{\alpha}\frac{d\alpha}{dx}, \tag{203}$$

where
$$\alpha = \left(\frac{\partial^2 f}{\partial \eta^2}\right)_{\eta=0} = \frac{1}{u_1}\left(\frac{\nu x}{u_1}\right)^{\frac{1}{2}}\left(\frac{\partial u}{\partial y}\right)_{y=0}. \tag{204}$$

Now (200) reduces to equation (69) with

$$[G_2(x)]^{\frac{1}{2}}\eta = Y, \qquad [G_2(x)]^{\frac{1}{2}}f = F, \qquad \beta = G_1/G_2, \tag{205}$$

so that, in particular, if α is taken from the solution of the approximate equation (200),

$$\frac{\alpha}{M^{\frac{1}{2}}} = \left(\frac{G_2}{M}\right)^{\frac{1}{2}}\left(\frac{d^2F}{dY^2}\right)_{Y=0} = \frac{1}{\beta^{\frac{1}{2}}}\left(\frac{d^2F}{dY^2}\right)_{Y=0}. \tag{206}$$

G_1 being equal to M. Since corresponding values of $(d^2F/dY^2)_{Y=0}$ and β are known, corresponding values of β and $\alpha/M^{\frac{1}{2}}$ are known. Hence $G_2\ (= M/\beta)$ is equal to M divided by a known function of $\alpha/M^{\frac{1}{2}}$. Since M is a known function of x, (203) is an ordinary differential equation for α. When this is solved the solution is complete. Some corresponding values of β and $(d^2F/dY^2)_{Y=0}$ are shown for reference below:

$\beta =$	$-0{\cdot}1988$	$-0{\cdot}19$	$-0{\cdot}18$	$-0{\cdot}16$	$-0{\cdot}14$	$-0{\cdot}10$	0	$0{\cdot}1$	$0{\cdot}2$
$\left(\dfrac{d^2F}{dY^2}\right)_{Y=0} =$	0	$0{\cdot}086$	$0{\cdot}128_5$	$0{\cdot}190_5$	$0{\cdot}239_5$	$0{\cdot}319$	$0{\cdot}4696$	$0{\cdot}5870$	$0{\cdot}686_9$
$\beta =$	$0{\cdot}3$	$0{\cdot}4$	$0{\cdot}5$	$0{\cdot}6$	$0{\cdot}8$	$1{\cdot}0$	$1{\cdot}2$	$1{\cdot}6$	$2{\cdot}0$	$2{\cdot}4$
$\left(\dfrac{d^2F}{dY^2}\right)_{Y=0} =$	$0{\cdot}774_8$	$0{\cdot}854_2$	$0{\cdot}927_7$	$0{\cdot}996$	$1{\cdot}120$	$1{\cdot}2326$	$1{\cdot}336$	$1{\cdot}521$	$1{\cdot}687$	$1{\cdot}837$

The somewhat different application of (203), involving further approximations, which was made by Falkner and Skan, has been criticized by Howarth,† who also points out that (203) will fail in the neighbourhood of the separation point. For since $\beta = -0{\cdot}1988$ at the separation point, G_2 remains finite there. Since α vanishes, (203) would make $d\alpha/dx$ vanish also at the separation point, and this is not correct.

Fairly satisfactory results are obtained in a region of accelerated flow, and the method may be used as an alternative to Dryden's modification‡ to bridge over a region in which $\Lambda > 12$ when such a region occurs in an application of the momentum equation with a quartic velocity distribution (§ 60, p. 161).

The results obtained by Falkner and Skan for the skin-friction over the forward part of a circular cylinder with the use of a measured pressure distribution are shown together with those of Green in Chap. IX, Fig. 164.

Thom,‖ remarking that round the front of a circular cylinder u/u_1 is almost independent of x, writes $u/u_1 = f$, and seeks a first approximation with f a function of y by neglecting the term $v\,\partial u/\partial y$ in the equation of motion. Actually f is thus found as a function of $(u_1'/\nu)^{\frac{1}{2}}y$, and so is a function of y alone only when u_1' is constant. The first approximation is then used to evaluate the neglected term $v\,\partial u/\partial y$ and the neglected part $u_1\,\partial f/\partial x$ of $\partial u/\partial x$, and a second approximation is found, which results in the equation

$$\left.\begin{aligned} y &= \left(\frac{3\nu}{2u_1'}\right)^{\frac{1}{2}} \int\limits_0^f \left\{f^3-3f+2-\frac{3}{2}\frac{u_1 u_1''}{u_1'^2}\int\limits_f^1 \frac{fF(f)}{F'(f)}\,df+3\int\limits_f^1 \phi(f)\,df\right\}^{-\frac{1}{2}} df, \\ &\text{where} \\ F(f) &= \log_e \frac{(\sqrt{3}-\sqrt{2})\sqrt{(1-f)}}{\sqrt{3}-\sqrt{(2+f)}}, \\ \phi(f) &= \frac{1}{F'(f)}\int\limits_0^f fF'(f)\,df. \end{aligned}\right\} \quad (207)$$

The values obtained by Thom in this way for the skin-friction round the front of a circular cylinder are shown in Chap. IX, Fig. 164, together with those

† *A.R.C. Reports and Memoranda*, No. 1632 (1935), pp. 37–44.
‡ See footnote ‡, p. 161.
‖ *A.R.C. Reports and Memoranda*, No. 1176 (1928).

of Green and of Falkner and Skan. Up to 45° from the forward stagnation point the solution is satisfactory; beyond that it departs widely from the values obtained by other writers and from the observed values. Up to 45° the velocity distribution outside the boundary layer is approximately linear, and we have seen in § 54 that for a linear velocity distribution u/u_1 is a function of $(u_1'/\nu)^{\frac{1}{2}}y$ only. It is, in fact, only in an accelerated region with the velocity distribution approximately linear that we should expect Thom's approximation to give satisfactory results, and then its only advantage over the solution in series is that the formulae can be expressed in terms of simple quadratures.

A very laborious iterative process has also been suggested by Thom (*loc. cit.*). He shows that, if A, B, C, D are the vertices of a small rectangle with AD and BC of length $2x$ and parallel to the wall, and AB and CD of length $2y$ and perpendicular to the wall, and if P is the centre of this rectangle, then, approximately,

$$\left.\begin{aligned} u_P = \tfrac{1}{4}(u_A+u_B+u_C+u_D)-Y_1 u_P(u_A+u_B-u_C-u_D)& \\ -Y_2 v_P(u_A+u_D-u_B-u_C)+Y_3,& \\ \text{where}\qquad Y_1=\frac{y^2}{8\nu x},\qquad Y_2=\frac{y}{8\nu},\qquad Y_3=-\frac{y^2}{2\nu\rho}\frac{\partial p}{\partial x}.& \end{aligned}\right\} \tag{208}$$

The boundary layer having been divided into a rectangular net and plausible values of u assumed at the corners, the values of u at the centres are calculated from (208), the values of v being calculated from the equation of continuity. The centres of the new rectangular net at the corners of which the values of u are now known are the corners of the original net and new values at these points are calculated from (208). This iterative process has to be repeated many times before the values are repeated sufficiently accurately.

65. Boundary layer growth. Motion started impulsively from rest.

When relative motion of a viscous incompressible fluid of constant density and of an immersed solid body is started impulsively from rest, the initial motion of the fluid is irrotational, without circulation. This is shown by observation, and may be proved theoretically in the same way as for inviscid fluids,† since it may be assumed that the viscous stresses remain finite. The fluid in contact with the solid body is, however, at rest relative to the boundary, whilst the adjacent layer of fluid is slipping past the boundary with a velocity determined from ideal fluid theory. There is thus initially a surface of slip, or vortex-sheet, in the fluid, coincident with the surface of the solid body. In other words there is a boundary layer of zero thickness. The vorticity in the sheet diffuses from the boundary into the fluid and is convected by the stream. The boundary layer

† Lamb's *Hydrodynamics* (1932), p. 11. It is assumed that any extraneous impulsive body forces acting on the fluid are conservative.

grows in thickness. (The same results follow from a consideration of the equations for the vorticity components in a viscous incompressible fluid, or of the equation for the circulation in a circuit moving with the fluid.†)

In any region along the boundary where the fluid is flowing against a pressure gradient, the forward stream will, after a time, leave the boundary if the pressure gradient extends far enough. Up to the time when separation begins, the velocity and pressure just outside the boundary layer may be taken to be the same as those at the surface in the irrotational motion without circulation, since this assumption provides a very close approximation to the facts. The pressure may also, as in boundary layer theory generally, be taken as constant across any section of the boundary layer.

Separation begins when the velocity gradient normal to the boundary vanishes at the boundary. For two-dimensional motion, the time, T, that elapses before separation begins, and the distribution of velocity in the boundary layer, may be approximately calculated. For an impulsive start, the second approximation to the velocity distribution, sufficient to give a first approximation to T, was calculated by Blasius.‡ The third approximation to the velocity distribution, and the second approximation to T, have been calculated by Goldstein and Rosenhead.||

After separation has once begun, the position of separation moves upstream. The movement could be followed theoretically on the assumption that the velocity and pressure outside the boundary layer continue to be the same as in the irrotational motion without circulation; but this assumption is no longer valid, and the results would have at best only a qualitative value,—and then only for flow past a symmetrical cylinder, since for an asymmetrical cylinder a circulation begins to grow as soon as separation starts. Even for a symmetrical cylinder, the thickening of the boundary layer beyond the position of separation—or, rather, its projection into the main body of the fluid—and the consequent formation of a wake deprive results obtained on the above assumption of any quantitative value.

We assume that at time $t = 0$ a cylinder starts to move in a straight line with velocity u_0, and that this velocity remains constant

† Chap. III, § 36. See also Jeffreys, *Proc. Camb. Phil. Soc.* 24 (1928), 477–479.
‡ *Zeitschr. f. Math. u. Phys.* 56 (1908), 20–37.
|| *Proc. Camb. Phil. Soc.* 32 (1936), 392–401.

thereafter. We take a frame of reference fixed relative to the cylinder. If x is distance along a section of the cylinder from the forward stagnation point and y distance normal to the surface of the cylinder, the approximate equation of motion in the boundary layer is

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=u_1\frac{du_1}{dx}+\nu\frac{\partial^2 u}{\partial y^2}, \tag{209}$$

where u_1 is the velocity just outside the boundary layer, as before. Initially the boundary layer has zero thickness, and at the beginning of the motion the diffusion far outweighs the convection and the influence of the pressure gradient,—i.e. the convection terms in the acceleration on the right can be neglected compared with $\partial u/\partial t$, and the term $u_1 du_1/dx$ neglected compared with $\nu\partial^2 u/\partial y^2$. The equation for the first approximation to u is

$$\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2}, \tag{210}$$

and the solution required is

$$\left.\begin{aligned} u &= u_1 \operatorname{erf}\eta, \\ \text{where}\qquad \eta &= \frac{y}{2(\nu t)^{\frac{1}{2}}}, \end{aligned}\right\} \tag{211}$$

and $\operatorname{erf}\eta$ is defined in equation (160). This solution makes $u=0$ when $\eta=0$, u practically equal to u_1 when η is large and theoretically equal to u_1 when $\eta=\infty$, and makes the thickness of the boundary layer zero when $t=0$.

The first approximation to v must satisfy the equation of continuity and must vanish when $\eta=0$. It is therefore given by

$$v=-2\sqrt{(\nu t)}u_1'[\eta\operatorname{erf}\eta-\pi^{-\frac{1}{2}}(1-e^{-\eta^2})], \tag{212}$$

where the dash denotes differentiation with respect to x.† When $\eta\to\infty$,

$$v\sim-2(\nu t)^{\frac{1}{2}}\eta u_1'=-yu_1',$$

and becomes infinite. The solution therefore fails theoretically for infinite values of η. But for moderate values of η, at which u is practically equal to u_1, v is of order $(\nu t)^{\frac{1}{2}}u_1'$.

To find a second approximation, denote by u', v' the terms that must be added to the first approximations given by (211) and (212). u'/u and v'/v are of order tu_1', and to find u' it is sufficient to solve the equation

$$\nu\frac{\partial^2 u'}{\partial y^2}-\frac{\partial u'}{\partial t}=-u_1u_1'+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y},$$

† In this section the dash is used to denote differentiation on u_1 only.

where on the right u and v have their values as given by (211) and (212) for the first approximation. Write

$$u' = tu_1 u_1' f(\eta). \tag{213}$$

Then

$$\frac{d^2f}{d\eta^2}+2\eta\frac{df}{d\eta}-4f = 4[\operatorname{erf}^2\eta-2\pi^{-\frac{1}{2}}\eta e^{-\eta^2}\operatorname{erf}\eta-1+2\pi^{-1}(e^{-\eta^2}-e^{-2\eta^2})]. \tag{214}$$

The solution of this equation is

$$\left.\begin{aligned} f = \tfrac{1}{2}(2\eta^2-1)\operatorname{erf}^2\eta+3\pi^{-\frac{1}{2}}\eta e^{-\eta^2}\operatorname{erf}\eta+1 \\ -\tfrac{4}{3}\pi^{-1}e^{-\eta^2}+2\pi^{-1}e^{-2\eta^2}+\alpha(2\eta^2+1) \\ +\beta[\tfrac{1}{2}\pi^{-\frac{1}{2}}(2\eta^2+1)\operatorname{erf}\eta+\eta e^{-\eta^2}], \end{aligned}\right\} \tag{215}$$

where α and β are constants to be chosen so that $u' = 0$ at $\eta = 0$ and at $\eta = \infty$. These conditions require

$$\left.\begin{aligned} \alpha &= -\left(1+\frac{2}{3\pi}\right) = -1{\cdot}21221, \\ \beta &= \frac{1}{\sqrt{\pi}}\left(1+\frac{4}{3\pi}\right) = 0{\cdot}80364. \end{aligned}\right\} \tag{216}$$

Then u is the sum of the expressions given by (211) and (213), i.e.

$$u = u_1 \operatorname{erf}\eta+tu_1 u_1' f(\eta). \tag{217}$$

The position of separation of forward flow from the wall is given by $\partial u/\partial y = 0$, i.e. $\partial u/\partial\eta = 0$, at $\eta = 0$. The time at which separation occurs at any particular place is hence found to be given by

$$1+\left(1+\frac{4}{3\pi}\right)u_1' t = 0. \tag{218}$$

Separation will occur first where u_1' has its greatest negative value The interval to separation is given by

$$T = 0{\cdot}70205/(-u_1')_{\max} \tag{219}$$

The rather complicated calculation of the third approximation to the velocity has been carried out. It is found that the next approximation to the time at which separation occurs at any particular place is given by

$$t^{-1} = -0{\cdot}7122u_1'+\sqrt{\{0{\cdot}7271u_1'^2+0{\cdot}05975u_1 u_1''\}}. \tag{220}$$

t has its least value where $-u_1'$ is greatest if and only if u_1 is zero there.

For a circular cylinder $-u_1'$ is greatest at the rear stagnation point,

and separation begins there both on the first and second approximations. If a is the radius of the cylinder, $u_1 = 2u_0 \sin x/a$, and the time that elapses from the commencement of the motion until separation first begins is given by $u_0 T_1 = 0{\cdot}35a$ for the first approximation and by $u_0 T_2 = 0{\cdot}32a$ for the second approximation. These expressions give the distance travelled by the cylinder from the commencement of the motion, and the second approximation is about 9 per cent. less than the first.

For a symmetrical cylinder of any section, it is to be remarked that whether $-u_1'$ attains its greatest value at the rear stagnation point or not depends on the shape of the section; and consequently separation may not begin at the rear stagnation point even according to the first approximation to T (equation (219)). This is especially the case for a bluff cylinder. Thus Tollmien† has pointed out that for an elliptic cylinder with its major axis across the stream separation begins at the rear stagnation point only if the ratio of the squares of the axes does not exceed $\frac{4}{3}$. As this ratio is further increased the positions of initial separation move symmetrically round towards the ends of the major axis, and the time interval to separation continually decreases.

As an example of a cylinder of asymmetrical section, the case of an ellipse with axes in the ratio 1:6, and with its major axis at an angle of 7° to the stream, has been considered. For the irrotational motion without circulation the rear stagnation point is at a distance of $0{\cdot}0221a$ from the end of the major axis, towards the upper side of the ellipse, where $2a$ is the length of the major axis. For the first approximation separation begins at a distance of $0{\cdot}0173a$ from the rear stagnation point towards the lower side of the ellipse, after a time given by $u_0 T_1 = 0{\cdot}0158a$. For the second approximation separation begins at $0{\cdot}0170a$ from the rear stagnation point after a time given by $u_0 T_2 = 0{\cdot}0144a$. The position of initial separation is not much altered. The interval is again reduced by about 9 per cent.‡ Since the position of initial separation is not much altered, the term in $u_1 u_1''$ in (220) makes very little difference, and the same percentage reduction would always be found.

The second approximation to the velocity (corresponding to

† *Handbuch der Experimentalphysik*, **4**, part 1 (Leipzig, 1931), 274, 275.

‡ Goldstein and Rosenhead, *loc. cit.* The first approximation had by Howarth.

has been found by Tollmien† for flow past a rotating cylinder, the whole system being started impulsively from rest.

The growth of the boundary layer at the surface of a body of revolution has been studied by Boltze‡ and the results have been applied to a sphere. By numerical computation the valuc of $(\partial u/\partial y)_{y=0}$ was found up to the term involving t^3, and separation was found to begin at the rear stagnation point after the sphere has travelled (relatively to the undisturbed fluid) a distance equal to 0·39 times its radius.

66. Boundary layer growth. Uniformly accelerated motion.

For uniformly accelerated motion starting from rest, u_0 is proportional to t, and the velocity u_1 outside the boundary layer, which before separation is again to be found from ideal fluid theory, will be of the form $tw_1(x)$. Since

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial x},$$

the equation of motion is

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = w_1+t^2w_1\frac{dw_1}{dx}+\nu\frac{\partial^2 u}{\partial y^2}. \tag{221}$$

As before the equation may be solved by successive approximation (or by a series in t), the equation for the first approximation being

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}+w_1.$$

The solution for which $u = 0$ at $y = 0$, $u/tw_1 \to 1$ when $y \to \infty$ or $t \to 0$ is

$$u = tw_1(x)\{-2\eta^2+2\pi^{-\frac{1}{2}}\eta e^{-\eta^2}+(2\eta^2+1)\operatorname{erf}\eta\}, \tag{222}$$

where $\eta = \frac{1}{2}y/(\nu t)^{\frac{1}{2}}$, as before. The second approximation, for which $t^3w_1w_1'$ multiplied by a certain function of η must be added to the value of u in (222), was found explicitly by Blasius,|| who also obtained by numerical computation the next term (involving t^5) in $(\partial u/\partial y)_{y=0}$ and gave as the equation for the time at which separation begins at any particular place

$$1+0{\cdot}427w_1't^2-\{0{\cdot}026w_1'^2+0{\cdot}010w_1w_1''\}t^4 = 0. \tag{223}$$

† *Göttingen Dissertation*, 1924; *Handbuch der Experimentalphysik*, 4, part 1 (Leipzig, 1931), 276, 277.

‡ *Göttingen Dissertation*, 1908.

|| *Loc. cit.* in the footnote on p. 182. Fig. 23 on p. 60 was drawn from the second approximation.

For a circular cylinder of radius a separation begins again at the rear stagnation point according to either the first or second approximation, the calculated time intervals before separation begins being such that the distance travelled before separation is $0{\cdot}585a$ for the first approximation, and $0{\cdot}52a$ for the second. The second approximation is about 11 per cent. less than the first.

For the elliptic cylinder previously considered (incidence 7°, ratio of axes 6:1, length of major axis $= 2a$), separation begins at $0{\cdot}0173a$ from the rear stagnation point when the distance travelled is $0{\cdot}0264a$ for the first approximation, and at $0{\cdot}0169a$ from the rear stagnation point when the distance travelled is $0{\cdot}0234a$ for the second approximation. The position of initial separation is not much altered, and the term in $w_1 w_1''$ in (223) makes very little difference. The second approximation to the distance travelled is again 11 per cent. less than the first.

67. Boundary layers for periodic motion.

The existence of a boundary layer at an oscillating solid surface arises from the fact that the vorticity which is produced at the surface and diffuses into the body of the fluid changes sign periodically. (In previous cases boundary layers are produced because the vorticity produced at a solid surface, in addition to diffusing into the body of the fluid, is convected with the main stream.) The thickness of the boundary layer at an oscillating surface is proportional to the square root of the product of the kinematic viscosity and the period of the motion. The same results apply for a fixed surface and an oscillating stream.

The simplest example is an infinite lamina oscillating in its own plane in a viscous fluid in the absence of external pressure gradients: a solution of this problem was given by Stokes.† Due to a prescribed motion $u = \alpha\cos(\sigma t+\epsilon)$ at the boundary, a velocity distribution is produced in the fluid such that

$$u = \alpha e^{-\beta y}\cos(\sigma t-\beta y+\epsilon), \tag{224}$$

where
$$\beta = (\sigma/2\nu)^{\frac{1}{2}}, \tag{225}$$

and the plane of the lamina is taken as the (z, x) plane, the fluid being

† *Trans. Camb. Phil. Soc.* **9** (1851), [20], [21] or *Math. and Phys. Papers*, **3**, 19, 20. See also Lamb, *Hydrodynamics* (Cambridge, 1932), pp. 619, 620, where a number of similar examples are also considered.

on the side of the plane for which y is positive. The amplitude of the resulting oscillation is diminished in the ratio e^{-C} when $y = C(2\nu/\sigma)^{\frac{1}{2}}$.

The influence of a rigid boundary on standing wave motion has been investigated by Rayleigh† without, and by Schlichting‡ with, the approximations of boundary layer theory. The amplitude being supposed small, the 'first-order' motion, in which squares of the amplitude are neglected, is easily investigated. The investigation of the 'second-order' motion, in which squares of the amplitude are retained, yields results of more interest. The second-order motion contains a non-periodic part, and, corresponding to a 'first-order' velocity $\alpha \cos kx \cos \sigma t$ near the boundary just outside the 'thin frictional layer' (i.e. the boundary layer), Rayleigh finds that the components of this non-periodic velocity are, at distances from the boundary sufficient for $e^{-\beta y}$ to have become insensible,||

$$\left.\begin{array}{ll} & (3k/8\sigma)\alpha^2 \sin 2kx\, e^{-2ky}(1-2ky) \\ \text{and} & -(2k^2/\beta\sigma)\alpha^2 \cos 2kx\, e^{-2ky}(-\tfrac{13}{16}+\tfrac{3}{8}\beta y), \end{array}\right\} \quad (226)$$

parallel and perpendicular to the wall, respectively. The steady motion thus represented consists of a series of vortices periodic with respect to x in half a wave-length of the original standing wave. The fluid moves from the boundary at the nodes ($kx = \frac{1}{2}\pi, \frac{3}{2}\pi,...$) and towards the boundary at the loops ($kx = 0, \pi, 2\pi,...$). The horizontal motion is directed from the loops to the nodes near the boundary, and changes sign when $y = (2k)^{-1}$.

To ascertain the character of the motion in the frictional layer, the terms in $e^{-\beta y}$ which were omitted in (226) must be retained. When this is done it appears that the velocity parallel to the surface changes sign, as we go out from the wall, for a value of βy somewhat greater than $\frac{3}{4}\pi$, after which it stays of one sign until $2ky = 1$. The greatest magnitude of the velocity inside the layer for $\beta y < \frac{3}{4}\pi$ is found to be about $\frac{1}{7}$ of the velocity just outside the layer.

Rayleigh also investigated the circumstances when the motion has its origin in the assumed motion of a flexible plate, situated when in equilibrium at $y = 0$, which is such that to the second order the boundary conditions are $u = 0$, $v = \alpha \sin kx \cos \sigma t$, say, at

$$y = (\alpha/\sigma)\sin kx \sin \sigma t.$$

† *Phil. Trans.* A, **175** (1883), 1–21; *Scientific Papers*, **2**, 239–257. Rayleigh notes the existence of a 'thin frictional layer'.

‡ *Physik. Zeitschr.* **33** (1932), 327–335.

|| $\beta = (\sigma/2\nu)^{\frac{1}{2}}$ as in (225).

The results are rather similar to those above; but the fluid moves from the boundary at the loops and towards it at the nodes, with the horizontal motion directed from the nodes to the loops near the plate.†

It will be noted that according to (226) the velocity parallel to the boundary for small values of y (i.e. just outside the boundary layer) is equal to $(3k/8\sigma)\alpha^2 \sin 2kx$. Hence the effect of the condition of zero slip at the boundary is such that the assumed potential wave motion, $u = \alpha \cos kx \cos \sigma t$, produces, even outside the boundary layer, a steady second-order flow, with a magnitude independent of the viscosity. The same result was found by Schlichting (*loc. cit.*), who applied the approximations of boundary layer theory, and, for a velocity $w_1(x)\cos \sigma t$ outside the boundary layer, found for this steady second-order velocity component a limiting value $-(3/4\sigma)w_1 w_1'$ at the edge of the boundary layer. Since $w_1 = \alpha \cos kx$ in Rayleigh's investigation, the results are in agreement.

Flow in a long straight tube of radius a under the influence of a periodic pressure gradient has been investigated theoretically and experimentally by Richardson and Tyler‡ and theoretically by Sexl.‖ If the tube is long enough, the velocity (u) along the tube is independent of the distance (x) along the tube, and the velocity at right angles to the axis is zero. If r is radial distance from the axis of the tube, the exact equation of motion is

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \tag{227}$$

where
$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \alpha \cos \sigma t. \tag{228}$$

The solution can be obtained exactly in terms of Bessel functions of order zero. When $a(\sigma/\nu)^{\frac{1}{2}}$ is small it assumes the parabolic form

$$u = (\alpha/4\nu)(a^2 - r^2)\cos \sigma t. \tag{229}$$

When $a(\sigma/\nu)^{\frac{1}{2}}$ is large the solution is

$$u = \frac{\alpha}{\sigma}\sin \sigma t - \frac{\alpha}{\sigma}\left(\frac{a}{r}\right)^{\frac{1}{2}} e^{-\beta(a-r)} \sin\{\sigma t - \beta(a-r)\}, \tag{230}$$

where β is $(\sigma/2\nu)^{\frac{1}{2}}$ as before. In the central portion of the tube where $\beta(a-r)$ is large only the first term is important. The first term

† The systems of vortices described above find application in the explanation of certain observed phenomena in acoustics. For references to these and related phenomena Rayleigh's paper may be consulted.

‡ *Proc. Phys. Soc.* **42** (1929), 1–15.

‖ *Zeitschr. f. Phys.* **61** (1930), 349–362.

represents an oscillation of the same period as the pressure gradient but with a phase difference of a quarter of a period.

When the approximations of the boundary layer theory are applied to this problem the term $\nu r^{-1}\,\partial u/\partial r$ in equation (227) is dropped and the solution (230) emerges quite simply.

In the experiments $\overline{u^2}$, the temporal mean value of the square of the velocity, was measured. $\overline{u^2}$ has its maximum value in the boundary layer near the wall and not in the central portion of the tube, for from (230)

$$\overline{u^2} = \frac{\alpha^2}{2\sigma^2}\{1-2(a/r)^{\frac{1}{2}}e^{-\beta(a-r)}\cos\beta(a-r)+(a/r)e^{-2\beta(a-r)}\}, \qquad (231)$$

and the maximum of this expression is at $\beta(a-r) = 2{\cdot}28$. This result is in good agreement with the experiments of Richardson and Tyler.

V
TURBULENCE

68. The mean flow.

In the mathematical treatment of turbulent flow it is assumed that the motion can be separated into a mean flow whose components are U, V, W, and a superposed turbulent flow whose components are u, v, w, the mean values of which are zero.† In most cases these means may be taken with regard to time at a fixed point, or with regard to one of the coordinates at a given instant of time. Some discussion of the methods of taking means was given by Reynolds,‡ but there has been little subsequent discussion of this question.

In all cases of steady mean flow the means are taken over a long period of time at a fixed point. In other cases the appropriate method for taking means will depend on the particular problem which is being solved. If, for instance, the problem of the turbulent flow near an infinite plate moving with variable velocity were to be discussed, the mean values would be taken over planes parallel to the plate.

Difficulty occurs when the mean flow is variable. It is then necessary to assume that the fluctuations in u, v, w are so rapid that a significant mean velocity can be taken in an interval which is so short that the change in U, V and W during that interval can be neglected.

In taking averages the following principles will be adopted. If A and B are dependent variables which are being averaged and S is any one of x, y, z, t, then $\overline{\partial A/\partial S} = \partial \bar{A}/\partial S$,|| and $\overline{\bar{A}B} = \bar{A}\bar{B}$, where the bar denotes a mean value.

† In this chapter (except in equation (1)) U, V, W are the components of the mean velocity, u, v, w of the turbulent velocity, and $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are the root-mean-square values of u, v, w. A bar over the top denotes a mean value.

‡ 'On the Dynamical Theory of Incompressible Viscous Fluids and the Determination of the Criterion', *Phil. Trans.* A, 186 (1895), 123–164. See also Lamb's *Hydrodynamics* (1932), p. 674 *et seq.*

|| For example, with time means

$$\overline{\frac{\partial A}{\partial t}} = \frac{1}{2\tau}\int_{t-\tau}^{t+\tau} \frac{\partial A}{\partial t}\,dt = \frac{1}{2\tau}\{A(t+\tau)-A(t-\tau)\}$$

$$= \frac{1}{2\tau}\frac{\partial}{\partial t}\int_{t-\tau}^{t+\tau} A\,dt = \frac{\partial}{\partial t}\left\{\frac{1}{2\tau}\int_{t-\tau}^{t+\tau} A\,dt\right\} = \frac{\partial \bar{A}}{\partial t}.$$

If the method of averaging does not involve the variable of differentiation, no difficulty arises.

For a proof with a different method of averaging, see Taylor, *Proc. London Math. Soc.* (2), 20 (1922), 202, 203.

69. The Reynolds stresses.

The equation of motion of an incompressible fluid may be written†

$$\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(p_{xx}-\rho uu)+\frac{\partial}{\partial y}(p_{xy}-\rho uv)+\frac{\partial}{\partial z}(p_{xz}-\rho uw) \tag{1}$$

and two similar equations. If $U+u$ be substituted for u, $V+v$ for v, and $W+w$ for w, and the mean value taken, (1) becomes

$$\rho\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}(\overline{p_{xx}}-\rho UU-\rho\overline{uu})+\frac{\partial}{\partial y}(\overline{p_{xy}}-\rho UV-\rho\overline{uv})$$
$$+\frac{\partial}{\partial z}(\overline{p_{xz}}-\rho UW-\rho\overline{uw}). \tag{2}$$

This equation has the same form as (1) if the stress

p_{xx} is replaced by $\overline{p_{xx}}-\rho\overline{uu}$,
p_{xy} ,, ,, $\overline{p_{xy}}-\rho\overline{uv}$,
p_{xz} ,, ,, $\overline{p_{xz}}-\rho\overline{uw}$.

Thus the equations of the mean flow are the same as the ordinary equations of motion provided that stress components $-\rho\overline{u^2}$, $-\rho\overline{v^2}$, $-\rho\overline{w^2}$, $-\rho\overline{vw}$, $-\rho\overline{wu}$, $-\rho\overline{uv}$ are added to the mean values of the stresses p_{xx}, p_{yy}, p_{zz}, p_{yz}, p_{zx}, p_{xy} which are due to viscous forces. These virtual stresses are called the Reynolds stresses, and are the mathematical representations of the transport of momentum across a surface due to the velocity fluctuations.‡

The equation of continuity, when averaged, becomes

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z} = 0.$$

70. Example. The Reynolds shearing stress for pressure flow between parallel planes.

A simple example in which the Reynolds stresses are known is that of pressure flow between parallel planes.

Let the axis of x be parallel to the direction of mean motion, and denote by v the component perpendicular to the parallel planes. The average state of affairs may be supposed independent both of z and of x, so that $\partial(\rho\overline{u^2})/\partial x = 0$, $\partial(\rho\overline{uw})/\partial z = 0$, etc. Hence (2) becomes

$$\frac{\partial}{\partial x}\overline{p_{xx}}+\frac{\partial}{\partial y}(\overline{p_{xy}}-\rho\overline{uv}) = 0, \tag{3}$$

† These are equivalent to equations (19) and (20) of Chap. III in virtue of the equation of continuity. In equation (1), u, v, w are taken temporarily as the components of total velocity.

‡ Reynolds, *loc. cit.*; Lamb's *Hydrodynamics*, *loc. cit.*

and since
$$p_{xy} = \mu\left(\frac{\partial U}{\partial y}+\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),$$
$$\overline{p_{xy}} = \mu\frac{\partial U}{\partial y}. \tag{4}$$
The second equation of motion is
$$\frac{\partial}{\partial x}(\overline{p_{xy}}-\rho\overline{uv})+\frac{\partial}{\partial y}(\overline{p_{yy}}-\rho\overline{v^2})+\frac{\partial}{\partial z}(\overline{p_{yz}}-\rho\overline{vw}) = 0,$$
and since in this case the first and last terms vanish, $(\overline{p_{yy}}-\rho\overline{v^2})$ is independent of y. Now
$$\overline{p_{yy}} = -\bar{p}-2\mu\,\partial\bar{v}/\partial y,$$
and $\bar{v} = 0$, so that
$$(\bar{p}+\rho\overline{v^2}) \tag{5}$$
is independent of y. Since $\partial(\rho\overline{v^2})/\partial x = 0$, (5) shows that $\partial\bar{p}/\partial x$ is independent of y.

It has been shown above that $\overline{p_{yy}} = -\bar{p}$. Similarly, $\overline{p_{zz}} = -\bar{p}$. Hence since
$$3\bar{p} = -\overline{p_{xx}}-\overline{p_{yy}}-\overline{p_{zz}},$$
$\overline{p_{xx}} = -\bar{p}$. The integral of (3) is therefore
$$\rho\overline{uv} = -y\frac{\partial\bar{p}}{\partial x}+\mu\frac{\partial U}{\partial y}+\text{constant}. \tag{6}$$

71. Reynolds's equations of motion in cylindrical polar coordinates.

Reynolds's equations, expressed in cylindrical polar coordinates r, ϕ, z, are, with the viscous terms neglected,†
$$\frac{\partial V_r}{\partial t}+V_r\frac{\partial V_r}{\partial r}+\frac{V_\phi}{r}\frac{\partial V_r}{\partial \phi}+V_z\frac{\partial V_r}{\partial z}-\frac{V_\phi^2}{r}$$
$$= -\frac{1}{\rho}\frac{\partial\bar{p}}{\partial r}-\frac{\partial}{\partial r}\overline{v_r^2}-\frac{1}{r}\frac{\partial}{\partial\phi}(\overline{v_r v_\phi})-\frac{\partial}{\partial z}(\overline{v_r v_z})-\frac{\overline{v_r^2}}{r}+\frac{\overline{v_\phi^2}}{r},$$
$$\frac{\partial V_\phi}{\partial t}+V_r\frac{\partial V_\phi}{\partial r}+\frac{V_\phi}{r}\frac{\partial V_\phi}{\partial \phi}+V_z\frac{\partial V_\phi}{\partial z}+\frac{V_r V_\phi}{r}$$
$$= -\frac{1}{\rho}\frac{1}{r}\frac{\partial\bar{p}}{\partial\phi}-\frac{\partial}{\partial r}(\overline{v_r v_\phi})-\frac{1}{r}\frac{\partial}{\partial\phi}(\overline{v_\phi^2})-\frac{\partial}{\partial z}(\overline{v_\phi v_z})-\frac{2\overline{v_r v_\phi}}{r},$$
$$\frac{\partial V_z}{\partial t}+V_r\frac{\partial V_z}{\partial r}+\frac{V_\phi}{r}\frac{\partial V_z}{\partial \phi}+V_z\frac{\partial V_z}{\partial z}$$
$$= -\frac{1}{\rho}\frac{\partial\bar{p}}{\partial z}-\frac{\partial}{\partial r}(\overline{v_r v_z})-\frac{1}{r}\frac{\partial}{\partial\phi}(\overline{v_\phi v_z})-\frac{\partial}{\partial z}(\overline{v_z^2})-\frac{\overline{v_r v_z}}{r}.$$

† The components of mean velocity in cylindrical polar coordinates are here denoted by V_r, V_ϕ, V_z, and the turbulent velocity components by v_r, v_ϕ, v_z.

72. Coefficients of correlation.

Three of the Reynolds stresses depend only on the magnitude of one component of velocity, but the three components of shear stress depend on the magnitudes of two component velocities and on the correlation between them. The coefficient of correlation between u and v is defined as

$$R_{uv} = \frac{\overline{uv}}{\sqrt{\overline{u^2}}\sqrt{\overline{v^2}}} = \frac{\overline{uv}}{uv}. \tag{7}$$

u, v, w will be used to denote $\sqrt{\overline{u^2}}$, $\sqrt{\overline{v^2}}$, $\sqrt{\overline{w^2}}$ from now on.

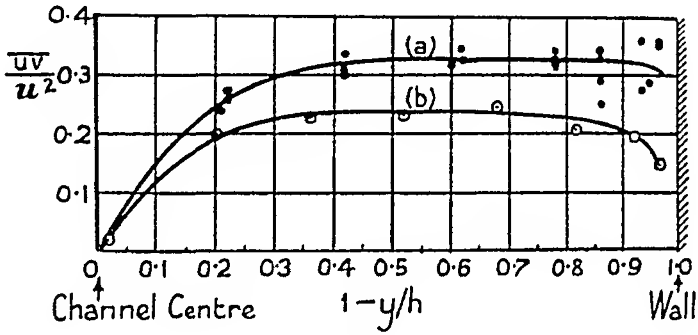


Fig. 47.

In the case of pressure flow between parallel plates $\rho\overline{uv}$ can be found from (6) by measuring the mean pressure gradient. To find R_{uv} it is necessary to measure u and v. Only provisional measurements of v in this case have been made,† but u has been measured by means of the hot wire technique.‡ Since in general u is nearly equal to v,|| $\overline{uv}/u^2$ is likely to be nearly equal to R_{uv}. Values of $\overline{uv}/u^2$ have been given by Kármán,†† who based his calculations (*a*) on the measurements of Wattendorf, (*b*) on those of Reichardt. These are shown in Fig. 47.

73. Reynolds's energy criterion.

Reynolds's experiments with flow in pipes showed that if the Reynolds number of the flow $U_m a/\nu$‡‡ is less than 1,000, the flow will become steady however large the disturbances at the entry may be.

† F. L. Wattendorf, *Journ. Aero. Sciences*, 3 (1936), 200–202.

‡ Chap. VI, §§ 117, 119.

|| See § 77, p. 200.

†† *Proc. Fourth Internat. Congress for Applied Mechanics, Cambridge*, 1934 (Cambridge, 1935), pp. 63, 64.

‡‡ U_m is the average velocity over a cross-section, and a the radius of the pipe.

Experiments with very carefully controlled conditions of entry have since shown that when the disturbances are very small the flow may remain steady when $U_m a/\nu$ is as high as 16,000.†

To account for the existence of a critical Reynolds number separating steady from turbulent conditions, Reynolds found the condition that the energy of the disturbed motion may increase. With any given form of small disturbance the criterion which distinguishes between an initial increase or an initial decrease in energy of the disturbed motion is a definite value for the Reynolds number of the motion. Thus for pressure flow with mean velocity U_m between parallel planes distant b apart Reynolds found that if $U_m b/\nu > 517$ the energy of the disturbed motion increases initially. This result was obtained by assuming a definite type of disturbance which satisfies the boundary conditions. Reynolds found that the calculated critical value of $U_m b/\nu$ depended on the form assumed for the disturbances. Orr‡ pursued the matter farther and found the form of disturbance which gives the value 117 for $U_m b/\nu$ below which all small disturbances initially decrease. It is certain therefore that for Reynolds numbers below this all possible small disturbances will continually decrease. Orr also calculated the criterion ($U_0 b/\nu < 177$) for initial decrease of disturbance when one plane moves with velocity U_0 relative to the other.

These minimum criteria are well below the observed lower criteria. It appears therefore that, when the Reynolds number of the motion is between Orr's number and the observed lower criterion, disturbances can be imposed which increase initially but subsequently die away.

Disturbances of pure laminar flow of uniform vorticity which increase very greatly initially and subsequently die away have been discussed by Orr.|| They are of the type

$$u = (A/a_0)\cos a_0 x \sin b_0 y, \qquad v = (A/b_0)\sin a_0 x \cos b_0 y,$$

where b_0 is large compared with a_0. If such a disturbance is superposed on the flow $U = d_0 y$, the vorticity of the disturbance, which is originally arranged as shown in Fig. 48 (a), is convected by the mean motion, and after time $t = b_0/a_0 d_0$ the areas of positive and negative vorticity are situated as in Fig. 48 (b). In the first position

† See Chap. VII, § 148.
‡ *Proc. Roy. Irish Acad.* 27 (1907), 69–138 (especially pp. 128, 134).
|| *Ibid.*, pp. 90–94.

(Fig. 48 (*a*)) the centres of positive and negative vorticity are close together in vertical lines, so that the velocities they produce are small. In the second position (Fig. 48 (*b*)) the centres of positive vorticity are close together on one set of vertical lines, while the centres of negative vorticity are on intermediate lines. This arrangement produces much greater velocities than that shown in Fig. 48 (*a*).

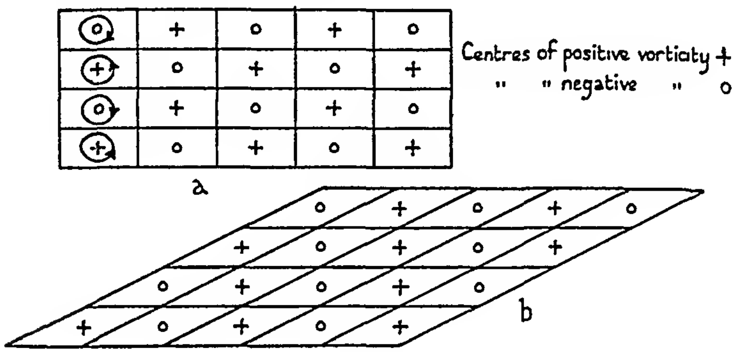


FIG. 48.

74. Stability for infinitesimal disturbances.

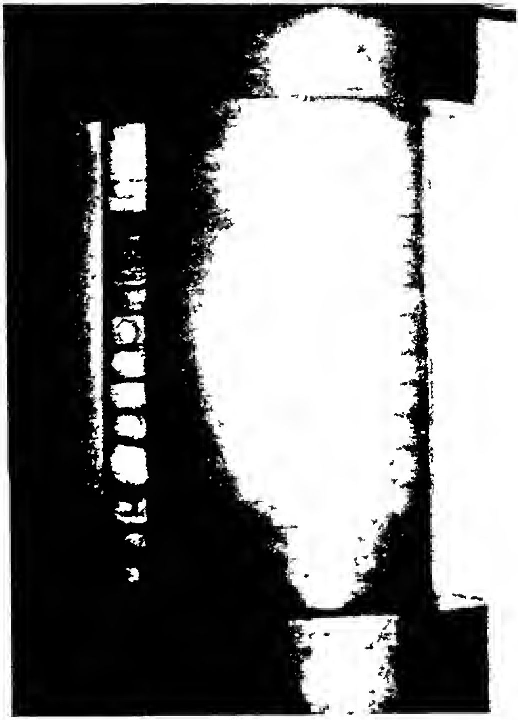
The importance of stability in connexion with turbulence arises because a motion which is definitely unstable for small disturbances cannot remain steady for speeds higher than that at which instability sets in. On the other hand, a motion which is definitely stable for small disturbances may become turbulent when finite disturbances are imposed on it. Perhaps the simplest case of steady motion is that of flow parallel to the axis of x between parallel planes. It seems now to be generally admitted that when there is no pressure gradient, the steady flow being due to relative motion of the two planes, the motion is stable; but there seems little doubt that in fact the flow would be turbulent when some definite Reynolds number is exceeded, provided a sufficiently large finite disturbance were applied.

75. The stability of flow between rotating cylinders.

The only case in which instability has been proved by calculation and verified experimentally is that of flow between rotating cylinders.† For given ratios of radii and of rotational speeds of the two cylinders a definite mode of disturbance appears when a calculable Reynolds number of the flow is just exceeded. This instability

† Taylor, *Phil. Trans.* A, 223 (1923), 289–343.

PLATE 22



consists of alternate ring-shaped vortices symmetrical about the axis of the cylinders and spaced a definite distance apart. By arranging that the inner cylinder is covered with a thin coat of coloured fluid, the annular space between the cylinders being filled with water, the vortices can be observed, the planes between them appearing as

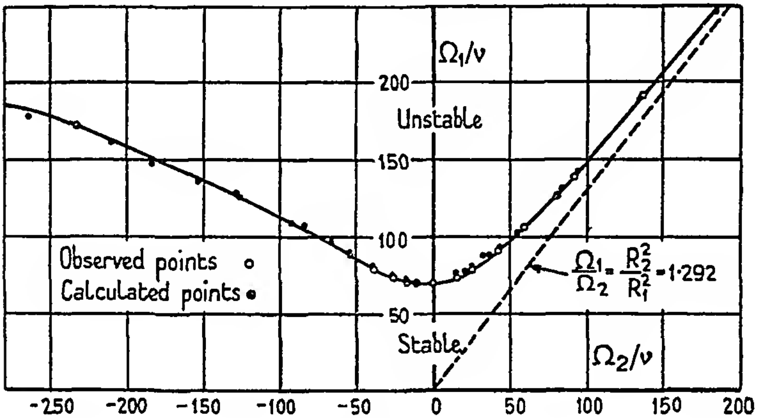


FIG. 49.

dark lines when viewed at right angles to the axis. A photograph of these lines is shown in Pl. 22, where regularity of the spacing may be seen. It appears that when the mean flow is such that only one mode of disturbance is just unstable, all others being stable, this mode immediately makes its appearance. The comparison between observed and calculated speeds at which instability sets in is shown in Fig. 49 for a particular pair of radii (3·55 and 4·035 cm.). The ordinates and abscissae are Ω_1/ν and Ω_2/ν, where Ω_1 and Ω_2 are the angular velocities of the inner and outer cylinders respectively.†

76. The stability of two-dimensional laminar flow.

The two-dimensional periodic disturbances of a field of flow in which U is a function of y only may be represented by a stream-function‡ $\psi = \phi(y)e^{i\alpha(x-ct)}$, and the differential equation for ϕ is

$$(U-c)(\phi''-\alpha^2\phi)-U''\phi = -\frac{i\nu}{\alpha}(\phi''''-2\alpha^2\phi''+\alpha^4\phi). \tag{8}$$

† This work has been extended by Dean to the case of pressure flow in the annular space between two cylinders, the pressure acting round the cylinder and therefore, strictly, many-valued. See *Proc. Roy. Soc.* A, **121** (1928), 402–420. See also Chap. VII, § 149.

‡ It is assumed that any initial disturbance may be analysed into periodic dis-

If all velocities are expressed as fractions of $U_{\max}$, the maximum velocity, and all lengths as fractions of some length b (e.g. the distance between two planes between which the flow is confined), (8) can be written

$$(U-c)(\phi''-\alpha^2\phi)-U''\phi = -\frac{i}{\alpha R}(\phi''''-2\alpha^2\phi''+\alpha^4\phi), \tag{9}$$

where $R = U_{\max}b/\nu$ and is defined as the Reynolds number of the flow, and the dashes refer to differentiation with respect to the new non-dimensional variable.

To explore the stability of flow between two planes it is necessary to write down the conditions that

$$\phi = \phi' = 0 \quad \text{at} \quad y = 0 \text{ and } y = b.$$

Since (9) has four independent solutions these four boundary conditions will lead to a period equation for determining a relationship between c and α. If the imaginary part of αc is positive, the disturbance is unstable.

The case when $U'' = 0$ (i.e. the flow is a uniform shearing) has been extensively explored over a large range of values of R. All oscillations appear to be stable, but it cannot be said that this has been definitely proved. For low values† of R the problem has been attacked by expansion in power series and for high values‡ by the use of asymptotic series.

A method for obtaining solutions of (9) for high values of αR has been developed by Heisenberg,|| Tietjens,†† and Tollmien.‡‡ These authors divide the four independent solutions into two classes,

turbances of this type, although this has never been rigorously proved except in the case of a simple shearing motion (Haupt, *Sitzungsber. d. k. bayr. Akad. d. Wiss., Math. Phys. Kl.* (1912), pp. 289–301). The reasons why the usual proof of the possibility of such an expansion fails have been set out by Southwell and Chitty, *Phil. Trans.* A, 229 (1930), 232–242.

On the assumption that any disturbance (possibly three-dimensional) may be analysed into constituents which are periodic in t, x, and z, it has been proved by Squire (*Proc. Roy. Soc.* A, 142 (1933), 621–628) that if instability arises for any Reynolds number, then it arises for the smallest Reynolds number when the motion is two-dimensional.

† Southwell and Chitty, *Phil. Trans.* A, 229 (1930), 205–253. (See also for the parabolic distribution *Proc. Camb. Phil. Soc.* 32 (1936), Goldstein, pp. 40–54, Pekeris, pp. 55–66.)

‡ Hopf, *Ann. d. Phys.* (4), 44 (1914), 1–60.

|| *Ann. d. Phys.* (4), 74 (1924), 577–627.

†† *Zeitschr. f. angew. Math. u. Mech.* 5 (1925), 200–217.

‡‡ *Göttinger Nachrichten, Math.-Phys. Klasse* (1929), pp. 21–44.

(a) solutions which are similar to those of an inviscid fluid, namely, solutions of

$$(U-c)(\phi''-\alpha^2\phi)-U''\phi = 0, \tag{10}$$

and (b) those which involve very rapid variations of ϕ and are similar to the solutions of

$$(U-c)\phi''+\frac{i\phi''''}{\alpha R} = 0. \tag{11}$$

At the point where $U = c$ the inviscid solution involves an infinite velocity and an infinite rate of shear, so that a solution which neglects viscosity (no matter how small) in the neighbourhood of this point is invalid. A finite viscosity prevents these infinite velocities from being attained. By superposing solutions of equations (10) and (11) it is possible to satisfy all the boundary conditions; but the full mathematical discussion is very complicated, the complications arising largely from the fact that any solution of (10) or (11) which is an asymptotic approximation to a solution of (9) for large values of αR, is not, in general, an approximation to the same solution of (9) on both sides of the point where $U = c$. Moreover, in the immediate neighbourhood of this point the method of approximation (approximating to solutions of (9) by solutions of (10) and (11)) breaks down.

Tollmien discussed the stability of an approximation to the Blasius distribution of velocity near a flat plate. He found that unstable waves can exist when $U_1\delta_1/\nu$ is greater than 420, where δ_1 is the displacement thickness of the boundary layer (defined by $\delta_1 = \int (1-U/U_1)\,dy$) and U_1 is the velocity outside the layer. At this calculated initial speed waves of length $17{\cdot}1\delta_1$ should become unstable, so that definite waves of this length might be expected to appear at the appropriate distance down the plate. It is a serious criticism of this result that no such definite waves have been observed even at speeds more than twice as great as the calculated critical speed.† All the available experimental work seems to show that the boundary layer of a flat plate becomes turbulent at a value of $U_1\delta_1/\nu$ which depends on the amount of turbulence in the main stream of air outside the layer.‡ There is therefore no experimental evidence that the Blasius régime is unstable.

† Tollmien (*Handbuch der Experimentalphysik*, 4, part 1 (Leipzig, 1931), 306) cites a photograph by Prandtl (*Zeitschr. f. angew. Math. u. Mech.* 1 (1921), 435) as evidence that such waves are produced, but the disturbances in the figure can hardly be said to look like definite waves.

‡ Cf. Dryden, *Proc. Fourth Internat. Congress for Applied Mechanics, Cambridge*, 1934 (Cambridge, 1935), p. 175; *N.A.C.A. Report* No. 562 (1936). See Chap. VII, § 151.

Apart from criticisms that may be made about the validity of Tollmien's method, it may be pointed out that it assumes that the velocity in the undisturbed motion is a function of y only. Except when the velocity distribution is parabolic (as it is for motion under the action of a uniform pressure gradient) or when the motion is a uniform shearing, it is necessary to apply body forces to the fluid in order to maintain the undisturbed motion. In Tollmien's case the disturbances so maintained are not necessarily the same as the disturbances in a free Blasius boundary layer which increases in thickness downstream.†

When the method is applied to shearing flow between parallel plates the difficulty just mentioned does not arise. The investigation of Hopf‡ seems to show that shearing flow is stable, and though Rayleigh|| cast some doubt on the validity of Hopf's work, Southwell and Chitty†† believe that 'it reveals with sufficient accuracy all the main features of the problem'.

77. Isotropic turbulence.

In all cases of turbulent motion there seems to be a strong tendency for the mean-square values of the three components of turbulent motion to become equal to one another. Observations made in a natural wind near the ground show that the transverse and vertical components are unequal near the ground but tend to equality at greater heights.‡‡ Ultramicroscopic and other observations of the turbulent components in a pipe show that they tend to become equal to one another near the centre of the pipe.||||

In a wind tunnel where turbulence is formed or controlled by a honeycomb, turbulence rapidly settles down to a condition for which the average-square values of the three components are equal to

† Tollmien has also discussed the stability of velocity distributions in which the curve of U against y has a point of inflexion, and has shown that in such cases the motion is unstable for infinitely large Reynolds numbers (*Göttinger Nachrichten, Math.-Phys. Klasse*, New Series, **1** (1935), 79–114). A critical Reynolds number has, however, not yet been calculated.

‡ *Ann. d. Phys.* (4), **44** (1914), 1–60.

|| *Phil. Mag.* (6), **28** (1914), 619; *Scientific Papers*, **6**, 275.

†† *Phil. Trans.* A, **229** (1930), 208.

‡‡ Taylor, *Quarterly Journ. of the Roy. Meteorological Soc.* **53** (1927), 201–211.

|||| Fage and Townend, *Proc. Roy. Soc.* A, **135** (1932), 656–677; Townend, *ibid.* **145** (1934), 180–211; Fage, *Phil. Mag.* (7), **21** (1936), 80–105; Chap. VIII, § 172.

one another.† It seems certain that the turbulence is then isotropic in the sense that the mean-square value of any component of turbulence is independent of the direction in which the component is taken.

A statistically isotropic condition of turbulence might be expected to arise when the time that has elapsed since the turbulence was formed is so great that there is no correlation between the motion of a particle and its initial motion. With this consideration in view, we might expect the turbulence behind a grid to become truly isotropic in the sense that the average value of any function of the turbulent velocity components or their space derivatives is unaltered if the axes of reference are rotated.‡

78. The effect of contraction on turbulence in a wind tunnel.

In a wind tunnel the air comes to the working section through a contracting entrance in which the mean speed is greatly increased. The longitudinal component of turbulence decreases through the contraction. The effect of the contraction on turbulence may be regarded as due partly to the extension of the fluid parallel to the axis of the tunnel, with corresponding contraction in perpendicular directions, and partly to the readjustment of the components of turbulent velocity which takes place when the normal isotropic condition is upset. Though both these causes are operating simultaneously in the contracting entrance to a wind tunnel, some insight into the effect of contraction may be obtained by considering only the effect of the first. An instantaneous or impulsive change in the dimensions of a volume of fluid containing turbulent motions, the principal axes of the strain being parallel to the coordinate axes, causes the components of vorticity ξ_0, η_0, ζ_0 to change to ξ_1, η_1, ζ_1, where

$$\xi_1 = l\xi_0, \qquad \eta_1 = m\eta_0, \qquad \zeta_1 = n\zeta_0,\| \tag{12}$$

l, m, n being the expansion or contraction ratios in the directions of the axes. The condition of continuity for an incompressible fluid is $lmn = 1$.

When ξ_0, η_0, ζ_0 are known, equations (12) give ξ_1, η_1, ζ_1, and the

† Taylor, 'Statistical Theory of Turbulence', Part 4, *Proc. Roy. Soc.* A, **151** (1935), 465–478. See also § 88 (p. 219) *infra*.

‡ For further discussion of this definition of isotropy, see § 91.

|| This is a direct application of Cauchy's equations for the vorticity. **Lamb's** *Hydrodynamics* (1932), pp. 204, 205.

corresponding velocities can be found.† An example in which the complete solution of the problem has been obtained is that of the motion represented by

$$\left.\begin{aligned} u_0 &= A_0 \cos ax_0 \sin by_0 \sin cz_0, \\ v_0 &= B_0 \sin ax_0 \cos by_0 \sin cz_0, \\ w_0 &= C_0 \sin ax_0 \sin by_0 \cos cz_0, \end{aligned}\right\} \tag{13}$$

with $A_0 a + B_0 b + C_0 c = 0$ to satisfy the equation of continuity. This motion becomes

$$\left.\begin{aligned} u_1 &= A_1 \cos l^{-1}ax_1 \sin m^{-1}by_1 \sin n^{-1}cz_1, \\ v_1 &= B_1 \sin l^{-1}ax_1 \cos m^{-1}by_1 \sin n^{-1}cz_1, \\ w_1 &= C_1 \sin l^{-1}ax_1 \sin m^{-1}by_1 \cos n^{-1}cz_1, \end{aligned}\right\} \tag{14}$$

where $A_1 = l\left(\dfrac{cm^2(A_0 c - C_0 a) - bn^2(B_0 a - A_0 b)}{a^2 l^{-2} + b^2 m^{-2} + c^2 n^{-2}}\right)$, etc.

In (13) and (14) (x_0, y_0, z_0) are the coordinates of a fluid particle before, and (x_1, y_1, z_1) its coordinates after the change in dimensions; and (u_0, v_0, w_0), (u_1, v_1, w_1) the corresponding turbulent velocity components.

When the contraction is large and symmetrical, so that

$$m = n = l^{-\frac{1}{2}},$$

$$\left.\begin{aligned} A_1 = A_0 l^{-1}\left(\frac{a^2+b^2+c^2}{b^2+c^2}\right), \qquad B_1 = l^{\frac{1}{2}}\left(\frac{c(B_0 c - C_0 b)}{b^2+c^2}\right), \\ C_1 = l^{\frac{1}{2}}\left(\frac{b(C_0 b - B_0 c)}{b^2+c^2}\right). \end{aligned}\right. \tag{15}$$

Thus the longitudinal component of turbulent velocity is inversely proportional to l while the lateral components increase in proportion to $l^{\frac{1}{2}}$.‡

The turbulence represented by (13) is not isotropic. If we suppose $a = b = c$ the initial turbulence is more nearly like isotropic turbulence than with any other choice of $a:b:c$. In this case (15) becomes

$$\frac{A_1}{A_0} = \frac{3}{2} l^{-1},$$

so that it is useful to compare the effect of contraction on the ratio

† By the use of a method due to Helmholtz (see Lamb's *Hydrodynamics* (1932), pp. 208–210). The investigation given here is due to Taylor, 'Turbulence in a Contracting Stream', *Zeitschr. f. angew. Math. u. Mech.* 15 (1935), 91–96.

‡ This idea was first put forward by Prandtl, *The Physics of Solids and Fluids* (London, 1930), p. 358.

of the longitudinal components of turbulence after and before contraction with $1{\cdot}5l^{-1}$. The comparison of observed and calculated components is given in Table 10.† ($u_{0_{max}}$ and $u_{1_{max}}$ denote observed maximum values.)

TABLE 10

l	$1{\cdot}5l^{-1}$	$(\overline{u_1^2}/\overline{u_0^2})^{\frac{1}{2}}$	$u_{1_{max}}/u_{0_{max}}$	*Authority*	*Method*
3·26	0·46	0·50	..	Simmons	Hot wiro
,,	,,	0·38	..	Townend	Heated spot
,,	,,	0·41			
,,	,,	..	0·38	Fage	Ultramicroscope
,,	,,	..	0·52		
13·2	0·114	..	0·12*	Simmons	Hot wire
,,	,,	..	0·15*	,,	,,
2·7	0·55	..	0·33	,,	,,
,,	,,	..	0·38	,,	,,

* Early measurements using uncompensated amplification.

79. Statistical theories of turbulence.

The object of a statistical theory of turbulence is to find methods of representing the turbulent field by considering the mean values and frequency distributions of quantities connected with the motion. Burgers‡ has attempted to apply to turbulence the statistical methods developed in connexion with the Kinetic Theory of Gases. For this purpose he considers a two-dimensional field of turbulence determined by a stream-function ψ. He then considers the values of ψ at a rectangular network of points with spacing ϵ. If ψ_A, ψ_B, ψ_C, ψ_D, ψ_O are the values at the corners A, B, C, D and the centre O of a square whose sides are 2ϵ, then, if ϵ is small,

$$u = \frac{\psi_B - \psi_A}{2\epsilon},$$

$$v = \frac{\psi_B - \psi_C}{2\epsilon},$$

$$\zeta' = \frac{1}{2\epsilon^2}(4\psi_O - \psi_A - \psi_B - \psi_C - \psi_D),$$

ζ' denoting the turbulent vorticity.

As in the Kinetic Theory of Gases any state of motion is represented by a point in N-dimensional space, where N is here the total

† Cf. Taylor, *loc. cit.* Tho comparison of observed and calculated lateral components is also given in the paper cited.

‡ *Proc. Roy. Acad. Sci. Amsterdam*, **32** (1929), 414–425, 643–657, 818–833; **36** (1933), 276–284, 390–399, 487–496.

number of points in the rectangular network. In order to apply this conception to the discussion of turbulence it is necessary to make some assumption in order to determine the frequency distribution of the representative point in the N-dimensional space, and it is here that the chief difficulty arises. Burgers attempts to use the dissipation function in this connexion in the same way that entropy is used in statistical mechanics. In so doing he leaves the equations of motion out of account. This theory seems promising, but it cannot be said that it has yet been developed far enough to be regarded as a definite theory of turbulence.

Another statistical representation of turbulent flow depends on the conception that the scale of turbulence can be described in terms of the correlation between the velocities u_A at a point A and u_B at another point B. If A and B are very close together, u_A and u_B are closely correlated: if they are far apart compared with the scale of the turbulence, this correlation may be expected to disappear. The coefficient of correlation R_y between u_A and u_B is

$$R_y = \frac{\overline{u_A u_B}}{u_A u_B}, \tag{16}$$

where y is the distance between the points A and B, and the axis of y is along AB.

If u represents the downstream component of turbulent velocity, $\overline{u_A u_B}$, u_A^2, u_B^2 can be measured by the hot wire technique. In such measurements it is convenient to fix one hot wire and to traverse the second wire perpendicular to the air-stream in the direction y. Correlations can be measured in other directions provided that one wire is not so nearly downstream of the other that the heat wake of the upstream wire falls on the downstream wire.

If the coordinates of B relative to A are x, y, z, the correlation coefficient between u_A and u_B may be represented by R_{xyz}; and the turbulence may be described statistically in terms of surfaces $R_{xyz} =$ const. The correlations between u_A and u_B at pairs of stations situated on the axes of reference will be denoted by R_x, R_y, R_z.

The relationship between R_y and y is shown by the points (and full-line curve) in Fig. 50 (p. 225) for turbulence produced in an air-stream by passage over a grid of square meshes 3 in. $\times$ 3 in. At a wind speed of 15 ft./sec., R_y tends to zero at $y = 2{\cdot}3$ inches. Measurements behind a similar screen of $M = 0{\cdot}9$ in. mesh show that, except

near $y = 0$, R_y seems to depend on y/M, i.e. the values of R_y at corresponding values of y/M in the two cases are the same. The scale of the turbulence produced by similar grids of different sizes may be expected to be proportional to the mesh of the grids.† This expectation is therefore satisfied if the scale of the R_y curve is taken as a measure of the scale of the turbulence. The scale of the R_y curve may conveniently be defined as

$$l_2 = \int_0^Y R_y\,dy, \tag{17}$$

where Y is the value of y above which R_y is sensibly zero.‡

80. Mixture length theories.

Up to the present time the centre of interest in turbulent motion has been its relationship to the mean flow. The statistical effect of turbulence on the mean flow has been regarded as similar to that of viscosity. Lumps of fluid are supposed to transfer the transferable properties from one layer to another just as molecular agitation transfers properties like heat and momentum in a non-turbulent fluid. In such theories a mixture length, l, plays a part analogous to the mean free path in molecular diffusion. The transfer of transferable properties is supposed to be effected by the motion of lumps of fluid which leave a layer in which their properties are those of the mean flow in the neighbourhood, and move in a direction transverse to the mean flow through a distance l. At this point they are supposed to mix with the surrounding fluid, so that their properties become identical with the average properties of the fluid in that region. The simplest case that can be discussed by this method is that of the transfer of a property θ in the direction of the axis y when the mean value of θ is constant over planes perpendicular to this direction. Suppose that a particle starts from a layer $y = h_1$ and that it carries with it the value $\theta(h_1)$, the mean value of θ at $y = h_1$. After moving to $y = h_2$, where the mean value of θ is $\theta(h_2)$, θ differs from the mean by an amount $\theta(h_1) - \theta(h_2)$. The mean rate of transfer of θ across a unit area perpendicular to y is

$$Q = \overline{v[\theta(h_1) - \theta(h_2)]}, \tag{18}$$

where the bar indicates that the mean value over $y = h_2$ is taken.

† Cf. § 94 (p. 227 *infra*).

‡ Taylor, 'Statistical Theory of Turbulence', *Proc. Roy. Soc.* A, **151** (1935), 421–454.

Expanding $\theta(h_1)-\theta(h_2)$ in a Taylor series we find that Q is the average value of

$$v\left\{-(h_2-h_1)\frac{d\theta}{dy}+\frac{1}{2}(h_2-h_1)^2\frac{d^2\theta}{dy^2}+...\right\},$$

so that $$Q = -\overline{v(h_2-h_1)}\frac{d\theta}{dy}+\frac{1}{2}\overline{v(h_2-h_1)^2}\frac{d^2\theta}{dy^2}+....$$

If the change in θ in the path h_2-h_1 is small, only the first term need be considered, so that

$$Q = -\overline{v(h_2-h_1)}\frac{d\theta}{dy}. \tag{19}$$

The meaning of the expression $\overline{v(h_2-h_1)}$ will be considered later in connexion with diffusion, but by analogy with the Kinetic Theory of Gases we may suppose that there is some mean distance l' between the beginning of a path and its final end by the process of mixture such that

$$\overline{v(h_2-h_1)} = l'\nu. \tag{20}$$

The length l' so defined may be called the mixture length.

The effect of turbulence on the transfer of a property is therefore represented according to mixture length theories by

$$Q = -l'\nu\frac{d\theta}{dy}. \tag{21}$$

81. The momentum transfer theory.

To account for the distribution of mean velocity in turbulent fields of flow the hypothesis that momentum is a transferable property in the sense of equation (18) has been put forward. If the mean velocity U is parallel to the axis of x, and U is a function of y only, then the assumption that momentum is a transferable property enables us to write ρU instead of θ in (21). The Q in (21) then represents the rate of transfer of momentum in the y direction, and is identical with the Reynolds stress $\rho\overline{uv}$. This will be represented by $-\tau$, so that (21) becomes

$$\tau = -\rho\overline{uv} = \rho l'\nu\frac{dU}{dy}. \tag{22}$$

It will be seen from (22) that $\rho l'\nu$ is virtually a coefficient of viscosity.

The rate at which momentum is communicated to unit volume by turbulence is therefore

$$M = \frac{d}{dy}\left(\rho l'v\frac{dU}{dy}\right). \tag{23}$$

The values of the coefficient of virtual viscosity $\rho l'v$ can be found by analysing cases where the distribution of mean velocity has been measured; but before it is possible to put forward a theory by the aid of which distributions of mean velocity can be predicted, it is necessary to find out how the virtual viscosity depends on the mean velocity and the boundary conditions. For this purpose Prandtl† has put forward the hypothesis that

$$\overline{uv} = -l^2\left(\frac{dU}{dy}\right)\left|\frac{dU}{dy}\right|.$$

The idea underlying Prandtl's hypothesis is that it has been observed that the mean values of the squares of the three components of turbulent velocity tend to be equal to one another.‡ If u and v were absolutely correlated and $\overline{u^2} = \overline{v^2}$, then $u = v$ and $|\overline{uv}| = \overline{uv} = \overline{u^2}$. Since the momentum is assumed to be transferable

$$u = (h_1-h_2)\frac{dU}{dy},$$

so that
$$\overline{u^2} = \overline{(h_1-h_2)^2}\left(\frac{dU}{dy}\right)^2,$$

and hence when u and v are absolutely correlated

$$\overline{uv} = -\overline{(h_1-h_2)^2}\left(\frac{dU}{dy}\right)\left|\frac{dU}{dy}\right|. \tag{24}$$

In fact u and v are not absolutely correlated, so that $|\overline{uv}|$ is less than $\overline{u^2}$ or $\overline{v^2}$; but the hypothesis that there is a length analogous to $\sqrt{\overline{(h_2-h_1)^2}}$, for which

$$\overline{uv} = -l^2\left(\frac{dU}{dy}\right)\left|\frac{dU}{dy}\right|, \tag{25}$$

opens up the possibility of a partial explanation of the effect of turbulence on the mean flow of fluids.

With this hypothesis equations (22) and (23) become

$$\tau = \rho l^2\left(\frac{dU}{dy}\right)\left|\frac{dU}{dy}\right| \tag{26}$$

† *Zeitschr. f. angew. Math. u. Mech.* 5 (1925), 137, 138; *Verhandlungen des 2. internationalen Kongresses für technische Mechanik, Zürich*, 1926, pp. 62–74.

‡ See § 77 (p. 200).

and
$$M = \frac{d}{dy}\left\{\rho l^2\left(\frac{dU}{dy}\right)\left|\frac{dU}{dy}\right|\right\}. \tag{27}$$

It will be noted that the mixture length l defined in this way is not identical with the mixture length l'. The former can be evaluated in cases where τ is known (e.g. flow through a pipe), but l' can be evaluated only when both τ and ν are known.

Mixture length theories cannot be subjected to complete experimental verification. Their usefulness must be judged either by comparing the values of l obtained from (26), using experimental values of τ and U, with what might be expected from *a priori* considerations, or by making further assumptions about l and comparing the distributions of mean velocity calculated from (26) with those observed experimentally.

A generalized version of the momentum transfer theory applicable to cases where neither the mean nor the turbulent motions are confined to two dimensions has been given by Prandtl, who suggests, particularly when one component of the mean rate-of-deformation tensor is much greater than the others, the substitution

$$-\rho\overline{u^2} = 2\rho l^2 J\frac{\partial U}{\partial x}, \qquad -\rho\overline{uv} = \rho l^2 J\left(\frac{\partial V}{\partial x}+\frac{\partial U}{\partial y}\right), \quad \text{etc.,}$$

where

$$J^2 = 2\left(\frac{\partial U}{\partial x}\right)^2+2\left(\frac{\partial V}{\partial y}\right)^2+2\left(\frac{\partial W}{\partial z}\right)^2 +\left(\frac{\partial W}{\partial y}+\frac{\partial V}{\partial z}\right)^2+\left(\frac{\partial U}{\partial z}+\frac{\partial W}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}+\frac{\partial U}{\partial y}\right)^2. \tag{28}$$

82. Hypotheses for predicting l.

The simplest hypothesis for predicting l is that of Prandtl—that near a plane wall $l = By$, where y is the distance from the wall and B is a constant.† In a jet or wake Prandtl assumes that at any section l is proportional to the breadth of the section.‡ Another hypothesis is that of Kármán—that l depends not directly on the distance from the wall but on the distribution of mean velocity. If l is to depend on the mean flow in the neighbourhood it must, in the case of two-dimensional mean flow parallel to the axis of x, depend on dU/dy, d^2U/dy^2, etc. The simplest length that can be derived from

† *Zeitschr. des Vereines deutscher Ingenieure*, 77 (1933), 105–113. See also Chap. VIII, § 153.

‡ *Verhandlungen des 2. internationalen Kongresses für technische Mechanik, Zürich*, 1926, pp. 62–74. See also Chap. XIII, § 252.

a measured distribution of U is $(dU/dy)/(d^2U/dy^2)$, and Kármán takes

$$l = K\frac{dU}{dy}\Big/\frac{d^2U}{dy^2}, \tag{29}$$

where K is a constant.† When there is no pressure gradient Prandtl's and Kármán's hypotheses come to the same thing close to a wall, for the value of τ is then independent of y. Prandtl's hypothesis gives

$$\tau = B^2\rho y^2\left(\frac{dU}{dy}\right)^2, \tag{30}$$

while Kármán's gives

$$\tau = K^2\rho\frac{(dU/dy)^4}{(d^2U/dy^2)^2}. \tag{31}$$

The solution of (30) is

$$U = \frac{\sqrt{(\tau/\rho)}}{B}\log y + \text{const.}$$

The solution of (31)‡ is

$$U = \frac{1}{K}\sqrt{\left(\frac{\tau}{\rho}\right)}\log y + \text{const.}$$

These are identical if $B = K$.

83. The vorticity transfer theory.§

The assumption that momentum is a transferable property necessarily involves the assumption that the fluctuating variations in pressure, which certainly exist in a turbulent field of flow, are ineffective so far as the mean transport of momentum is concerned. The only case in which this can be proved to be true is when the momentum in the direction x is transferred in the plane yz by turbulent motion in which lines of particles parallel to the x-axis remain parallel to this axis throughout the motion. On the other hand, if the turbulent motion is two-dimensional in the plane xy, the ζ-component of vorticity is conserved, so that ζ is a transferable property.

Taking the case when the mean velocity U is in the x direction and

† *Göttinger Nachrichten, Math.-Phys. Klasse* (1930), pp. 58–76. See also Chap. VIII, § 153.

‡ For large speeds and small viscosity, dU/dy takes very large values at the wall. In solving (31) dU/dy has been taken as infinite at the wall. The solution of (30) automatically makes dU/dy infinite at the wall.

§ Taylor, *Phil. Trans.* A, 215 (1915), 1–26; *Proc. Roy. Soc.* A, 135 (1932), 685 *et seq.*

is a function of y only, the full equation of motion, neglecting viscosity, may be written

$$-\frac{\partial}{\partial x}\left(\frac{p}{\rho}+\tfrac{1}{2}(U+u)^2+\tfrac{1}{2}v^2\right)=\frac{\partial}{\partial t}(U+u)-v\left(\zeta'-\frac{dU}{dy}\right),$$

where ζ' is the turbulent vorticity. Taking the mean value we get

$$\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}=\overline{v\zeta'}, \tag{32}$$

since the average value of u^2+v^2 will not alter with x. It appears, therefore, that in this case the effect of turbulence is to communicate momentum at rate $\rho\overline{v\zeta'}$ to unit volume per unit time. Hence in the notation of (23)

$$M=\rho\overline{v\zeta'}. \tag{33}$$

Since ζ is a transferable property,

$$\overline{v\zeta'}=-\overline{v(h_2-h_1)}\frac{d\bar{\zeta}}{dy}=-l'v\frac{d\bar{\zeta}}{dy}, \tag{34}$$

and since $\bar{\zeta}=-dU/dy$,

$$M=\rho l'v\frac{d^2U}{dy^2}. \tag{35}$$

Comparison of (35) with (23) shows that they are identical only when $l'v$ is independent of y.

Prandtl's hypothesis may be applied to the vorticity transfer theory by taking $l'v=l^2|dU/dy|$. The equation analogous to (27) is then

$$M=\rho l^2\left|\frac{dU}{dy}\right|\frac{d^2U}{dy^2}. \tag{36}$$

The expressions (27) and (36) may be used in comparing the results of the vorticity and momentum transfer theories in cases when M is known, as it is for instance in the turbulent flow through a pipe.†

84. The generalized vorticity transfer theory.‡

Averaging the equations of motion with the last expression in equation (20) of Chapter III for the acceleration, we see that, if viscosity is neglected, the equations of steady mean motion may be put in the form

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+W\frac{\partial U}{\partial z}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}-\frac{\partial}{\partial x}\overline{(\tfrac{1}{2}q^2)}+\overline{v\zeta'-w\eta'}. \tag{37}$$

† See Chap. VIII, §§ 156, 157.

‡ Taylor, *Proc. Roy. Soc.* A, 135 (1932), 697–700.

with two similar equations: or, in the vector notation of Chapter III,

$$\operatorname{grad}\tfrac{1}{2}V^2-V\times\omega = -\operatorname{grad}\left(\frac{\bar{p}}{\rho}+\overline{\tfrac{1}{2}q^2}\right)+\overline{v\times\omega'}. \tag{38}$$

In these equations (U, V, W) are the components of the mean velocity, denoted by V: $\bar{p}$ is the mean pressure; (u, v, w) are the turbulent velocity components, with a resultant of magnitude q denoted, when considered as a vector, by v; ω is the vorticity of the mean motion, with components (ξ, η, ζ), and ω' the vorticity of the superposed turbulent motion, with components (ξ', η', ζ').

(The equations in this form are exactly the same as Reynolds's equations, since the turbulent velocity satisfies the equation of continuity.)

When the motion is not confined to two dimensions the vorticity components are not conserved, and so the vorticity components are not transferable in the sense that heat is a transferable property. On the other hand, in a non-viscous fluid the components of vorticity at any point depend only on the vorticity of the same element of fluid at some initial time and on the nine components of strain and rotation which transform the element from its initial to its final state. This may be expressed in the Lagrangian system by Cauchy's equations†

$$\xi+\xi' = \xi_0\frac{\partial x}{\partial a}+\eta_0\frac{\partial x}{\partial b}+\zeta_0\frac{\partial x}{\partial c} \tag{39}$$

and two similar equations, where (a, b, c) are the initial positions of the element whose coordinates are (x, y, z), (ξ_0, η_0, ζ_0) are its initial, and $(\xi+\xi', \eta+\eta', \zeta+\zeta')$ its final components of vorticity (in accordance with the notation above).

With the assumptions previously made, ξ_0, η_0, ζ_0 are also the components of the *mean* vorticity at (a, b, c). Thus if the mean motion is steady, and if only the first-order terms in a Taylor series are retained,

$$\xi_0 = \xi-(x-a)\frac{\partial\xi}{\partial x}-(y-b)\frac{\partial\xi}{\partial y}-(z-c)\frac{\partial\xi}{\partial z}, \tag{40}$$

with two similar expressions for η_0 and ζ_0. The substitution of these expressions in (39) provides formulae for (ξ', η', ζ'), and hence for $\overline{v\zeta'-w\eta'}$, etc. These formulae will be found in the paper by G. I. Taylor cited above.

If, with $(x-a)$, $(y-b)$, $(z-c)$ denoted by L_1, L_2, L_3, we are content

† Lamb, *Hydrodynamics* (1932), pp. 204, 205.

to neglect not only the squares of the L's, as in (40), but all terms quadratic in the L's and their derivatives with respect to x, y, or z, then the formulae for ξ', η', ζ' may be considerably simplified. It is convenient to start, not from the final form of Cauchy's equations for the vorticity, but from certain equations that occur in their derivation (equations (2) of § 146 of Lamb's *Hydrodynamics* or p. 42 of Cauchy's memoir, *Théorie de la propagation des ondes*). In the present notation these equations are

$$\xi_0 = (\xi+\xi')\frac{\partial(y,z)}{\partial(b,c)}+(\eta+\eta')\frac{\partial(z,x)}{\partial(b,c)}+(\zeta+\zeta')\frac{\partial(x,y)}{\partial(b,c)} \tag{41}$$

and two similar equations. The equation of continuity in this, the Lagrangian, system is†

$$\frac{\partial(x,y,z)}{\partial(a,b,c)} = 1 \quad \left(\text{or} \quad \frac{\partial(a,b,c)}{\partial(x,y,z)} = 1\right),$$

and if, with the help of the equation of continuity, we express the derivatives of a, b, c with respect to x, y, z in terms of those of x, y, z with respect to a, b, c, we find

$$\frac{\partial a}{\partial x} = \frac{\partial(y,z)}{\partial(b,c)}, \qquad \frac{\partial a}{\partial y} = \frac{\partial(z,x)}{\partial(b,c)}, \qquad \frac{\partial a}{\partial z} = \frac{\partial(x,y)}{\partial(b,c)}, \quad \text{etc.}$$

But $$\frac{\partial a}{\partial x} = 1-\frac{\partial L_1}{\partial x}, \qquad \frac{\partial a}{\partial y} = -\frac{\partial L_1}{\partial y}, \qquad \frac{\partial a}{\partial z} = -\frac{\partial L_1}{\partial z},$$

and so, if second-order terms are neglected, the equation (41), together with (40), gives

$$\xi' = \xi\frac{\partial L_1}{\partial x}+\eta\frac{\partial L_1}{\partial y}+\zeta\frac{\partial L_1}{\partial z}-L_1\frac{\partial \xi}{\partial x}-L_2\frac{\partial \xi}{\partial y}-L_3\frac{\partial \xi}{\partial z}, \tag{42}$$

with two similar expressions for η' and ζ'. Since

$$\frac{\partial \xi}{\partial x}+\frac{\partial \eta}{\partial y}+\frac{\partial \zeta}{\partial z} = 0,$$

and in terms of the L's the equation of continuity reduces to

$$\frac{\partial L_1}{\partial x}+\frac{\partial L_2}{\partial y}+\frac{\partial L_3}{\partial z} = 0$$

if second-order terms are neglected, (42) is equivalent to

$$\xi' = \frac{\partial}{\partial y}(L_1\eta-L_2\xi)-\frac{\partial}{\partial z}(L_3\xi-L_1\zeta) \tag{43}$$

† Lamb, *op. cit.*, p. 14.

and two similar equations. In vector notation

$$\boldsymbol{\omega}' = \operatorname{curl}(\mathbf{L}\times\boldsymbol{\omega}), \tag{44}$$

where $\mathbf{L}$ has the components L_1, L_2, L_3.†

When the mean motion is confined to the direction of x, and U is a function of y only,

$$\zeta = -\frac{dU}{dy}, \qquad \xi = \eta = 0.$$

Then from (42)

$$\overline{v\zeta'-w\eta'} = \overline{L_2 v}\frac{d^2U}{dy^2}-\left(\overline{v\frac{\partial L_3}{\partial z}-w\frac{\partial L_2}{\partial z}}\right)\frac{dU}{dy}, \tag{45}$$

or from (43), on the assumption that $\overline{L_1 v'}$ does not vary with x or $\overline{L_2 w'}$ with z,

$$\overline{v\zeta'-w\eta'} = \frac{d}{dy}\left(\overline{L_2 v}\frac{dU}{dy}\right)-\left(\overline{L_1\frac{\partial v}{\partial x}-L_2\frac{\partial u}{\partial x}}\right)\frac{dU}{dy}. \tag{46}$$

If now the turbulent motion is two-dimensional in the (x, y) plane, the last term on the right in (45) goes out, and (37) becomes

$$0 = -\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x}+\overline{L_2 v}\frac{d^2U}{dy^2}, \tag{47}$$

which is identical with (32) and (34), since $\overline{L_2 v}$ here has the same meaning as $\overline{v(h_2-h_1)}$ in (34). On the other hand, if

$$\partial u/\partial x = \partial v/\partial x = \partial w/\partial x = 0,$$

so that lines of particles parallel to the axis of x move as a whole, the last term on the right in (46) goes out, so that (37) becomes

$$0 = -\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x}+\frac{d}{dy}\left(\overline{L_2 v}\frac{dU}{dy}\right), \tag{48}$$

which is the equation of the momentum transfer theory. It is easy to show that when $\partial u/\partial x = \partial v/\partial x = \partial w/\partial x = 0$ (37) always reduces to the momentum transfer equation for mean motion in one direction, whether U is a function of y only or not.

85. The modified vorticity transfer theory.‡

The intractability of equations (42) makes it desirable to introduce some further assumptions with a view to simplification. One such assumption is that the components of vorticity are transferable in the sense that heat is transferable. With this assumption

$$\xi+\xi' = \xi_0, \qquad \eta+\eta' = \eta_0, \qquad \zeta+\zeta' = \zeta_0.$$

† Goldstein, *Proc. Camb. Phil. Soc.* 31 (1935), 351–359.

‡ Taylor, *Proc. Roy. Soc.* A, 151 (1935), 494–497; 159 (1937), 499–502.

These equations are satisfied if

$$\frac{\partial x}{\partial a}=\frac{\partial y}{\partial b}=\frac{\partial z}{\partial c}=1$$

and
$$\frac{\partial x}{\partial b}=\frac{\partial x}{\partial c}=\frac{\partial y}{\partial c}=\frac{\partial y}{\partial a}=\frac{\partial z}{\partial a}=\frac{\partial z}{\partial b}=0.$$

Under these conditions

$$\overline{v\zeta'-w\eta'}=\overline{(x-a)w}\frac{\partial\eta}{\partial x}+\overline{(y-b)w}\frac{\partial\eta}{\partial y}+\overline{(z-c)w}\frac{\partial\eta}{\partial z}-\overline{(x-a)v}\frac{\partial\zeta}{\partial x}-\overline{(y-b)v}\frac{\partial\zeta}{\partial y}-\overline{(z-c)v}\frac{\partial\zeta}{\partial z}. \quad (49)$$

Equation (49), with the two equations formed by cyclic permutation of xyz, are the equations of the modified vorticity transfer theory.

If in addition we assume that the turbulent motion is statistically isotropic,

$$\overline{(x-a)w}=\overline{(y-b)w}=\overline{(x-a)v}=\overline{(z-c)v}=\overline{(y-b)u}=\overline{(z-c)u}=0$$

and
$$\overline{(x-a)u}=\overline{(y-b)v}=\overline{(z-c)w}=K \text{ (say)},$$

so that
$$\overline{v\zeta'-w\eta'}=K\left(\frac{\partial\eta}{\partial z}-\frac{\partial\zeta}{\partial y}\right)=K\nabla^2U \text{ simply.}\dagger \quad (50)$$

86. Diffusion in turbulent motion.

Diffusion by turbulent motion is related to the mixture length in somewhat the same way that molecular diffusion is related to the mean free path. The coefficient of diffusion in the direction y is $\overline{v(h_2-h_1)}$ or $\overline{l'v}$, where l' is the mixture length (equation (20), p. 206).

If a diffusable property starts from a concentrated plane source the concentration after time T is proportional to $T^{-\frac{1}{2}}e^{-Y^2/4KT}$, K being the coefficient of diffusion and Y the distance from the source. If it starts from a line source the concentration is proportional to $T^{-1}e^{-Y^2/4KT}$, and if it starts from a point source the concentration is proportional to $T^{-\frac{3}{2}}e^{-Y^2/4KT}$. According to the mixture length theory K may be taken as $\overline{l'v}$.

Measurements of the diffusing power of turbulence can be made in a wind tunnel by exploring the distribution of temperature downstream from a line or point source of heat. Taking the case

† When the mean velocity U is in the x-direction and is a function of y only, Prandtl's hypothesis is $K = l^2|dU/dy|$.

of a line source (e.g. an electrically heated wire) placed along the axis of z in a wind tunnel the centre line of which is the axis of x, the heat will diffuse, according to the mixture length theory, in the same way that it would under the influence of molecular conductivity in a non-turbulent stream, but the coefficient of conductivity will be much greater than in the molecular case. Except at points very close to the source the diffusion of heat in the wake behind a heated wire placed in a stream of velocity U_0 is nearly identical with the diffusion of heat from a concentrated plane source. Thus if the decay of turbulence down the wind tunnel is neglected, so that $l'v$ may be taken as constant, the temperature in the wake due to turbulent diffusion is

$$\theta = \frac{A}{\sqrt{x}}\exp\left(-\frac{Y^2}{4l'vT}\right);$$

and T, the time of diffusion, is x/U_0, so that

$$\theta = \frac{A}{\sqrt{x}}\exp\left(-\frac{Y^2U_0}{4xl'v}\right) = \frac{A}{\sqrt{x}}\exp\left(-\frac{Y^2}{2\overline{Y^2}}\right), \tag{51}$$

where $$\overline{Y^2} = 2l'vT = 2xl'v/U_0. \tag{52}$$

$\overline{Y^2}$ is the mean square of the distances of heated particles from the middle of the heat wake.

87. Discontinuous diffusion from a source in one dimension.

To simplify the calculation of diffusion we may suppose that a large number of particles start at time $T = 0$ from the origin $Y = 0$. We may suppose that they move with velocity v through a distance d and that another path also of length d then starts, the direction of motion being independent of the initial direction, so that, at the end of a path of length d, there is no correlation of the velocity with its value at the beginning of the path. After n flights, i e. after time $T = nd/v$, a fraction $(\frac{1}{2})^n$ of the total number of particles will be at distances $\pm nd$ from the origin. The proportions at distances $(n-2)d$, $(n-4)d$, etc., are the successive terms of the binomial expansion of $(\frac{1}{2}+\frac{1}{2})^n$. Thus if $n = 4$, $\frac{1}{16}$th will be at a distance $\pm 4d$, $\frac{1}{4}$ at $\pm 2d$, and $\frac{3}{8}$ at the origin. In general, the proportion at distance $(n-2s)d$ is $\dfrac{1}{2^n}\dfrac{n!}{(n-s)!\,s!}$.

In the limit when n is large the distribution tends to become Gaussian. To prove this Stirling's formula for large n,

$$n! \sim e^{-(n+1)}(n+1)^{n+\frac{1}{2}}(2\pi)^{\frac{1}{2}},$$

may be used, so that

$$\frac{1}{2^n}\frac{n!}{(n-s)!\,s!} \sim \frac{1}{2^n}\frac{(n+1)^{n+\frac{1}{2}}(2\pi)^{-\frac{1}{2}}}{(n-s+1)^{n-s+\frac{1}{2}}(s+1)^{s+\frac{1}{2}}}. \tag{53}$$

The maximum value of the right-hand side of (53) occurs when $s = \frac{1}{2}n$, i.e. near the origin. If we put $m = s-\frac{1}{2}n$ and take logarithms, we find

$$\log\left(\frac{1}{2^n}\frac{n!}{(n-s)!\,s!}\right) \sim f(n)-(\tfrac{1}{2}n-m+\tfrac{1}{2})\log(\tfrac{1}{2}n-m+1)$$
$$-(\tfrac{1}{2}n+m+\tfrac{1}{2})\log(\tfrac{1}{2}n+m+1). \tag{54}$$

If m is small compared with n,

$$\log(\tfrac{1}{2}n-m+1) = \log\tfrac{1}{2}n-\frac{2(m-1)}{n}-\tfrac{1}{2}\left(\frac{m-1}{\frac{1}{2}n}\right)^2-\dots. \tag{55}$$

The largest terms containing m in the right-hand side of (54) are

$$-m\left(\frac{m-1}{\frac{1}{2}n}\right)-m\left(\frac{m+1}{\frac{1}{2}n}\right)+\tfrac{1}{4}n\left(\frac{m-1}{\frac{1}{2}n}\right)^2+\tfrac{1}{4}n\left(\frac{m+1}{\frac{1}{2}n}\right)^2.$$

If m is large compared with unity, these reduce to $-2m^2/n$, so that

$$\log\left(\frac{1}{2^n}\frac{n!}{(n-s)!\,s!}\right) \sim f(n)-2\frac{m^2}{n}.$$

The frequency of particles at distance $Y = (n-2s)d = -2md$ from the origin is therefore proportional to $e^{-2m^2/n}$, or

$$\exp\left(-\frac{Y^2}{2dTv}\right) = \exp\left(-\frac{Y^2}{2\overline{Y^2}}\right), \tag{56}$$

where T $(= nd/v)$ is the total time of diffusion, and

$$\overline{Y^2} = dvT. \tag{57}$$

Comparison of (56) and (57) with the expressions (51) and (52) previously obtained in terms of the mixture length l' shows that they are identical if $d = 2l'$. Since $l'v$ is the *average value* of $v\times$(the distance moved by the particle since the beginning of its flight), it will be seen that l' might be expected *a priori* to be equal to $\frac{1}{2}d$.

The formula (57) could have been proved directly, since

$$\overline{Y^2} = \overline{(y_1+y_2+y_3+\dots+y_n)^2}, \tag{58}$$

where $y_1, y_2,\dots, y_n$ are each numerically equal to d but may be either positive or negative. There is no correlation between the directions of successive jumps, so that

$$\overline{y_1y_2} = \overline{y_2y_3} = \overline{y_1y_3} = \dots = 0.$$

Hence
$$\overline{Y^2} = nd^2 = dvT. \tag{59}$$

88. Diffusion by continuous movements.†

The diffusion theory outlined above depends on a physical conception very like that of the mixture length theory. The process is carried out in definite jumps of length d, and at the end of each jump the previous history of any particle is, as it were, completely wiped out. In the discontinuous diffusion theory this idea is introduced by assuming that there is no correlation between the direction of any flight and the directions of previous flights. In the mixture length theory the particle is supposed to mix with its surroundings and to lose its identity at the end of each flight.

It is difficult to form a concept of any definite physical process equivalent to mixture in this sense. The processes involved are not in fact discontinuous as is assumed in these theories. The velocities in turbulent motion are continuous and the motions of particles are continuous.

To discuss diffusion of particles in continuous movement it is necessary to find some method for defining statistically the velocity of a particle and its variation with time. For simplicity we may consider a field of turbulent flow which is statistically uniform,‡ so that the value of v^2 and the mean squares of all the derivatives of v with respect to time are the same at all points. We shall consider diffusion from a plane xz where all the diffusing particles are concentrated at time $t = 0$. If Y is the coordinate of a particle at time T, $Y = \int\limits_0^T v\,dt$, and

$$\frac{1}{2}\frac{d}{dT}\overline{Y^2} = \overline{Y\frac{dY}{dT}} = \overline{Yv_T} = \overline{v_T\int\limits_0^T v\,dt}. \tag{60}$$

Now in finding the average value of $v_T\int\limits_0^T v\,dt$ we may imagine the whole time from 0 to T divided into n intervals. In each of these intervals the summation over all particles is made. Thus in the sth interval the contribution to the average value of $v_T\int\limits_0^T v\,dt$ is $(\overline{v_T v_{Ts/n}})(T/n)$. Since the value of v^2 at time T is equal to that at Ts/n, $\overline{v_T v_{Ts/n}} = v^2R$, where R is the coefficient of correlation

† Taylor, *Proc. London Math Soc* (2), 20 (1922), 196–212.

‡ For a theoretical discussion of the case where the turbulence is decreasing, as it is down a wind tunnel, see Taylor, 'Statistical Theory of Turbulence', *Proc. Roy Soc.* A, **151** (1935), 429.

between v at time T and v at time Ts/n. It is clear that in continuous motion R becomes equal to 1 as Ts/n approaches T. If ξ is the time interval between T and Ts/n, we may use the symbol R_ξ to represent the coefficient of correlation between v at time T and v at time $T-\xi$. Then if $d\xi$ is the interval T/n, $(\overline{v_T\, v_{Ts/n}})(T/n) = \nu^2 R_\xi\, d\xi$. Hence (60) becomes

$$\frac{1}{2}\frac{d}{dT}\overline{Y^2} = \overline{Yv_T} = \nu^2 \int_0^T R_\xi\, d\xi \tag{61}$$

and

$$\tfrac{1}{2}\overline{Y^2} = \nu^2 \int_0^T\int_0^t R_\xi\, d\xi dt. \tag{62}$$

In general we may expect R_ξ to decrease with increasing ξ. Suppose that for all times greater than $\xi = T_1$, $R_\xi = 0$. Then $\int_0^T R_\xi\, d\xi = \int_0^{T_1} R_\xi\, d\xi$, so that when $T > T_1$,

$$\frac{1}{2}\frac{d}{dT}\overline{Y^2} = \text{constant} = \nu^2 \int_0^{T_1} R_\xi\, d\xi \tag{63}$$

and

$$\tfrac{1}{2}\overline{Y^2} = \nu^2 T \int_0^{T_1} R_\xi\, d\xi + \text{constant}. \tag{64}$$

Equations (64) and (52) may be identified except for the constant if

$$l' = \nu \int_0^{T_1} R_\xi\, d\xi. \tag{65}$$

The length $\nu \int_0^{T_1} R_\xi\, d\xi$ is therefore analogous, so far as diffusion is concerned, to a mixture length, but no assumption has been made about mixture in deriving it; indeed this theory of diffusion by continuous movements is equally valid if mixture never takes place.

When T is small $R_\xi = 1$, and in these circumstances the diffusion formula (62) gives

$$\overline{Y^2} = \nu^2 T^2, \quad \text{or} \quad \sqrt{\overline{Y^2}} = \nu T. \tag{66}$$

It appears therefore that, when T is small, $\sqrt{\overline{Y^2}}$ is proportional to T. This is clearly so, because over the time interval in which R_ξ is nearly equal to 1 the velocities of particles are nearly constant, so that for each particle $Y = vT$. In this case therefore not only is $\sqrt{\overline{Y^2}} = \nu T$, but the frequency distribution of Y is the same as the frequency distribution of v, which has been shown experimentally

to be the error distribution.† Measurements by Schubauer‡ and Simmons|| confirm this distribution at points near the source. Farther from the source the distribution seems to depend on the turbulence upstream of the grid.

The above-mentioned experiments were carried out by examining the distribution of temperature at various sections downstream from a heated wire placed across a wind tunnel down which a turbulent stream of mean velocity U_0 was blowing. If all particles leaving the heated source are supposed to have acquired the temperature of the source, the difference in temperature between any point in the heated region and that of the main stream is a measure of the frequency at which heated particles pass the point in question. The distribution of temperature near the source was found to be representable by the formula $\theta = Ax^{-1}\exp(-Y^2/2\overline{Y^2})$, where $\sqrt{\overline{Y^2}}$ was proportional to the distance downstream from the source and A is a constant.

It is a little confusing that the distribution of temperature close to the source is the error curve, just as it is when the distribution is due either to molecular diffusion or to the fact that a large number of uncorrelated paths have been traversed since the heated particles left the source. The distribution close to the source is the same as the frequency distribution of turbulent velocity, which happens to be the error distribution. If the frequency distribution of velocities had obeyed some other law, the distribution of temperature near the source would not have fitted an error curve. On the other hand, the temperature distribution very far from the source must necessarily fit an error curve whatever be the frequency distribution of velocities.

Since in the experiments the turbulent velocities were small compared with U_0, the time of diffusion T was x/U_0. Hence from (66)

$$\sqrt{\overline{Y^2}}/x = v/U_0. \tag{67}$$

The value of v found in this way from Schubauer's experimental results was very nearly the same as the value of u found at the same point in the air stream by means of a hot wire. These measurements therefore confirm the idea that turbulence produced, as in this case, by grids with regular spacing is isotropic.

The analysis of the diffusion of heat at greater distances down-

† Simmons and Salter, *Proc. Roy. Soc.* A, **145** (1934), 212–234. See also Townend, *ibid.*, pp. 180–211.

‡ *N.A.C.A. Report* No. 524 (1935).

|| Taylor, 'Statistical Theory of Turbulence', *op. cit.*, pp. 468–470.

stream, where R_ξ differs appreciably from 1, has been carried out, but is complicated by the fact that the turbulence dies away downstream, so that v^2 is not constant down a wind tunnel.†

89. Atmospheric turbulence.

Most of the earlier discussions of the effect of turbulence on the temperature and velocity of the atmosphere were based on *mixture length* theories in which, for lack of information and for simplicity, a virtual coefficient of viscosity and of conductivity was assumed which was constant at all heights. These approximate theories yielded some useful results when applied to large scale phenomena, such as the distribution of mean velocity in the lower layers of the atmosphere. They are not applicable to small-scale phenomena such as the diffusion of concentrated puffs of smoke. To discuss this the theory of diffusion by continuous movements has been adopted.‡ Taking

$$R_\xi = (a/v\xi)^n,$$

when $v\xi$ is large, (62) leads to

$$\overline{Y^2} = \tfrac{1}{2}c^2(vT)^{2-n}, \tag{68}$$

where $c^2 = \dfrac{4a^n}{(1-n)(2-n)}$.

For diffusion in the lower atmosphere it has been found that (68) fits the observations made with smoke clouds provided $2-n = 1{\cdot}75$, i.e. $n = 0{\cdot}25$.

The frequency distribution of velocities in the atmosphere is approximately Gaussian,|| as it is in the turbulent air of a wind tunnel.

90. The dissipation of energy in turbulent motion.

When air flows through a pipe in turbulent motion at high Reynolds numbers it is known†† that if the surface stress is expressed in the form $\tau_0 = \rho U_\tau^2$, then there is a universal velocity distribution of the form

$$\frac{U_c - U}{U_\tau} = f\left(\frac{r}{a}\right),$$

† Taylor, 'Statistical Theory of Turbulence', *op. cit.*, p. 420.

‡ O. G. Sutton, 'Eddy Diffusion in the Atmosphere', *Proc. Roy. Soc.* A, 135 (1932), 143–165.

|| Hesselberg and Bjorkdal, *Beiträge zur Physik der freien Atmos.* 15 (1929), 121–133; Graham, *A.R.C. Reports and Memoranda*, No. 1704 (1936).

†† See Chap. VIII, §§ 154, 156, 157, 159.

where U_c is the maximum velocity along the axis of the pipe, U is the velocity at a distance r from the axis, and a is the radius of the pipe. Now τ, the Reynolds stress at radius r, is equal to $r\tau_0/a$, so that at any point $\tau/\rho U_\tau^2$, and therefore also $\tau/\rho(U_c-U)^2$, are constant if the speed varies. On the assumption that the root-mean-square value of each turbulent component of velocity is proportional to the observed maximum value, Fage† has shown that in a pipe u/U_τ, v/U_τ, w/U_τ stay constant as the speed changes at high Reynolds numbers, so that at any point τ/ρ (or $\overline{uv}$), $\overline{u^2}, \overline{v^2}, \overline{w^2}$ are all proportional to $(U_c-U)^2$. These conditions would be satisfied if, when U_c-U is increased in any ratio, the field of turbulent flow is increased at every point in the same ratio.

This hypothetical relationship between the turbulence patterns at different speeds would, however, be inconsistent with the condition that the dissipation of energy by viscosity must be equal to the work done. The rate of dissipation of energy per unit volume in geometrically similar fields is proportional to $\mu u_1^2 l^{-2}$, where u_1 is some typical velocity in the field, which may be either a mean velocity (e.g. U_c-U) or a turbulent component, and l is some typical length. Since the Reynolds stresses are proportional to ρu_1^2, the rate at which work is done per unit volume is proportional to $\rho u_1^3 l^{-1}$.

It is impossible therefore to account for the dissipation of energy in turbulent motion by imagining that a series of fields of flow which are possible at one speed can be repeated at a higher speed—though this hypothesis would account for other observed phenomena.

On the other hand, the fact that, when $U_c a/\nu$ is sufficiently high (a being the radius of the pipe), τ is proportional to $\rho(U_c-U)^2$ while u, v, w are proportional to $U-U_c$, shows that the rate of dissipation of energy per unit volume is proportional to $\rho u^3 a^{-1}$, even when there is no geometrical similarity between the flow patterns at different speeds.

91. Dissipation in isotropic turbulence.‡ The length λ.

The simplest case in which the decay of energy can be discussed by statistical methods is that of isotropic turbulence, i.e. turbulence in which the average value of any function of the velocity components or their space derivatives is unaltered if the axes of reference are

† *Proc. Roy. Soc.* A, **155** (1936), 576–596

‡ Taylor, *Proc. Roy. Soc* A, **151** (1935), 430–454.

rotated in any manner or are reflected.† Turbulent fields of this type can be produced in a stream of air by passing it through a regularly spaced grid of parallel bars.

The general expression for the mean rate of dissipation is

$$\overline{W} = \mu\left\{2\overline{\left(\frac{\partial u}{\partial x}\right)^2}+2\overline{\left(\frac{\partial v}{\partial y}\right)^2}+2\overline{\left(\frac{\partial w}{\partial z}\right)^2}\right.$$
$$\left.+\overline{\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2}+\overline{\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2}+\overline{\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2}\right\}. \tag{69}$$

For isotropic turbulence

$$\overline{\left(\frac{\partial u}{\partial x}\right)^2} = \overline{\left(\frac{\partial v}{\partial y}\right)^2} = \overline{\left(\frac{\partial w}{\partial z}\right)^2} = a_1,$$

$$\overline{\left(\frac{\partial u}{\partial y}\right)^2} = \overline{\left(\frac{\partial u}{\partial z}\right)^2} = \overline{\left(\frac{\partial v}{\partial x}\right)^2} = \overline{\left(\frac{\partial v}{\partial z}\right)^2} = \overline{\left(\frac{\partial w}{\partial x}\right)^2} = \overline{\left(\frac{\partial w}{\partial y}\right)^2} = a_2,$$

and
$$\overline{\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}} = \overline{\frac{\partial w}{\partial y}\frac{\partial v}{\partial z}} = \overline{\frac{\partial u}{\partial z}\frac{\partial w}{\partial x}} = a_3,$$

where a_1, a_2, a_3 are symbols introduced for brevity. Hence

$$\overline{W} = \mu(6a_1+6a_2+6a_3). \tag{70}$$

The quantities a_1, a_2, a_3 are not independent. Since the fluid is assumed incompressible

$$\overline{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)^2} = 0,$$

so that in isotropic turbulence

$$a_1+2a_4 = 0, \tag{71}$$

where
$$a_4 = \overline{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}} = \overline{\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}} = \overline{\frac{\partial w}{\partial z}\frac{\partial u}{\partial x}}.$$

Another relationship between a_1, a_2, a_3, a_4 can be obtained by turning the axes through 45° about the z-axis. The transformation is

$$\left.\begin{aligned}\sqrt{2}x' &= x+y\\ \sqrt{2}y' &= -x+y\end{aligned}\right\} \qquad \left.\begin{aligned}\sqrt{2}u' &= u+v\\ \sqrt{2}v' &= -u+v\end{aligned}\right\}.$$

The transformation for $\partial u/\partial x$ is

$$\frac{\partial u}{\partial x} = \frac{1}{2}\left(\frac{\partial u'}{\partial x'}-\frac{\partial v'}{\partial x'}-\frac{\partial u'}{\partial y'}+\frac{\partial v'}{\partial y'}\right).$$

† This definition of isotropy implies strictly that there is no mean motion relative to the frame of reference. The results will be unaltered if a constant mean motion is superposed. If applied to other cases the results can, at best, be only approximate.

Hence $$\overline{\left(\frac{\partial u}{\partial x}\right)^2} = a_1 = \frac{1}{4}\left\{\overline{\left(\frac{\partial u'}{\partial x'}\right)^2}+\overline{\left(\frac{\partial v'}{\partial x'}\right)^2}+\ldots-2\overline{\frac{\partial u'}{\partial x'}\frac{\partial v'}{\partial x'}}+\ldots\right\}. \quad (72)$$

The conditions of isotropy necessitate that terms like

$$\overline{(\partial u'/\partial x').(\partial v'/\partial x')}$$

shall vanish, for they would change sign by a rotation of the axes through 180°

Collecting together such terms as $\overline{(\partial u'/\partial x')^2}$ and $\overline{(\partial v'/\partial y')^2}$ which are equal to one another in isotropic turbulence, and remembering that isotropy also necessitates that $\overline{(\partial u'/\partial x')^2} = a_1$, etc., we find from (72) that

$$a_1 = \tfrac{1}{2}(a_1+a_2+a_3+a_4),$$

or $$a_1-a_2-a_3-a_4 = 0. \quad (73)$$

Yet another relation between a_1, a_2, a_3, a_4 can be obtained by considering the mean value of $\nabla^2 p$. The equations of motion of an incompressible viscous fluid are

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-\nu\nabla^2 u$$

with two similar equations.† Differentiating these three equations by x, y, z respectively, and adding, we get

$$-\frac{1}{\rho}\nabla^2 p = \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2+2\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial w}{\partial y}\frac{\partial v}{\partial z}+\frac{\partial u}{\partial z}\frac{\partial w}{\partial x}\right), \quad (74)$$

and when we take the mean value of both sides of (74) we arrive at the equation

$$-\frac{1}{\rho}\overline{\nabla^2 p} = 3a_1+6a_3.$$

In a uniform field of turbulence $\overline{\nabla^2 p} = 0$,‡ so that

$$a_1+2a_3 = 0. \quad (75)$$

Combining (71), (73), and (75) we find that

$$a_1 = \tfrac{1}{2}a_2 = -2a_3 = -2a_4. \quad (76)$$

† These are the equations if there is no mean flow. If there is a constant mean velocity, the average value of the right-hand side of (74) is unaltered.

‡ Take integrals over a large volume V_1. Then

$$V_1\overline{\nabla^2 p} = \iiint \nabla^2 p\, dxdydz = \iint \left(l\frac{\partial p}{\partial x}+m\frac{\partial p}{\partial y}+n\frac{\partial p}{\partial z}\right) dS,$$

where the surface integral is over the boundary of V_1, and l, m, n are the direction cosines of the outward normal. Since the integrand of the surface integral does not continually increase as the volume increases, the surface integral divided by the volume tends to zero. Hence $\nabla^2 p = 0$.

Hence (70) may be written

$$\overline{W} = 15\mu a_1 \quad \text{or} \quad 7{\cdot}5\mu a_2 \quad \text{or} \quad 7{\cdot}5\mu\overline{(\partial u/\partial y)^2}. \tag{77}$$

The value of $\overline{(\partial u/\partial y)^2}$ is closely related to the manner in which R_y† falls off from its initial value, 1·0, as y increases from zero. It has been shown in fact‡ that

$$R_y = 1-\frac{1}{2!}\frac{y^2}{\overline{u^2}}\overline{\left(\frac{\partial u}{\partial y}\right)^2}+\dots. \tag{78}$$

The curvature of the R_y curve at the origin is therefore a measure of $\overline{(\partial u/\partial y)^2}$, and

$$\overline{\left(\frac{\partial u}{\partial y}\right)^2} = 2\overline{u^2}\lim_{y\to 0}\left(\frac{1-R_y}{y^2}\right). \tag{79}$$

The significance of (79) can be appreciated by defining a length λ such that

$$\frac{1}{\lambda^2} = \lim_{y\to 0}\left(\frac{1-R_y}{y^2}\right). \tag{80}$$

λ is then the intercept on the axis of y of the parabola drawn to touch the (R_y, y) curve at its vertex (see Fig. 50).

From (77), (79), and (80)

$$\overline{W} = 15\mu\overline{u^2}/\lambda^2. \tag{81}$$

Since $\overline{u^2}$ and R_y can be measured by the hot wire technique, the relationship (81) can be verified if $\overline{W}$ can be measured by other methods. In the case of turbulence in a wind stream behind a grid, $\overline{W}$ can be found by measuring the rate of decay of turbulence downstream from the grid. The mean rate of loss of kinetic energy per unit volume is

$$-\tfrac{1}{2}\rho U_0\frac{d}{dx}(\overline{u^2}+\overline{v^2}+\overline{w^2}),$$

or $-\frac{3}{2}\rho U_0\, d\overline{u^2}/dx$ in the case of isotropic turbulence.

† For the definition of R_y see equation (16), p. 204.

‡ Cf. Taylor, *Proc. Lond. Math. Soc.* (2), 20 (1922), 205, equation (14). Taking $\overline{u^2}$ as independent of y, and expanding u_y in a Taylor series, we have

$$R_y = \frac{\overline{uu_y}}{\overline{u^2}} = \frac{1}{\overline{u^2}}\left\{\overline{u^2}+y\overline{u\frac{\partial u}{\partial y}}+\frac{y^2}{2!}\overline{u\frac{\partial^2 u}{\partial y^2}}+\dots\right\}.$$

But since $\overline{u^2}$ is independent of y,

$$\overline{u\frac{\partial u}{\partial y}} = 0, \quad \text{and} \quad \overline{u\frac{\partial^2 u}{\partial y^2}} = -\overline{\left(\frac{\partial u}{\partial y}\right)^2}.$$

Hence $$R_y = 1-\frac{1}{2!}\frac{y^2}{\overline{u^2}}\overline{\left(\frac{\partial u}{\partial y}\right)^2}+\dots.$$

This must be equal to $\overline{W}$, so that

$$-\tfrac{3}{2}\rho U_0 \frac{d}{dx} u^2 = 15\mu \frac{u^2}{\lambda^2}, \tag{82}$$

and all the quantities in this equation can be measured. The value of λ calculated in this way for the stream in which R_y was measured

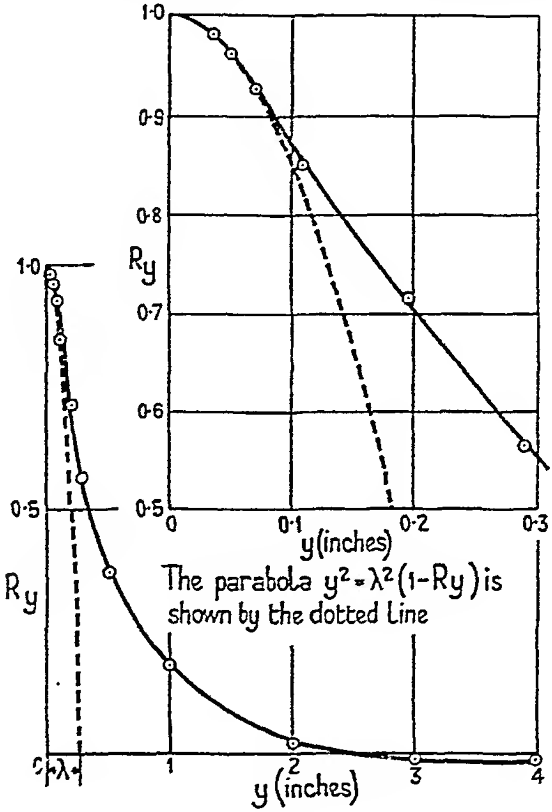


FIG. 50.

(see Fig. 50) was 0·26 in. The parabola $y^2/\lambda^2 = (1-R_y)$ is shown in Fig. 50. It will be seen that it lies close to the observed points near the apex of the R_y curve.

92. Relationship between λ and the scale of the turbulence.

We now consider how the length λ, which determines the dissipation, is related to the scale of the eddy-producing system. It has been pointed out that in a pipe the dissipation of energy is propor-

tional to $\rho u^3 a^{-1}$, where a is the radius of the pipe. The scale of the turbulence produced is clearly limited by the diameter of the pipe. It seems therefore that, in comparing the dissipation in the turbulence produced by geometrically similar turbulence-producing mechanisms on different scales and at different speeds, we may suppose that the dissipation is proportional to $\rho u^3 l^{-1}$, where l is any linear dimension which defines the scale of the turbulence-producing mechanism.

It has been shown (equation (81)) that in isotropic turbulence the rate of dissipation is $15\mu u^2/\lambda^2$, so that, when geometrically similar systems are compared on different scales and at different speeds, then at any point $15\mu u^2/\lambda^2$ is proportional to $\rho u^3 l^{-1}$. Thus

$$\frac{\lambda}{l} \propto \sqrt{\left(\frac{\nu}{lu}\right)}. \tag{83}$$

When the turbulence is produced by a grid of regularly spaced bars distant M apart (i.e. of mesh M) placed across a stream of wind, each bar leaves a wake in the stream. This wake disappears some way downstream, leaving turbulence the scale of which must be determined in some way by the mesh (M) or by the diameter (D) of the bars. The distance downstream at which the wake disappears depends on the ratio D/M; when D/M is as great as $\frac{1}{5}$ the wake disappears within a length of about $20M$. Farther downstream we may suppose that the turbulence is statistically uniform across the stream, and we may take the mesh length M as the typical length l.†
Thus (83) becomes

$$\frac{\lambda}{M} = A\sqrt{\left(\frac{\nu}{Mu}\right)}, \tag{84}$$

where A is a constant to be determined by experiment.

A may be expected to be constant only when geometrically similar grids are used: it is found experimentally to be practically constant for all square-mesh grids, whatever the ratio D/M may be (p. 228).

93. The Reynolds number of turbulence.

It will be remembered that the considerations on which (84) is based are derived from observations on the proportionality of $\sqrt{(\tau/\rho)}$, u, v, w, and $U-U_c$ in a pipe, and that these hold only when $U_c a/\nu$, and hence ua/ν, are greater than certain numbers. It follows that (84) can be expected to hold only when uM/ν exceeds some definite number. This quantity, uM/ν, may be called the Reynolds

† In some cases the typical length must be taken as increasing with distance downstream from the grid. See the reference to *N.A.C.A. Report* No. 581 on p. 233.

number of turbulence. The lowest value for which (84) holds must be determined by experiment.

94. The law of decay of turbulence behind grids.

With the expression (84) for λ, (82) may be integrated: the integral is

$$\frac{U_0}{u} = \frac{5x}{A^2 M} + \text{constant}. \tag{85}$$

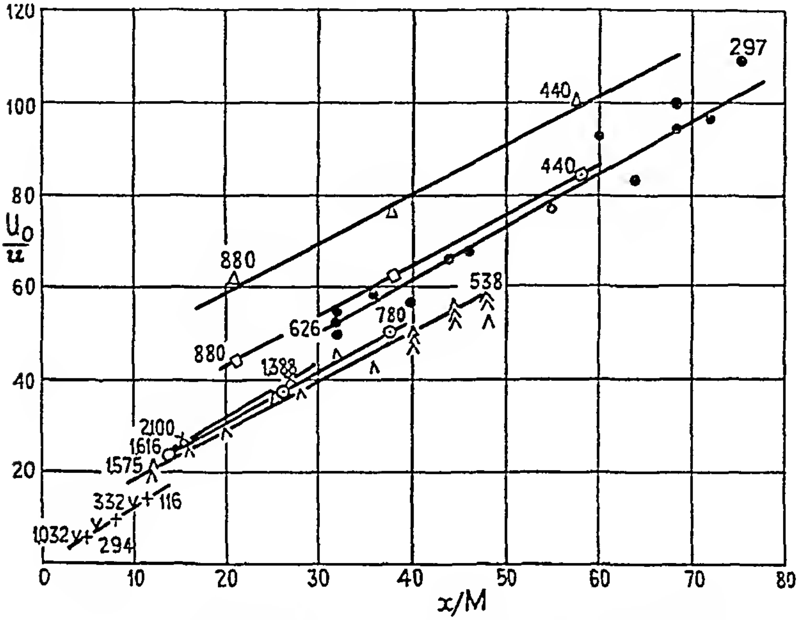


FIG. 51. Collected results of turbulence behind grids and honeycombs.

Dryden:
- × 5-inch grid, $M = 5$ inches, $D = 1$ inch.
- ○ $3\frac{1}{4}$-inch grid, $M = 3{\cdot}25$ inches, $D = 0{\cdot}65$ inch.
- ⊡ Hexagonal honeycomb, $M = 3$ inches.
- △ Honeycomb of 3-inch tubes, $M = 3$ inches.

Simmons and Salter:
- ● 3-inch square-mesh honeycomb, $M = 3$ inches.
- ∧ 3-inch grid of circular rods, $M = 3$ inches, $D = \frac{3}{8}$ inch.
- + grid, $M = 0{\cdot}02$ inch in 4-inch pipe.
- ∨ grid, $M = 1{\cdot}5$ inches in 1-foot tunnel.

Figures shown in the diagram give values of $u/M\nu$.

Thus if u is measured at different distances behind a grid, U_0/u should increase linearly. In Fig. 51 are shown measurements taken behind a square-mesh grid which verify this prediction.

If U_0/u is plotted as ordinate and x/M as abscissa, each set of observations taken at various distances down a wind tunnel behind

a square-mesh grid should be on a straight line whose slope is $5/A^2$. In this way, with a range of grids from $M = 5''$ down to $M = 0{\cdot}62''$, values of A were obtained varying only between the limits 1·95 and 2·20. Since the values of D/M were not the same in all the experiments, it seems that the scale of the turbulence depends on M, and not on D.

95. An experimental verification of isotropy in turbulence behind grids.

The comparison between the observed values of u^2 and the values of v^2 found by analysis of Schubauer's diffusion measurements (§ 88, p. 219) has shown that in a turbulent stream behind regularly spaced grids $u = v = w$. These conditions, however, do not form a complete verification of isotropy. A more complete verification can be obtained by measuring the correlation (R) between the values of u at pairs of points distant d apart along lines at various inclinations (θ) to the axis of a wind tunnel. If $\partial u/\partial x'$ is the gradient of u in a direction inclined at an angle θ to the tunnel axis (which is taken as the axis of x), then in the same way as (79) was found it may be shown that

$$\lim_{d\to 0}\frac{1-R}{d^2} = \frac{1}{2u^2}\overline{\left(\frac{\partial u}{\partial x'}\right)^2}.$$

But
$$\frac{\partial u}{\partial x'} = \cos\theta\frac{\partial u}{\partial x}+\sin\theta\frac{\partial u}{\partial y}.$$

Also $\overline{(\partial u/\partial x).(\partial u/\partial y)}$ vanishes (since it would change sign with a reversal of the direction of the y-axis), so that

$$\overline{\left(\frac{\partial u}{\partial x'}\right)^2} = \cos^2\theta\overline{\left(\frac{\partial u}{\partial x}\right)^2}+\sin^2\theta\overline{\left(\frac{\partial u}{\partial y}\right)^2}.$$

From (79) and (80)
$$\frac{1}{2u^2}\overline{\left(\frac{\partial u}{\partial y}\right)^2} = \frac{1}{\lambda^2},$$

and from (76)
$$\overline{\left(\frac{\partial u}{\partial x}\right)^2} = \frac{1}{2}\overline{\left(\frac{\partial u}{\partial y}\right)^2},$$

so that
$$\lim_{d\to 0}\frac{1-R}{d^2} = \frac{1}{\lambda^2}(\tfrac{1}{2}\cos^2\theta+\sin^2\theta). \tag{86}$$

Hence, if $d/\sqrt{(1-R)}$ is taken as the radius vector (r) in polar co-

ordinates (r, θ), and plotted against θ, the resulting curve for sufficiently small values of d should be the ellipse

$$\frac{1}{r^2} = \frac{\cos^2\theta}{2\lambda^2} + \frac{\sin^2\theta}{\lambda^2},$$

with semi-axes $\sqrt{2}\lambda$ and λ.

Measurements of $(1-R)/d^2$ have been made by Simmons in a stream of wind rendered turbulent by a square-mesh grid of mesh 3 inches, and are shown in Fig. 52. The full-line curve in Fig. 52 is the theoretical ellipse, with axes in the ratio $\sqrt{2}:1$. It will be seen

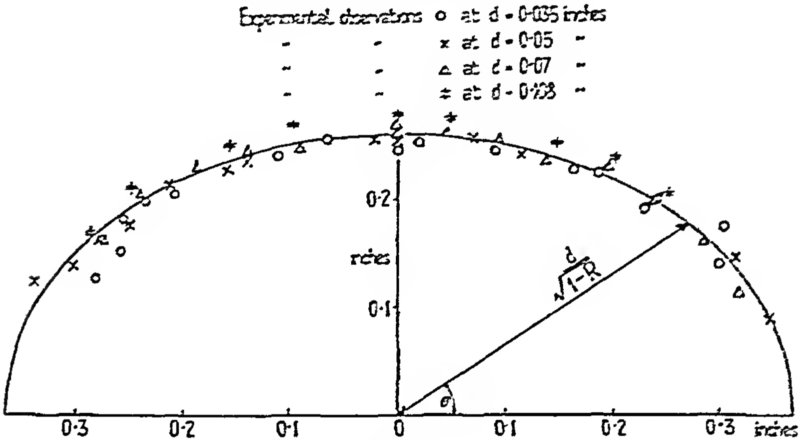


FIG. 52.

that the theory, which is based on the conception of isotropic turbulence, is very well confirmed.

96. The effect of a density gradient on stability.

The effect of gravity in suppressing turbulence in a fluid of variable density has been discussed from various points of view. Richardson and Prandtl† calculated the work done by turbulence against gravity in a fluid whose density decreases upwards. If Y is the height to which any particle in the fluid has risen above its original level, the work done per unit volume against gravity by the turbulence is

$$\tfrac{1}{2}\rho g \beta \overline{Y^2}, \quad \text{where} \quad \beta = \frac{1}{\rho}\left|\frac{d\rho}{dy}\right|. \tag{87}$$

† Richardson, *Proc. Roy. Soc.* A, 97 (1920), 354–373; *Phil. Mag.* (6), 49 (1925), 81–90. Prandtl, 'Einfluss stabilisierender Kräfte auf die Turbulenz', *Vorträge aus dem Gebiete der Aerodynamik und verwandter Gebiete, Aachen*, 1929 (Berlin, 1930), pp. 1–7.

The rate at which work is done is therefore

$$\tfrac{1}{2}\rho g\beta\frac{d}{dt}\overline{(Y^2)}.$$

When the turbulent energy is not decreasing this work must be supplied by the Reynolds stresses. If the mean motion is a uniform shearing parallel to the axis of x, so that $dU/dy = \alpha$, the rate at which the Reynolds stresses are doing work is $|\tau\alpha|$. Thus

$$|\tau\alpha| > \tfrac{1}{2}\rho g\beta\frac{d}{dt}\overline{(Y^2)}. \tag{88}$$

Now Y in (88) is identical with the Y used in the theory of diffusion by continuous movements, so that from (63) and (65)

$$\frac{1}{2}\frac{d}{dt}\overline{Y^2} = v^2\int\limits_0^{T_1} R_\xi\,d\xi = l'v,$$

where l' is the mixture length. Hence

$$|\tau\alpha| > g\rho\beta l'v. \tag{89}$$

If it is assumed that the momentum transfer theory of turbulent motion holds, then

$$\tau = \rho l'v\alpha, \tag{90}$$

so that the momentum transfer theory necessarily gives the relationship†

$$\frac{\alpha^2}{g\beta} > 1. \tag{91}$$

This is Richardson's relationship. If therefore a motion is established such that $\alpha^2/g\beta < 1$, it cannot, according to the momentum transfer theory, be turbulent.

Taylor‡ has calculated the stability equation for a non-viscous fluid of small uniform density gradient in uniform shearing motion. He finds that there is stability if $\alpha^2/g\beta < 4$.

Consideration of the stability of a system consisting of three superposed fluids moving with uniform shear led to the result that unstable oscillations can occur when

$$\alpha^2 > 2g\frac{\Delta\rho}{\rho h},$$

where $\Delta\rho$ is the small change in density at each interface, and h is the thickness of the central layer. Similar calculations for four fluids gave the lower limit for unstable waves as $\alpha^2 > 2{\cdot}11 g\beta$.

Similar results were obtained from calculations‖ for the case when

† Prandtl's version gives $\alpha^2/g\beta > \tfrac{1}{2}$, but the factor $\tfrac{1}{2}$ appears to be due to a mistake (see Taylor, *Rapports et Procès-Verbaux du Conseil Permanent International pour l'Exploration de la Mer*, 76 (1931), 35–43).

‡ *Proc. Roy. Soc.* A, 132 (1931), 499–523.

‖ *Ibid.*, p. 509, and Goldstein, *ibid.*, pp. 524–548.

upper and lower fluids, each of infinite extent and moving with uniform velocity, are separated by a layer of uniform vorticity and intermediate density.

The above stability results apply only to fluids of infinite extent. Schlichting† has extended the calculations of Tollmien on the stability of the boundary layer in a viscous fluid, and has found that the effect of gravity is to make all oscillations stable provided $\alpha^2 < 25g\beta$, where α now represents the rate of shear at the wall, and it is assumed that the density distribution is such that β is constant in the boundary layer and zero outside it. Measurements by Reichardt‡ in a wind tunnel heated at the top seem to confirm Schlichting's theoretical result. It was to be expected that this result would differ from that which is valid in an infinite fluid.

97. Diffusion in a turbulent field with a density gradient.

It is not possible from the stability calculations to say anything about the possibility or otherwise of turbulent motion with any given density gradient. On the other hand, Richardson's criterion—that turbulence is possible only when $\alpha^2/g\beta > 1$—must be valid if the momentum transfer theory of turbulent motion is valid. Hydrographic measurements by Jacobsen of the current and density at all depths at various stations near Denmark have been analysed (partly by Jacobsen and partly by Taylor‖) to find the transport of momentum and of salt in a vertical direction, together with values of $\alpha^2/g\beta$. The values of $\alpha^2/g\beta$ ranged from 0·008 to 0·38, yet the calculated rate of diffusion of salt and the calculated shear stress were thousands of times as great as could be accounted for by molecular diffusion and viscosity. It is clear therefore that turbulence can exist even when $\alpha^2/g\beta$ is as small as 0·008. For this reason it seems that the momentum transfer theory must be very far from the truth when there is a density gradient.

If we abandon the momentum transfer theory we can still get some information from the energy relation (88). If K_S is defined as the virtual coefficient of diffusion of salt, then $K_S = \frac{1}{2}d\overline{Y^2}/dt$, so that (88) becomes

$$|\tau\alpha| > g\rho\beta K_S. \tag{92}$$

† *Zeitschr. f. angew. Math. u. Mech.* **15** (1935), 313–338.

‡ Prandtl and Reichardt, 'Einfluss von Wärmeschichtung auf die Eigenschaften einer turbulenten Strömung', *Deutsche Forschung*, Part 21 (1934), 110–121.

‖ *Rapports et Procès-Verbaux du Conseil Permanent International pour l'Exploration de la Mer*, 76 (1931), 35–43.

Hence the energy relationship can be expressed in the form

$$\frac{\alpha^2}{g\beta} > \frac{K_S}{\tau/\rho\alpha}. \tag{93}$$

τ/α is a virtual coefficient of viscosity, so that $\tau/\rho\alpha$ may be regarded as a coefficient of diffusion of momentum. If we put† $\tau/\rho\alpha = K_u$, (93) becomes

$$\frac{\alpha^2}{g\beta} > \frac{K_S}{K_u}. \tag{94}$$

This equation can be verified, because both K_S and K_u can be calculated from the distributions of velocity and density in a stream where fresh water is flowing over salt water. The results of these calculations in two such cases (at Schultz's Grund and Randers Fjord) are given in Table 11.

TABLE 11

Schultz's Grund.

Depth, metres	$\frac{K_S}{K_u}$	$\frac{\alpha^2}{g\beta}$
2·5	0·09	0·14
5·0	0·13	0·26
7·5	0·067	0·17
10·0	0·023	0·098
12·5	0·021	0·035
15·0	0·05	0·008

Randers Fjord.

Depth, metres	$\frac{K_S}{K_u}$	$\frac{\alpha^2}{g\beta}$
1	0·17	0·17
2	0·20	0·38
3	0·15	0·26

It will be seen that all the observations, except that at 15 metres at Schultz's Grund, satisfy (94). The exceptional observation is near a velocity maximum, so that α is nearly zero.

ADDITIONAL REFERENCES

Taylor and Green (*Proc. Roy. Soc.* A, **158** (1937), 499–521) show that when a special type of initial motion is given to a viscous fluid the rate of dissipation increases until it reaches a maximum, at which its value is in fair agreement with the formula (84) (p. 226) when A is given its experimentally determined value 2·0.

Kármán, *Proc. Nat. Acad. Sci.* **23** (1937), 98–105; *Journ. Aero. Sciences*, **4** (1937), 131–138. The correlation between the velocity components in fixed directions at two points is expressed as a tensor. If R_1 is the correlation between the velocity components in the direction AB at the two points A and B at a distance r apart, R_2 the correlation between the components at right angles to AB, it is shown that

$$r\frac{dR_1}{dr} + 2(R_1 - R_2) = 0. \tag{95}$$

(This is a generalization of (86) (p. 228).) On the assumption that the mean

† The momentum transfer theory assumes that $K_u = K_S$.

values of all triple products of components of velocities at A and B vanish, a theory is developed for the decay of turbulence behind a grid. With this assumption the (R_1, r) curve may be of constant shape; in such a case the decay of turbulence is expressed by the formula

$$\frac{U_0}{u} = \text{constant}\left(1+\frac{x}{U_0 t_0}\right)^{5\alpha}, \tag{96}$$

which may be compared with (85) (p. 227): according to (96) it is only when $5\alpha = 1$ that U_0/u is a linear function of x.

Taylor (*Journ. Aero. Sciences*, 4 (1937), 311–315) shows that the special form of the (R_1, r) curve necessitated by Kármán's assumption (see above) is not in agreement with observation.

Taylor, *Proc. Roy. Soc.* A, **164** (1938), 15–23. In Kármán's expression for the rate of change of mean-square vorticity (*Journ. Aero. Sciences, op. cit.*) the terms neglected on the assumption that the mean values of triple products vanish (see above) can be evaluated from measured R_x curves. This is done in a particular case, and it is found that the value of the term which has been neglected is three times as great as the one which is not neglected.

Kármán and Howarth (*Proc. Roy. Soc.* A, **164** (1938), 192–215) show that a law of decay similar to (96) can arise when the triple correlations do not vanish, if further assumptions are made.

Dryden, Schubauer, Mock, and Skramstad, *N.A.C.A. Report* No. 581. In the Bureau of Standards tunnel the scale of turbulence increases with distance from the grid. The rate of decay is consistent with the formula

$$\frac{\lambda}{L} = B\sqrt{\left(\frac{\nu}{Lu}\right)}, \tag{97}$$

where L is the observed scale of turbulence defined by

$$L = \int_0^\infty R_y\,dy. \tag{98}$$

Some measurements are described in which band filters are inserted in hot wire anemometer circuits, thus eliminating all turbulence the frequency of which falls outside the band. (Cf. Chap. VI, § 121.)

Simmons and Salter (*Proc. Roy. Soc.* A, **165** (1938), 73–89) have constructed a set of high and low pass filters by means of which the turbulence behind a grid has been analysed into a spectrum. (Cf. Chap. VI, § 121.)

Taylor (*Proc. Roy. Soc.* A, **164** (1938), 476–490) shows that the spectrum of turbulence at a fixed point is connected with R_x by the pair of relations

$$R_x = \int_0^\infty F(n)\cos\frac{2\pi n x}{U_0}\,dn, \tag{99}$$

$$F(n) = \frac{4}{U_0}\int_0^\infty R_x\cos\frac{2\pi n x}{U_0}\,dx, \tag{100}$$

where $F(n)\,dn$ is the proportion of $\overline{u^2}$ which is due to components with frequencies between n and $n+dn$.

Dryden (*Journ. Applied Mechanics*, 4 (1937), 105–108) gives a bibliography of 31 papers and an account of developments between 1935 and 1937.

VI

EXPERIMENTAL APPARATUS AND METHODS OF MEASUREMENT

98. Introduction.

THIS chapter is intended to give the reader a sufficiently detailed account of the apparatus and of the methods of measurement used in aerodynamic experiments to enable him to appreciate the general nature of the experimental investigations to which reference is made elsewhere in the volumes, and to have some idea of the accuracy and also of the limitations of present-day experimental technique.

SECTION I

WIND TUNNELS, WATER TANKS, AND WHIRLING ARMS

99. Wind tunnels.

A wind tunnel is an apparatus for producing a uniform air-stream in which the aerodynamic properties of bodies can be observed and measured. There are three main types of wind tunnel: (1) open circuit tunnels, (2) closed circuit (return flow) tunnels, and (3) compressed air (variable density) tunnels. In each of these types the stream at the working section may be either free (open jet type) or bounded by rigid walls (closed jet). A brief outline of the characteristics of each of these types follows: details can be found in reports published by aerodynamic institutions throughout the world.†

100. Open circuit tunnels.

The open circuit type (Fig. 53 (*a*) and (*b*), p. 236) consists essentially of a duct, usually of square or rectangular cross-section, through which air is sucked by a fan at the outlet end. The air is afterwards discharged into the room, and returns slowly to the bell-mouthed inlet. The fan is of the airscrew type, which gives a steadier flow than a centrifugal fan. Even so, the eddies created by the rotating blades

† See Eiffel, *Nouvelles Recherches sur la Résistance de l'Air et l'Aviation* (Paris, 1914); *Ergebnisse der Aerodynamischen Versuchsanstalt zu Göttingen*; *A.R.C. Reports and Memoranda*; *N.A.C.A. Reports*. For a summary see Hoerner, *Zeitschr. des Vereines deutscher Ingenieure*, 80 (1936), 949–957.

The area of the working section of a wind tunnel is usually of the order of 50 sq. ft., but one or two existing tunnels can house full-size aeroplanes. The biggest, at Langley Field, U.S.A., has an oval jet 60 ft. broad by 30 ft. high, in which a wind speed of about 120 m.p.h. can be reached with an expenditure of 8,000 horse-power. (See De France, *N.A.C.A. Report* No. 459 (1933).)

constitute a serious source of disturbance in the flow, and measures must be taken to minimize their effect. Behind the airscrew in the N.P.L.† type a 'distributor', consisting of a large rectangular compartment perforated on all sides with fairly small holes or slots, serves to return the air to the tunnel room at a fairly low speed and over a considerable area, and so to break up the violently disturbed flow behind the airscrew into reasonably small eddies which have time to die away during their slow passage through the room back to the intake. Alternatively, the distributor may be considerably reduced in size or even entirely dispensed with, a honeycomb wall being built across the tunnel room (Fig. 53 (*b*)) to break up the large disturbances discharged by the fan.

If the jet is open, as in the tunnels designed by Eiffel in France, it is necessary to surround it by an air-tight working chamber, since the pressure at the working section is necessarily below atmospheric. In either the Eiffel or N.P.L. type the room containing the tunnel should have a cross-sectional area many times that of the tunnel itself, in order that the return flow in the room may be very slow. Disturbances leaving the outlet are thus given time to die away, and the tunnel virtually takes its supply from still air at the intake end. In addition, there is fitted near the intake end, to prevent swirl about the tunnel axis, a honeycomb, i.e. a bank of thin-walled tubes of fairly small cross-section. Its chief function is to maintain the direction of flow parallel to the axis of the tunnel. It also serves to break up any occasional large eddies which may reach the intake.

101. Closed circuit tunnels.

The chief disadvantages of the open circuit type are the large room-space it needs and its low efficiency, consequent upon the waste of practically the whole of the kinetic energy of the air at the discharge end. By the continual circulation of the air in a closed circuit much of this loss is avoided, and a given wind speed is obtained for an expenditure of much less power than in the open circuit type. The arrangement has the disadvantages that the return flow is now not slow enough to allow disturbances from the airscrew to die away, and that the air current has to be turned smoothly through four right angles in its passage from outlet to inlet. The first difficulty is largely overcome by the gradual expansion of the return-flow ducts to about

† National Physical Laboratory, Teddington.

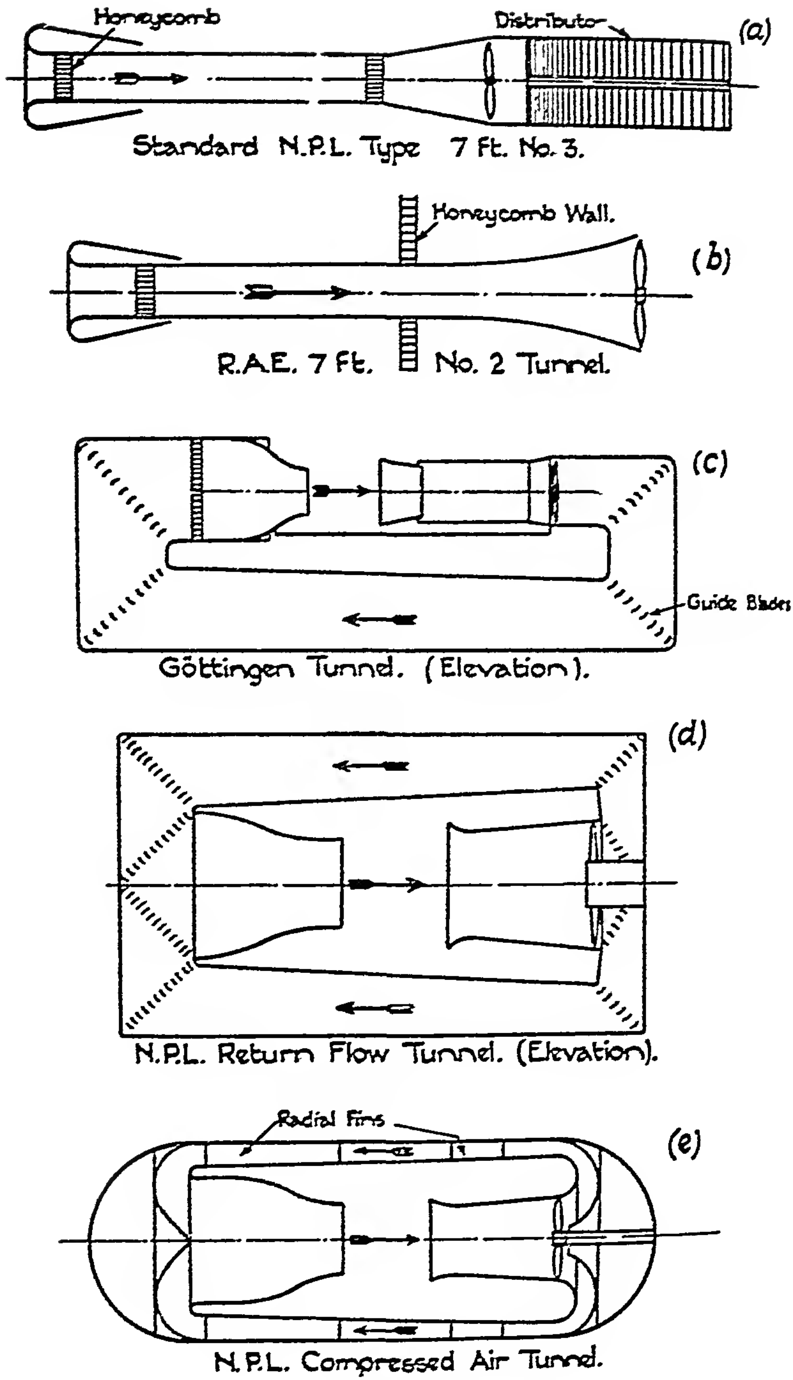

Honeycomb
Distributor
(a)
Standard N.P.L. Type 7 Ft. No. 3.
Honeycomb Wall.
(b)
R.A.E. 7 Ft. No. 2 Tunnel.
(c)
Guide Blades
Göttingen Tunnel. (Elevation).
(d)
N.P.L. Return Flow Tunnel. (Elevation).
Radial Fins
(e)
N.P.L. Compressed Air Tunnel.

Fig. 53.

four times the area of the working section, followed by a rapid contraction of the stream just before the working section is reached; the second is effectively surmounted by the provision of suitable guide vanes to turn the air smoothly round each right-angled bend.† Both the rapid contraction at the inlet and the guide vanes, which are the two essential features in the design of a successful closed circuit tunnel, were first used by Prandtl at Göttingen.

A honeycomb is usually placed just before the rapid contraction, but does not appear to be always essential; it has, for example, been omitted in the two new tunnels at the N.P.L. A definite improvement in both velocity distribution and power efficiency is obtained by placing radial aerofoils either behind or in front of the airscrew, set in such a way as to remove the rotation from the slipstream of the screw, and so to allow the air-current to reach the aerofoil cascade at the first bend behind the screw with only an axial velocity component.

The flow may be returned either by a single duct, or by two ducts symmetrically placed one on each side of the jet. Given a free choice, the single return duct is somewhat simpler to construct. The closed circuit open jet tunnels at Göttingen and the R.A.E.‡ are of this type; two at the N.P.L. have double return passages (Fig. 53 (*c*) and (*d*)).

The closed circuit type lends itself equally well to either a closed or open working section; as the return circuit is completely airtight the pressure at an open jet may be the atmospheric pressure, with free access to the jet from the tunnel room. The wind speed obtainable for a given power expenditure is appreciably lower with an open jet than with a closed one, but this is offset by greater accessibility and by smaller corrections for constraint of the tunnel flow in some experiments, e.g. airscrew tests.

102. Compressed air tunnels.

This type of wind tunnel, of which at present only two examples exist,|| enables the Reynolds numbers of flight to be reached in a

† Klein, Tupper, and Green, *Canadian Journ. of Research*, 3 (1930), 272–285; Frey, *Forsch. Ingwes.* 5 (1934), 105–117; Collar, *A.R.C. Reports and Memoranda*, No. 1768 (1937); *Patterson, ibid.*, No. 1773 (1937); *Aircraft Engineering*, 9 (1937), 205–208.

‡ Royal Aircraft Establishment, Farnborough.

|| A modified type of compressed air tunnel has recently been built at Göttingen. The pressure can be raised to 3 atmospheres, but it can also be reduced to 0·3 atmosphere. The latter condition enables high jet speeds to be reached without an excessive expenditure of power. This feature is of use mainly in connexion with tests

model test by the use of air at high pressure. It consists of a closed circuit wind tunnel, with jet open or closed, entirely enclosed in a steel shell capable of withstanding the requisite pressure. To economize space and to fit the tunnel neatly into the shell the return duct takes an annular form surrounding the working section (Fig. 53 (*e*)). Tests on a model at the N.P.L. showed that guide blades at the bends could be dispensed with, if a number of straight vanes was placed in the annular return duct to prevent a general swirl and a honeycomb employed immediately before the contracting jet. This is presumably because the return duct, completely surrounding the tunnel proper, has a comparatively small width perpendicular to the flow, so that its boundaries have a sufficient directive influence at the bends without intermediate guiding aerofoils.

The compressed air tunnel at the N.P.L.,† which has a jet 6 feet in diameter and a pressure range of 1 to 25 atmospheres, was first used for tests of wings and complete aeroplane models to provide data applicable to full-scale machines. It was later used to investigate certain fundamental problems at high Reynolds numbers, including the effects of surface roughness on the drag of aerofoils, the behaviour of flaps and other high-lift devices, and the drag of stream-line bodies. By the adoption of the momentum method of drag measurement (§ 115), the measurements of aerofoil drag in the tunnel have recently been extended to Reynolds numbers of the order of 24×10^6, the highest yet reached in wind tunnel tests.‡

103. Turbulence in wind tunnels and its effects.

Turbulence in the air-stream has an important influence on the nature of results obtained in wind tunnels, especially in certain kinds of measurement, of which the drag of stream-line bodies is a long-known example and the maximum lift of aerofoils a more recent one. In ordinary atmospheric tunnels the interpretation of results and their application to design is greatly complicated if

of airscrews or, with sufficient power and reduction of pressure, for the investigation of compressibility effects. See H. Winter, *Aircraft Engineering*, 8 (1936), 335, 336; *Luftwiss.* 3 (1936), 237–241.

† See Relf, *Engineering*, **131** (1931), 428–433.

‡ Relf, *Journ. Roy. Aero. Soc.* 39 (1935), 1–28; Relf, Jones, and Bell, *A.R.C. Reports and Memoranda*, No. 1706 (1936); Relf, Bell, and Smyth, *ibid.*, No. 1636 (1935); Jones and Williams, *ibid.*, No. 1708 (1936); No. 1710 (1936); No. 1804 (1937); Williams, Brown, and Smyth, *ibid.*, No. 1717 (1936); Williams and Brown, *ibid.*, No. 1772 (1937).

turbulence effects are present, since the results are then functions both of the Reynolds number and of the turbulence, and it is exceedingly difficult to separate the two effects.† In the compressed air tunnel, where full-scale Reynolds numbers are obtained, turbulence effects may prevent a direct application of results to full-scale prediction. Further, in order to study turbulence effects it is often desirable to be able to vary the degree of turbulence in a wind tunnel. It is thus of great importance in wind tunnel technique to be able to estimate the degree of turbulence present and to form some idea of the effects of such turbulence on the application of tunnel results to the problems confronting the designer.

Attempts to specify the degree of turbulence in a wind tunnel have been made in two different ways.† In one method a hot wire anemometer‡ is used to define the ratio of the root-mean-square longitudinal velocity fluctuation to the mean wind speed. In the second method the Reynolds number at which the drag coefficient $\left(\dfrac{D}{\frac{1}{2}\rho U_0^2 \pi d^2/4}\right)$ of a sphere is 0·30 is used as a measure of turbulence.‖ In certain tests in America the two methods were compared, and it was shown that a unique relation exists between the measured values of R_{crit} and $(\sqrt{\overline{u^2}}/U_0)(d/M)^{\frac{1}{5}}$,†† where R_{crit} is the Reynolds number for the sphere defined above, and M is the cross-dimension of the mesh used to introduce the turbulence. These tests were made in a tunnel of N.P.L. type at different distances from the honeycomb, and the result was confirmed by similar tests at the N.P.L. This result is, however, not general: a unique relation is to be expected only if the turbulence is isotropic. As regards turbulence not produced by grids or honeycombs, it may be remarked that in tests of a sphere in flight made in America‡‡ it was found that even in the disturbed air near the ground in a wind the critical Reynolds number was practically the same as that on a calm day higher up, when conditions must have been almost non-turbulent. There is little doubt that the value of $\sqrt{\overline{u^2}}/U_0$ was appreciable in the disturbed conditions

† See, for example, Dryden, *Journ. Aero. Sciences*, 1 (1934), 67–75.

‡ See §§ 117 and 119.

‖ For some results of measurements of this kind see Platt, *N.A.C.A. Report* No. 558 (1936).

†† Dryden, Schubauer, Mock, and Skramstad, *N.A.C.A. Report* No. 581 (1937). The existence of such a relation was first suggested by Taylor (see Chap. XI, § 219).

‡‡ Millikan and Klein, *Aircraft Engineering*, 5 (1933), 167–174.

close to the ground, and it must be concluded that the eddies present were so large that they affected the sphere as variations of total relative velocity and not as disturbances to the boundary layer. The effect of non-isotropic turbulence has not yet been investigated, but it has great practical interest, since most modern wind tunnels are of the return flow type with a large contraction ratio, and the transverse turbulent velocity components are in this case certainly not equal to the longitudinal one.

Very little can be said at present in regard to the correlation of turbulence measurements by methods suggested above with the effects of turbulence on the aerodynamic behaviour of various kinds of models tested in wind tunnels. The problem is under investigation, but so far the relevant results seem to indicate no general correlation. For example, the compressed air tunnel has given maximum lift results on certain aerofoils which agree well with full-scale observations, although the critical turbulence number for a sphere is 225,000 for the tunnel (6 in. sphere) and about 365,000 for the free air.† It would appear that in this instance the sphere drag is more sensitive to small degrees of turbulence than is the maximum lift of these aerofoils. On the other hand, tests in the compressed air tunnel at high Reynolds numbers, with the turbulence considerably augmented by means of screens, showed that different aerofoils react very differently as regards maximum lift variations. On the aerofoil section R.A.F. 28 a moderate increase of maximum lift with turbulence occurs at all Reynolds numbers, and appears to be roughly proportional to the degree of turbulence; but with the section Göttingen 387 the effect is small and indefinite at low Reynolds numbers, but very great at high Reynolds numbers (see Chap. X, § 198). These observations show the complexity of the subject, and suggest that much further research is required on the connexion between turbulence in the air-stream and the aerodynamic effects it produces by modifying the flow in the boundary layer.

104. The augmentation of turbulence in wind tunnels.

Screens or grids may be used to augment the turbulence in a wind tunnel. Such screens are often used when turbulence effects are being studied, since they are the only convenient way of producing varying degrees of turbulence in the same wind tunnel: varying

† Millikan and Klein *loc. cit.* (15 cm. sphere).

turbulence on the model may be attained both by altering its distance from the screen and by altering the spacing of the cords or strips composing the screen. In practice the method is complicated by the difficulty in defining the mean speed behind the screen without very detailed velocity explorations,† and by the fact that the turbulence decreases with distance from the screen so that with a model of any length, such as a stream-line body, the turbulence is by no means constant along the body. Another method which has been used to render the boundary layer of a body turbulent is to attach excrescences to the body itself, e.g. to put one or more rings of fine wire around the nose of a stream-line body.‡ This method is open to the objection that, in addition to producing changes in the drag by making the boundary layer turbulent, it may alter the form drag; but this effect can be separated if the form drag is determined by pressure plotting. When this is done it is found that in the case of a stream-line body a sufficient number of rings near the nose will render the whole boundary layer turbulent behind them, and the addition of further rings does not then affect the skin-friction drag.

105. The degree of turbulence desirable in a wind tunnel.

Opinion is at present divided on the degree of turbulence desirable in a wind tunnel. It may logically be argued that an atmospheric wind tunnel ought to be fairly turbulent—not only because increased turbulence often simulates the effects of an increased Reynolds number, but because in a turbulent stream the boundary layer of a body is more easily rendered turbulent and the conditions are more definite than in a non-turbulent stream. Consider, for example, the measurement of the drag of a stream-line body. If the tunnel stream is very turbulent, the boundary layer of the body will become turbulent fairly near the nose at all reasonably high Reynolds numbers, and a consistent variation of drag with Reynolds number will be measured, similar to the variation obtained on a flat plate in turbulent flow. There is therefore a possibility of extrapolation to higher Reynolds numbers, where the boundary layer would be turbulent even in a non-turbulent flow, by using flat plate data as a guide. If, however, the body is tested in a tunnel of low turbulence, the drag results will in general lie on some transition curve, i.e.

† Ower and Warden, *A.R.C. Reports and Memoranda*, No. 1559 (1934).
‡ Ower and Hutton, *ibid.*, No. 1409 (1931).

the boundary layer will be partly laminar and partly turbulent, and any rational basis of extrapolation becomes impossible. On the other hand, it may be argued that since it is at present impossible to correlate the measurement of turbulence with the effects it produces on different aerodynamic phenomena, and since the free air is believed to be effectively non-turbulent, tunnels should have as low a degree of turbulence as possible. Moreover, in a non-turbulent tunnel any desired degree of turbulence can be introduced by a grid. Whichever of these views ultimately proves to be the best as regards the practical use of atmospheric wind tunnels, there is no doubt that the different turbulence characteristics of the various types of present-day tunnel are a great handicap in the comparison of results from such tunnels, and that a complete knowledge of the effects of turbulence would clear up many discrepancies at present existing between results from different sources. Only in the particular case of the compressed air tunnel is the position clear. Here the full-scale Reynolds number is reached, and it is obviously desirable that the degree of turbulence should be that appropriate to the free air, which, as far as boundary layer flow is concerned, is believed to be very small. Since it is difficult to make a non-turbulent tunnel of compact design, the only question which arises in practice is the definition of a minimum tunnel turbulence which is sufficiently low to satisfy the above requirement.

106. Force measurements.

The number of methods which have been used in this and other countries to support models in a wind tunnel and to measure the forces and moments acting upon them is very large, and it is impossible to deal with them all. In general, it may be said that the method adopted in any particular experiment depends very much upon the nature of the force to be measured and on the ultimate accuracy required. The chief concern in accurate work is to avoid undue interference between the supporting members and the model itself, and to devise the system so that any interference which may unavoidably be present is easy to determine accurately. For example, in measuring the lift and drag of a complete aeroplane model supported on wires from roof balances, both the wire drag and the interference effects of the wires are small compared with the forces on the model, and no difficulty is experienced. On the other hand, in measuring the drag of a body of very good form, such as

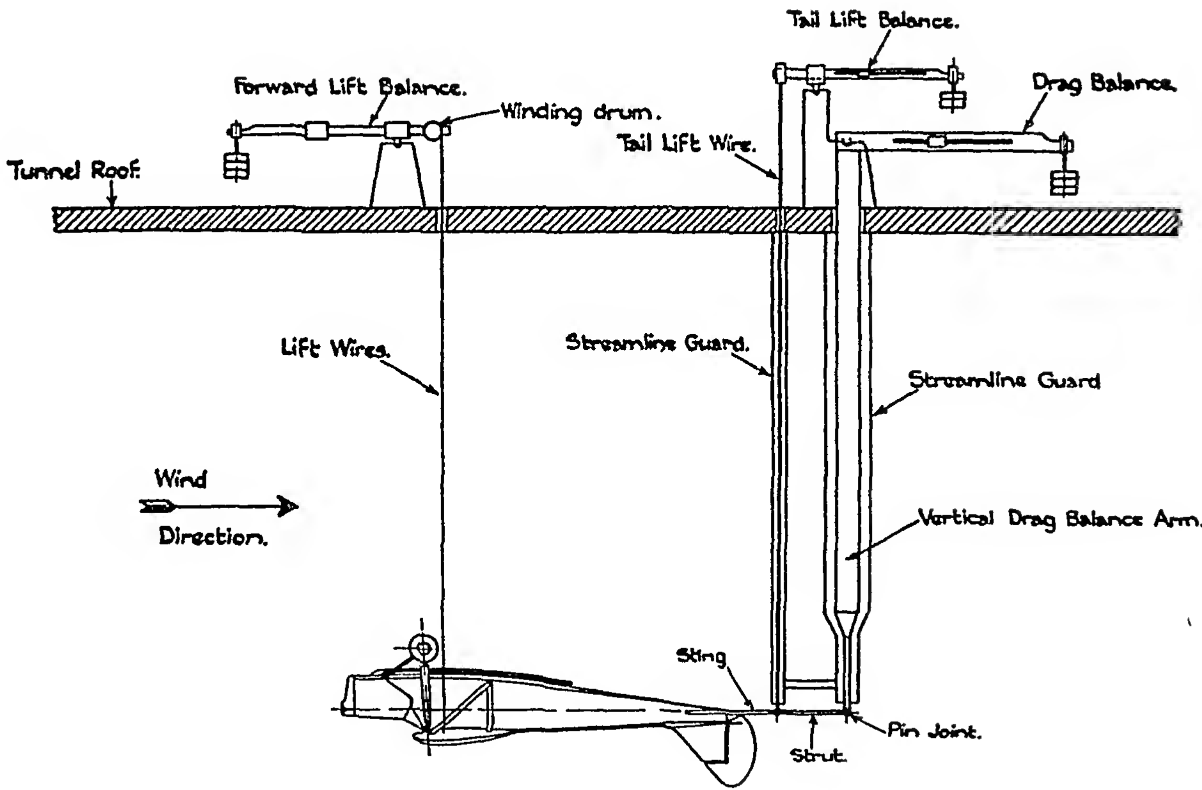
a model airship hull, the wire drag may be greater than that of the model, and the interference of even very fine wires with the flow near the model may introduce serious errors, making an accurate determination of the drag of the model alone a very difficult matter. A few notes are given below on the more commonly employed methods of force measurement, with particular reference to those which bear most directly on experiments relating to the study of fluid flow.

107. Forces on a model aerofoil or complete aeroplane.

This class forms by far the largest group of wind tunnel measurements. Generally, such work is primarily undertaken to provide practical data for the designer, but the results obtained, particularly the maximum lift and minimum drag of aerofoils, are of great interest in connexion with the study of the flow near the model. In one method of making tests of this kind the model is supported in an inverted position in the tunnel by two wires from the wings, while the tail of the aeroplane model, or a short 'sting' at the trailing edge of an aerofoil, is attached by a pin joint to a vertical arm, shielded as far as possible from the wind (Fig. 54). The wing wires, generally vertical, are attached to a balance, enabling the tension in them to be measured, while the tail arm forms part of a composite balance which measures both the vertical and horizontal components of the force transmitted through the pin joint at the tail of the model. From a knowledge of the vertical reaction at the forward wires and at the tail, of the horizontal reaction at the latter point, and of the geometrical dimensions of the system, the lift, drag, and pitching moment about any chosen axis can be evaluated. The wire drag is found by repeating the measurements with two dummy wires added, and the drag of the exposed part of the tail support by detaching the model from that support and holding it rigidly by wires as close to the support as possible.

Other methods of measuring the forces and moments on complete model aeroplanes have been devised. At Göttingen† there is a six-wire suspension system in use which enables the three forces and three moments that completely define the force system on an asymmetrical body to be determined at one setting of the model. The total lift is given by the sum of the tensions in three vertical

† *Ergebnisse der Aerodynamischen Versuchsanstalt zu Göttingen*, 4 (1932), 8–12.



Tail Lift Balance.
Forward Lift Balance.
Winding drum.
Drag Balance.
Tail Lift Wire.
Tunnel Roof.
Lift Wires.
Streamline Guard.
Streamline Guard
Wind
Direction.
Vertical Drag Balance Arm.
Sting
Pin Joint.
Strut.

FIG. 54.

wires, the drag by the sum of the tensions in two horizontal wires, and the lateral force by the tension in a third horizontal wire perpendicular to the drag wires. The latter are led forwards in the horizontal plane and are attached to two rings, each of which is tied down by a wire inclined forwards. A third vertical wire passes from each ring to a roof balance, the tension in these vertical wires being defined by the unknown drag and the known or measurable inclination of the wires by which the rings are anchored. A similar arrangement is used for measuring the lateral force. Thus, the three moments and forces can be measured with the model supported entirely by means of a wire suspension. The absence of rigid supports reduces the interference with the natural flow near the model to a minimum.

For special experiments, special balances are often used. Thus an addition to an existing balance at the N.P.L. enabled yawing and rolling moments to be measured on a complete model aeroplane when yawed.† Another system of suspension has been used to measure separately the forces on the wings and bodies of a number of body-wing combinations. Automatically recording balances have been installed in some modern aeronautical laboratories.

108. Drag of stream-line bodies.

The method illustrated in Fig. 54 has been frequently used to determine the drag of stream-line bodies. Two wires, as fine as possible, are attached at the ends of a horizontal diameter of the body near its centre of gravity to form a V in a vertical plane perpendicular to the wind direction, with their upper ends attached to the tunnel roof. A short spike in the tail of the model is attached by a free joint to the end of the arm of a drag balance placed either above or below the tunnel. Wire drag is measured by attaching two extra wires to the body, at the same point as the main supporting wires, and taking them to the floor of the tunnel so as to form a V of the same dimensions as that formed by the supporting wires. It is arguable that the interference of two wires attached at the same point of the model but leaving it in different directions might not be twice that of one wire, and that there might be an error in the drag correction in consequence. This was examined at one time by

† Lavender, Fewster, and Henderson, *A.R.C. Reports and Memoranda*, No. 822 (1923).

supporting the model also by a single vertical wire and then adding successively one vertical wire underneath, and two forming a V. It was found that the values of the drag of the model alone, measured in several ways involving different wire arrangements, were in reasonably good agreement, and the conclusion was drawn that the standard method of test was reasonably accurate. In the light of later knowledge of boundary layer flow it would be expected that the wires would produce only a small effect due to interference if they were well behind the region of transition to turbulent flow in the boundary layer. This would usually be the fact with normal-sized models in a 7-foot tunnel at high speeds, but not at the lower speeds.

109. Force measurement in the compressed air tunnel.

It will be seen from the above that the general principle followed has been to support the model with as few wires or spindles as possible while allowing it the requisite freedom of movement in the direction of the force component to be measured. In the N.P.L. compressed-air tunnel the procedure is different, and the balance takes the form of a ring-frame surrounding the jet and shielded from stray air-currents. The model is attached to this ring by any convenient system of wiring or spindles, the sole requisite being that the attachment must be rigid so that the model cannot move relatively to the ring-frame. The aerodynamic reactions on the model are thus transferred to the ring-frame, and are determined by measuring successively the moments produced about three parallel horizontal axes perpendicular to the wind direction. The corrections for the drag of the supporting wires or spindles are determined as in other tunnels either by the method of duplication, or by separately supporting the model by wires from the balance guard, whichever is more convenient.

110. Water tanks and whirling arms.

Most of the aerodynamic data of experimental origin available at the present time have been obtained from work carried out in wind tunnels. There are, however, other possible methods of experiment, of which the two that are most widely used are the towing of models through still water or air. In the former method the water is generally contained in a long tank and the model is attached to a carriage

which carries the necessary measuring apparatus, spans the width of the tank, and travels along its length. When models are to be moved through still air, they are usually attached to the end of a long arm capable of rotation about a fixed axis (whirling arm).

Tank tests are used mainly in connexion with the design of ships and the hulls of flying boats or the floats of seaplanes. They have also provided some interesting information on surface friction.† Higher Reynolds numbers can generally be more easily reached in a large tank than in an ordinary wind tunnel, both because the kinematic viscosity of water is only about one-thirteenth of that of air and because larger models can be used.

Whirling arms are not much used to-day, except for special classes of experiments such as fundamental calibrations of anemometers or investigations of the effects of a steady rotation about an axis. The measurement of forces on models carried by a whirling arm is obviously much more difficult than the corresponding measurement in wind tunnel work.

It will be seen that the fundamental difference between the two methods of experiment described, viz. wind tunnel work and towing models through stationary air or water, is that in the one case the relative translational velocity between model and fluid is obtained by moving the fluid and in the other by moving the model. Theoretically, if we allow for effects due to the fact that on the whirling arm the motion of the model is not rectilinear, this difference in technique should make no difference to the fluid forces acting on the model But in practice it is found impossible to generate a wind tunnel stream without imparting turbulence to the air (see above). Turbulence is absent, or at least widely different in character, in the fluid in the tank or the whirling shed. Hence differences in the results may be

† Froude, 'Experiments on the Surface Friction Experienced by a Plane moving through Water', *Report of the British Association, 42nd Meeting* (1872), pp. 118–124. See also *Report of the 44th Meeting* (1874), pp. 249–255.

Gebers, 'Das Ähnlichkeitsgesetz für den Flächenwiderstand im Wasser geradlinig fortbewegter polierter Platten', *Schiffbau*, **22** (1921), 687–690, 713–717, 738–741, 767–770, 791–795, 842–845, 899–902, 928–930.

Kempf, 'Über den Reibungswiderstand von Flächen verschiedener Form', *Proc. 1st Internat. Congress for Applied Mechanics, Delft*, 1924, pp. 439–448.

Perring, 'Some Experiments upon the Skin Friction of Smooth Surfaces', *Trans. Inst. Naval Arch.*, 68 (1926), 91–103.

Kempf, 'Neue Ergebnisse der Widerstandsforschung', *Werft, Reederei, Hafen*, **10** (1929), 234–239, 247–253.

Gebers, 'Einige Versuche über den Einfluss der Flächenform auf den Flächenwiderstand', *Schiffbau*, **34** (1933), 18–20.

anticipated in certain cases where the particular reaction to be measured is sensitive to changes of turbulence, even though the Reynolds numbers of two experiments are identical.

SECTION II
VELOCITY AND PRESSURE MEASUREMENTS

111. The pitot-static tube. Total-head and static-pressure tubes.

The pitot-static tube is the standard instrument for measuring air speed. It consists in effect of two tubes each of which includes a right-angled bend forming two branches, one usually shorter than the other. The head or shorter branch of one of the tubes is aligned with its axis along the wind direction and terminates in an open end. This tube—the total-head or pitot tube—measures the sum of the kinetic and static pressures acting in the air at its open end. The other tube, which in a suitably designed instrument measures the static pressure, is open to the stream through a number of small orifices in the walls of the head, with their axes normal to the axis of the tube, which is itself aligned with the stream. When the open ends of the stems are connected to opposite sides of a differential manometer the kinetic pressure is measured.

In the most convenient form of instrument the two tubes are arranged concentrically, with the static tube outermost, as shown in Fig. 55, which represents the N.P.L. standard pitot-static tube. Another form, shown in Fig. 56, differs from the first mainly in that the thin edge in which the tapered head of the standard terminates is replaced by a hemispherical nose. This facilitates manufacture and also makes the instrument more robust, for a thin edge is rather liable to damage. A round-nosed instrument (see Fig. 57) has also been designed by Prandtl, who has replaced the more usual static holes by an annular slit in the head of the static tube.

It is a matter for experiment whether the differential pressure measured by a pitot-static combination is in fact equal to the kinetic pressure. In general, the relation between the differential pressure and the velocity and density of the fluid can be expressed in the form

$$p = k \cdot \tfrac{1}{2}\rho q^2.$$

where k is a numerical factor which has to be determined by experiment. Apart from its dependence on the compressibility of the

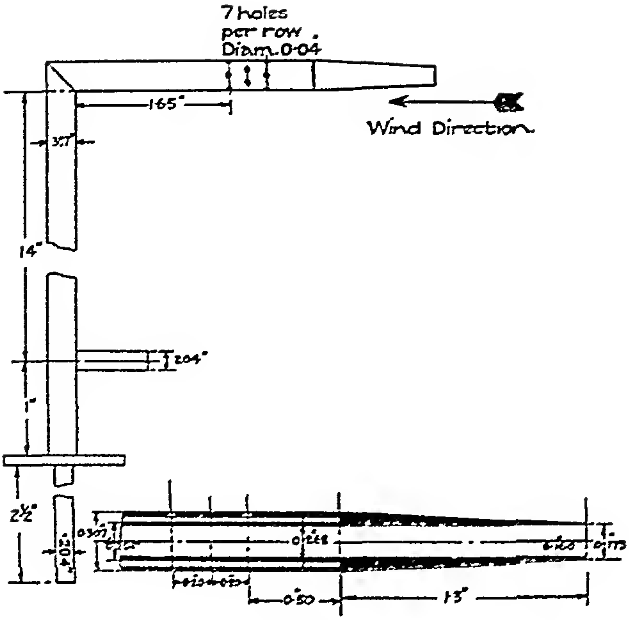

7 holes
per row
Diam. 0·04
·165"
Wind Direction
14"
1"
2½"

FIG. 55.

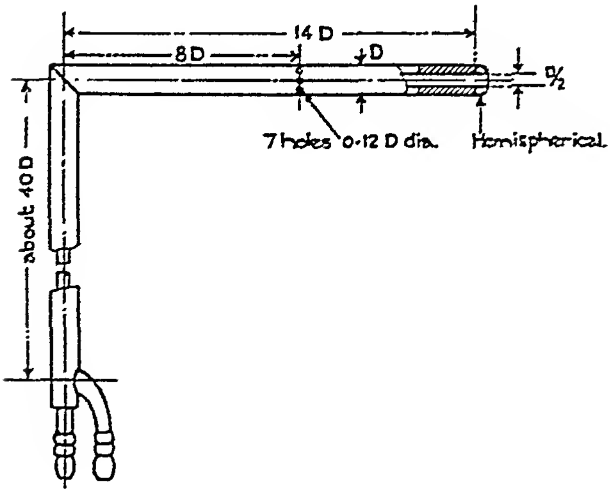

14 D
8 D
D
D/2
7 holes 0·12 D dia.
Hemispherical
about 40 D

FIG. 56.

fluid, which for our purposes may usually be ignored, k will depend on the form of the instrument, on the turbulence in the stream, and on the Reynolds number of the flow past the tube. In those forms of tube which are commonly used k is nearly constant and equal to unity.

A careful calibration of the N.P.L. standard instrument was made at the N.P.L. in 1912.† For this purpose the instrument was attached to the end of a whirling arm (see § 110) of about 30-foot radius and moved through the initially still air in a large shed at speeds ranging

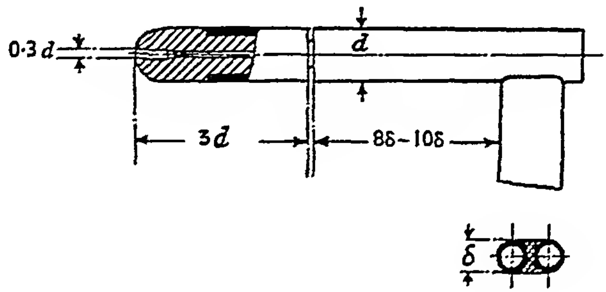


FIG. 57.

from 20 to 60 feet per second. After allowance was made for the 'swirl' speed of the air set up by the motion of the arm, the value of k was found to be unity within $\pm 0{\cdot}1$ per cent. over this speed range.‡ Subsequent experiments in a water tank at equivalent water speeds showed that this value of k could be taken to hold up to air speeds of about 250 feet per second, while more recently a model twenty-five times the linear dimensions of the original was also found to have a factor of 1 between 20 and 90 feet per second. The instrument shown in Fig. 56 has been calibrated in a wind tunnel against the N.P.L. standard and found to have a mean factor of 1·000 over the speed range 20 to 70 feet per second. Prandtl's tube also has a factor very close to 1.||

Calibrations of the two British instruments have been made at air speeds below 20 feet per second.†† Although pitot-static tubes are not often used at such low speeds in view of the difficulty of

† Bramwell, Relf, and Fage, *A.R.C. Reports and Memoranda*, No. 71 (1912).

‡ This close approach to unity must be regarded as fortuitous, for it was not until some years later (see p. 252) that the distribution of pressure along the static tube was investigated experimentally.

|| Kumbruch, *Forschungsarbeiten des Ver. deutsch. Ing.*, No. 240 (1921), 29, 30.

†† Ower and Johansen, *A.R.C. Reports and Memoranda*, No. 1437 (1932); *Proc. Roy. Soc.* A, 136 (1932), 153–175.

measuring accurately the very small kinetic pressures that are set up, yet very sensitive manometers (see footnote †, p. 276) have been designed for this purpose, so that the pitot-static combination can, if desired, be used at a speed of about 2 feet per second with 1 per cent. accuracy on speed. The following table gives the mean value of the factor k for the N.P.L. standard and for the round-nosed instrument over the range 2 to 20 feet per second.

Air speed 15° C. and 760 mm. (ft./sec.)	*Mean k N.P.L. standard*	*Mean k Round-nosed instrument*
2	1·020	1·055
4	0·989	1·006
6	0·995	1·001
8	0·992	0·996
10	0 991	0·992
12	0·992	0·991
14	0 995	0·992
16	0·998	0·996
18	0·999	0·999
20	1·000	1·001

It should be remarked that the accuracy of the results for the round-nosed instrument is probably inferior to that for the standard, but it is considered that the values given will enable an accuracy of 1 per cent. on air speed to be obtained even with this instrument at 2 feet per second. With the standard instrument the accuracy at 3 feet per second is believed to be within 0·5 per cent. on speed.

In use the head of the pitot-static combination has to be aligned with the wind direction. Fig. 58 shows for the N.P.L. standard instrument the variation in kinetic pressure reading with rotation about the axis of the stem,† and it will be seen that the correct position with the tube pointing into the wind ($\theta = 0°$) corresponds to a minimum pressure reading. Hence, provided the wind direction is known approximately (as it generally is even when there is doubt as to the exact direction) the correct presentation of the instrument is easily obtained by means of a search for this pressure minimum. In Prandtl's tube, alignment with the wind direction coincides, according to Kumbruch,‡ with a pressure maximum, but a similar procedure can obviously be adopted.

An experimental investigation of the characteristics of pitot-

† The dotted curve shows the corresponding variation in total-head reading.
‡ *Op. cit*, pp. 7–14.

static tubes was carried out at the N.P.L. in 1925.† The most important outcome of this work was the information it provided to enable the position of the static holes to be adjusted with respect to

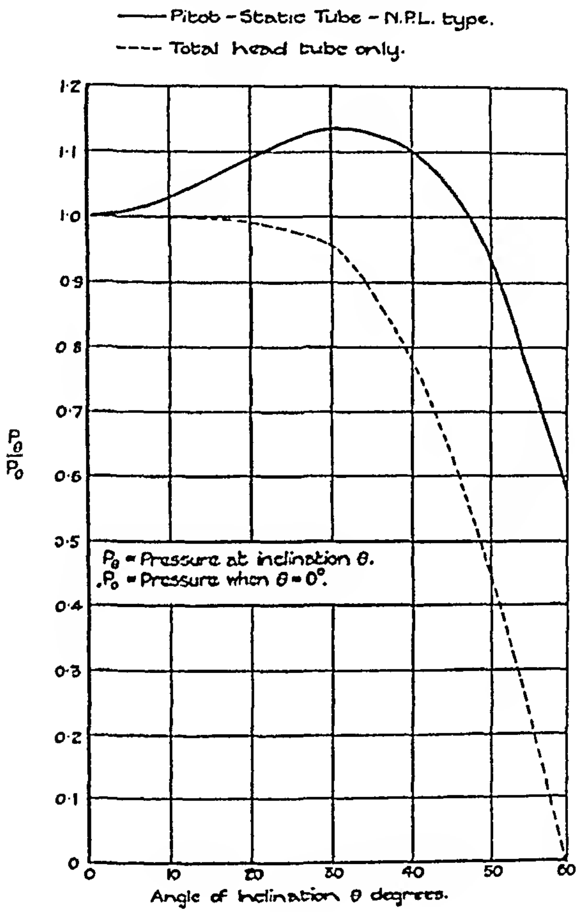


Fig. 58.

the nose and to the stem so that the negative pressure due to the former balances the positive pressure due to the latter and true static pressure is recorded at the static holes. Some previous work by Miss Barker had shown that the total-head tube indicates the true total head provided that qa/ν exceeds 30, where q is the air speed, a the radius of the mouth of the tube, and ν the kinematic

† Ower and Johansen, *A.R.C. Reports and Memoranda*, No. 981 (1926).

viscosity of the air.† Hence the total-head tube of an instrument of the dimensions shown in Figs. 55 and 56, when used in air at ordinary temperatures and pressures, will indicate true total head at all speeds above about 0·7 foot per second, and if in addition the position of the static holes is adjusted in the way just mentioned, it follows that the factor of such an instrument will be unity.

The fundamental calibrations of the pitot-static tube from which the factor k has been determined have invariably been made on whirling arms (§ 110). Thus apart from the swirls set up by the instruments themselves, the motion during calibration has taken place through stationary air. Calibration conditions have therefore been different from those encountered in moving air-streams, where there are, in general, turbulent components of velocity superimposed on the main translational motion. We must therefore consider the effect of these turbulent velocity components on the value of k.

In the first place, it appears that the total-head tube measures $\bar{p}+\frac{1}{2}\rho q^2+\frac{1}{2}\rho q^2$, where $\bar{p}$ is the mean static pressure, q^2 the square of the mean velocity, and q^2 the mean square of the turbulent velocity.‡

The reading of the static-pressure tube gives a measure of the average total pressure inside the tube, and differs from the true average static pressure by a pressure arising from the impact of the fluctuating cross velocities on the tube and its holes. The difference in reading due to this 'impact' pressure depends on the design of the tube, especially on the number, size, and arrangement of the static holes, and on the magnitude and frequency of the cross velocities. If a tube has a large number of small holes equally spaced around its periphery, the reading with the tube aligned in the mean direction of flow is independent of the azimuth position of the holes. It is to be expected that the relation between the reading of the tube $\bar{p}_m$ and the true average static pressure $\bar{p}$ can be written in the form

$$\bar{p}_m = \bar{p}+k_s\rho(v^2+w^2),$$

where k_s is a numerical factor which has a characteristic value for the same tube in turbulent streams of the same kind, and v^2 and w^2 are the mean squares of the cross components v and w of the turbulent

† *Proc. Roy. Soc.* A, **101** (1922), 435–445. According to F. Homann (*Forsch. Ingwes.* 7 (1936), 1–10) qa/ν must exceed 125. Corrections to be applied to measured values for small Reynolds numbers are given in the papers cited.

‡ For a theoretical discussion see Goldstein, *Proc. Roy. Soc.* A, **155** (1936), 570, 571.

velocity. Certain theoretical arguments indicate that k_s might be expected to have the value $\frac{1}{4}$, at any rate for isotropic turbulence,† but a reliable prediction of k_s can be obtained only by recourse to experiment. Fage‡ has determined k_s from values of $\bar{p}_m$, v^2 and w^2 measured in turbulent flow in two sets of pipes having in the one case a circular and in the other a very elongated rectangular cross-section (so that in the second case the flow was very nearly two-dimensional). In these cases theoretical relations for $\bar{p}$ in terms of ρv^2 and ρw^2 are known, on the assumption that the stresses due to viscosity are small compared with the Reynolds apparent stresses.

The relations are, for the flat rectangular pipe,

$$\bar{p}/\rho+v^2 = \text{constant},$$

where v^2 is the mean square of the turbulent velocity component at right angles to the wider wall (Chap. V, equation (5)); and for the circular pipe

$$r\frac{\partial}{\partial r}(\bar{p}/\rho+v_r^2) = v_\theta^2-v_r^2,$$

where v_r^2, v_θ^2 are the mean squares of the radial and circumferential turbulent velocity components at a distance r from the axis of the pipe (Chap. V, the first equation of § 71).

The conclusion obtained from Fage's experiments is that the reading of a static-pressure tube in fully developed turbulence exceeds the true static pressure by an amount given by $\frac{1}{4}\rho(v^2+w^2)$. In isotropic turbulent flow $v^2 = w^2$ and the measured pressure exceeds the true pressure by $\frac{1}{2}\rho v^2$ or $\frac{1}{6}\rho q^2$. Hence the differential pressure will be $\frac{1}{2}\rho q^2(1+\frac{2}{3}q^2/q^2)$.

112. Small total-head tubes.

Since very small combined pitot-static tubes are obviously difficult to construct, small total-head tubes are frequently employed for detailed explorations of the flow in the boundary layers of bodies such as cylinders or stream-line solids of revolution, and in transverse sections of pipes. The static pressure, a knowledge of which is required if velocities are to be deduced from the observations, is measured at holes in the surface of the body (see § 113) or in the

† Goldstein, *Proc. Roy. Soc.* A, 155 (1936), 571–575.

‡ *Ibid.*, pp. 576–596. Kumbruch has investigated the effect of large disturbances: *op. cit.*, pp. 19–24.

walls of the pipe. In the case of boundary layer flow along bodies possessing curvature in the direction of motion, the assumption is made that the static pressure is constant through the layer along any normal to the surface, an assumption that ceases to be valid for practical purposes only if the curvature is rapid or the boundary layer unduly thick. Total-head tubes used for explorations of this kind are generally made of hypodermic tubing. nickel being a more suitable metal than steel as it is not liable to become choked by rust after long use. Usually a small diameter of tube is required, and the only limitation to be observed in this respect is that Miss Barker's criterion, $qa/\nu > 30$ (see p. 252), is still fulfilled at the lowest speed it is proposed to measure in any particular case. The tubes are best operated by means of a micrometer arrangement which, for accurate work, should be carried by the body or the pipe itself. In this way the distance of the point of measurement from the surface of the body is easily determined from the micrometer reading and a single measurement of the distance corresponding to one particular reading. The static-pressure observation should be made before the total-head tube is in place, or, at all events, when it is sufficiently distant not to influence the pressure at the surface hole. This is particularly important in measurements of the velocity distribution in pipes of small diameter.

113. Measurement of the distribution of normal pressure on solid bodies.

It has been mentioned in § 112 that the static pressure at a point in the wall of a pipe or in the surface of a body can be measured by boring a small hole in the surface at the point in question and connecting a tube to it by which the pressure acting there is conveyed to a manometer. Such pressure holes should be small in diameter, particularly if they are located in regions of large pressure gradient. About 0·01 to 0·02 inch diameter is a useful average size. The edges of the hole should be flush with the surface at which the pressure is being measured,—it is very important that no protruding burrs be left,—and the axis should be approximately perpendicular to the surface. By using a number of such holes at various points on the body and measuring the pressure at each, the pressure distribution on the surface can be obtained with an accuracy depending on the number of holes.

A convenient practical method of forming the holes in the surface is as follows:—Several soft metal tubes about 0·05 inch internal diameter—'compo' tubing—are let into grooves cut in the surface of the model so that their outer surfaces protrude slightly above that of the model. They are held in place by wax run into the grooves in a molten state, and the whole is then made good by scraping to preserve the designed contours of the model. The tubes are soft and thick-walled, so that there is no difficulty in scraping their slightly projecting exteriors flush with the model surface. One end of each tube is sealed, the other being open and connected by rubber tubing to a manometer. The pressure on the surface of the model at any point along the length of the tube can then be obtained by piercing a hole with a fine drill or needle in the wall of the tube at the point in question. When the pressure there has been observed the hole can be sealed with a special grease mixture or covered with a small piece of thin paper stuck over it, and the pressure at any other point along the tube can then be measured in the same way.

It should perhaps be mentioned that in such work the pressures are almost invariably measured as differences from the static pressure at some conveniently situated section of the tunnel, which is taken as the datum pressure.

114. Forces due to normal pressures.

For a cylinder the lift and drag per unit length due to normal pressures only are given by

$$L = \int p\,dx, \qquad D = \int p\,dy,$$

where the integrations are taken right round the contour of the cylinder. These are the areas of the curves obtained by plotting p against x and y respectively right round the contour. The former gives a single closed curve, the latter two loops whose areas are to be counted of opposite sign. Fig. 59 shows the pressure distribution for an aerofoil plotted in this way.

Another application of the same method arises in the determination of the form drag of stream-line solids of revolution with their axes along the wind direction. In such cases the pressure holes are distributed along a generator, and the form drag—i.e. the resultant drag due to pressure distribution only—is obtained by graphical evaluation of the integral $\pi \int p\, d(r^2)$, where p is the pressure

measured at any pressure hole, r is the radius of the body at the hole, and the integration is along the length of the body. The percentage accuracy of the form drag of a stream-line body obtained

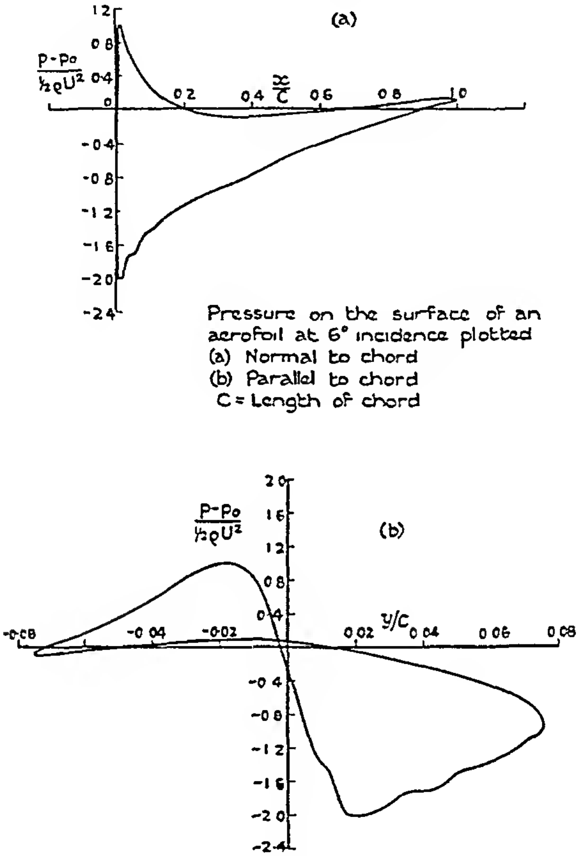


FIG. 59.

in this way is generally poor unless a very large number of pressure holes is used. The form drag of a stream-line body is generally small, so that the two loops of the curve of p against r^2 are usually of very nearly the same area. It is therefore difficult to obtain much accuracy in estimating the difference of these areas.

115. Prediction of drag from wake measurements.

The usual method of measuring the total drag of a stream-line body by balance measurements was referred to in § 108, but this force

can also be determined directly from total-head losses in the wake, measured at a section at right angles to the general direction of motion and in a region where the velocity changes due to the body are small.† This method has a number of advantages over direct force measurements: it can be used in flight on aeroplanes or airships; there is little interference from wires or supports; and it can be used to find the drag for two-dimensional motion unaffected by end effects.

For an aeroplane wing in flight, measurements have to be taken fairly close to the trailing edge, where the velocity changes are not small, and a method of predicting the drag for this more general case has been developed by Betz.‡ The flow between two parallel planes at right angles to the direction of motion, one in front and the other behind the wing, is considered. The velocities and pressures in the front plane are denoted by U_1, V_1, W_1 and p_1 respectively, and the corresponding values in the rear plane by U_2, V_2, W_2 and p_2. At infinity $U = U_0$, $V = W = 0$, and $p = p_0$. The relation for drag obtained from the momentum equation is

$$D = \iint (p_1+\rho U_1^2)\,dS - \iint (p_2+\rho U_2^2)\,dS, \tag{1}$$

where the integrals are surface integrals taken over the entire area of the infinite planes. The merit of Betz's analysis is that he transforms these integrals so that the integration is restricted to the wake.

The total-head relations

$$H_1 = p_1+\tfrac{1}{2}\rho(U_1^2+V_1^2+W_1^2)$$

and

$$H_2 = p_2+\tfrac{1}{2}\rho(U_2^2+V_2^2+W_2^2)$$

are introduced, and on substitution in (1) the drag relation becomes

$$D = \iint (H_1-H_2)\,dS + \tfrac{1}{2}\rho \iint (U_1^2-U_2^2)\,dS + \tfrac{1}{2}\rho \iint \{(V_2^2+W_2^2)-(V_1^2+W_1^2)\}\,dS. \tag{2}$$

The total head is constant along a stream-line when the effects of viscosity and apparent friction due to turbulence can be neglected, so (H_1-H_2) is zero except in the wake, and the first integral is confined

† Taylor, *Phil. Trans. Roy. Soc.* A, **225** (1925), 238–241; also Fage and Jones, *Proc. Roy. Soc.* A, **111** (1926), 592–603.

‡ *Zeitschr. f. Flugtechn. u. Motorluftschiffahrt*, 16 (1925), 42–44. See also Prandtl, *Aerodynamic Theory* (edited by Durand), 3 (Berlin, 1935), 202–206.

to the wake. To transform the second integral a hypothetical flow is taken, which is the same everywhere as the actual flow except in the wake. In the wake the total head is taken to be H_1, the same as that in the undisturbed stream, the pressure to be p_2, the same as that in the actual flow, and the x-component of the velocity is denoted by U_2^*. The assumption made implies the existence of a distribution of sources at the body and in the wake ahead of the rear measurement plane, of total strength $Q = \iint^{T} (U_2^* - U_2)\,dS$: the letter T denotes that the integration is confined to the wake, since everywhere outside $U_2^* = U_2$.† The second integral can be written

$$\tfrac{1}{2}\rho \iint (U_1^2 - U_2^2)\,dS = \tfrac{1}{2}\rho \iint (U_1^2 - U_2^{*2})\,dS + \tfrac{1}{2}\rho \iint^{T} (U_2^{*2} - U_2^2)\,dS,$$

and, on the assumptions made, it may be shown by the theory of the stream-line flow of an inviscid fluid that

$$\tfrac{1}{2}\rho \iint (U_1^2 - U_2^{*2})\,dS = -\rho Q U_0 = -\rho U_0 \iint^{T} (U_2^* - U_2)\,dS.$$

The second integral then becomes $\tfrac{1}{2}\rho \iint^{T} (U_2^* - U_2)(U_2^* + U_2 - 2U_0)\,dS$, and the drag relation (2) reduces to

$$D = \iint^{T} (H_1 - H_2)\,dS + \tfrac{1}{2}\rho \iint^{T} (U_2^* - U_2)(U_2^* + U_2 - 2U_0)\,dS$$
$$+ \tfrac{1}{2}\rho \iint \{(V_2^2 + W_2^2) - (V_1^2 + W_1^2)\}\,dS. \quad (3)$$

For the case of an aerofoil of finite span, the sum of the first two integrals gives the profile drag and the third integral gives the induced drag.

An experimental determination of profile drag by Betz's method involves therefore measurements of both total head and static pressure in the wake. From these measurements the values of U_2 and U_2^* can be deduced from the relations

$$U_2 = \sqrt{\left(\frac{2(H_2 - p_2)}{\rho}\right)}, \qquad U_2^* = \sqrt{\left(\frac{2(H_1 - p_2)}{\rho}\right)}.$$

A value of the profile drag can then be determined graphically from

† $\iint^{T} (U_2^* - U_2)\,dS$ will generally have different values at different sections of the wake, so in the hypothetical flow there must also be sources downstream of the measurement plane. The interaction between these sources and those upstream is simply neglected in Betz's method.

the sum of the first two integrals of relation (3). The contribution of the second integral is of the nature of a correction, which is small when the section taken is at great distance from the body, but is important near the body. The method has been used by Weidinger† and by M. Schrenk‡ to measure the profile drag of an aeroplane wing in flight.

An alternative formula in terms of the total head and static pressure in the wake has been obtained and used by B. M. Jones|| to obtain the profile drag of an aeroplane wing in flight. For an isolated stream-line body, experiencing a drag with no lift, a plane normal to the undisturbed velocity, U_0, can be taken far behind the body where the pressure is sensibly uniform and equal to the pressure in the undisturbed stream, p_0, the velocity being equal to U_0 everywhere in the plane except in the wake. If U_3 is the velocity in the wake and dS_3 an element of area in this plane, the drag, D, is given by the equation

$$D = \rho \iint U_3(U_0 - U_3)\, dS_3, \tag{4}$$

where the integration is taken over the wake. This relation gives a reliable measurement of drag when applied to a real fluid with a turbulent wake. In flight experiments, however, it is necessary to mount the measuring apparatus fairly close behind the wing in a plane where the static pressure is not equal to the undisturbed pressure. The wake, assumed for the moment to be non-turbulent, can be divided into stream tubes, stretching from the far distant plane to the plane of exploration close behind the body. If dS_2 denotes the element of area cut off from the latter plane by a tube, q_2 the velocity of flow through the element, and ϑ the inclination of the velocity to the perpendicular to the plane, the drag is given by

$$D = \rho \iint q_2 \cos\vartheta (U_0 - U_3)\, dS_2. \tag{5}$$

With no loss of total head in the tube of flow between the two planes all the quantities on the right-hand side of (5) can be determined from measurements taken in the plane close behind the body. In practice the flow in the wake is turbulent, and it is assumed that differences between the real flow and the assumed hypothetical flow

† *Jahr. Wiss. Gesellsch. Luftfahrt* (Munich, 1926), p. 112.

‡ *Luftfahrtforschung*, 2 (1928), 1–32.

|| 'Measurement of Profile Drag by the Pitot-Traverse Method' by the Cambridge University Aeronautics Laboratory, *A.R.C. Reports and Memoranda*, No. 1688 (1936).

do not affect the drag.† It is also assumed that the angle ϑ is small so that $\cos\vartheta$ can be taken as unity. It should be noted that when the mean direction of flow in the wake is inclined to the direction of the undisturbed stream, as in flow behind a wing exerting a lift, the measurement plane should be taken perpendicular to the mean direction of flow and not to the direction of the undisturbed stream. The relation (5) then gives the profile drag. For a stream-line body the assumption $\cos\vartheta = 1$ is probably sufficiently accurate, but close behind a bluff body may be a source of considerable error.

If H_2 and p_2 are the total head and pressure in the wake at the measurement plane,

$$H_2 = \tfrac{1}{2}\rho q_2^2 + p_2 = \tfrac{1}{2}\rho U_3^2 + p_0.$$

Write
$$g = \left[1 - \frac{(H_0 - H_2)}{\frac{1}{2}\rho U_0^2}\right].$$

Then
$$\frac{U_3}{U_0} = g^{\frac{1}{2}}, \qquad \frac{q_2}{U_0} = \left[g - \frac{(p_2 - p_0)}{\frac{1}{2}\rho U_0^2}\right]^{\frac{1}{2}},$$

and relations (4) and (5) reduce to the forms

$$D = \tfrac{1}{2}\rho U_0^2 \iint 2g^{\frac{1}{2}}(1 - g^{\frac{1}{2}})\, dS_3 \tag{6}$$

and
$$D = \tfrac{1}{2}\rho U_0^2 \iint 2\left[g - \frac{(p_2 - p_0)}{\frac{1}{2}\rho U_0^2}\right]^{\frac{1}{2}}[1 - g^{\frac{1}{2}}]\, dS_2 \tag{7}$$

respectively. Jones's relation (7) differs from Betz's relation (3) (in which the first two terms only are to be taken), but for the condition in which the method is likely to be used in practice the two relations become identical to first-order accuracy. The integrands may, in fact, be expanded in powers of $(p_2 - p_0)/(\frac{1}{2}\rho U_0^2 g)$, and are found to be identical as regards the first two terms of the expansion: they differ in the term involving the square of $(p_2 - p_0)/(\frac{1}{2}\rho U_0^2 g)$.

Values of the profile drag obtained at Cambridge from measurements at four distances behind a smooth aeroplane wing, and calculated according to relations (3), (6) and (7), respectively, are compared in Fig. 60.‡ It is seen that the differences between the drags given by Jones's relation (7) and Betz's relation (3) are negligible except for the observation made very close behind the trailing edge. The drag coefficients which would have been obtained

† Taylor (*A.R.C. Reports and Memoranda*, No. 1808 (1937)) has shown theoretically that in the worst case in Jones's measurements the error due to turbulent mixing is only $1\frac{1}{2}$ per cent.

‡ See footnote ||, p. 260.

if the rise in static pressure behind the wing had been neglected—i.e. from relation (6)—are given for comparison: the results are naturally more accurate the greater the distance behind the wing.

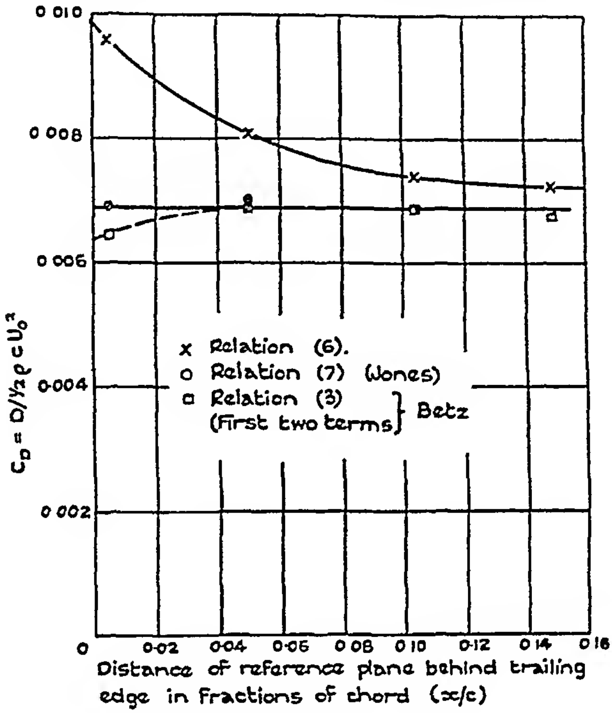


FIG. 60.

At some distance behind the body, g becomes very nearly equal to unity, and (6), which is equivalent to

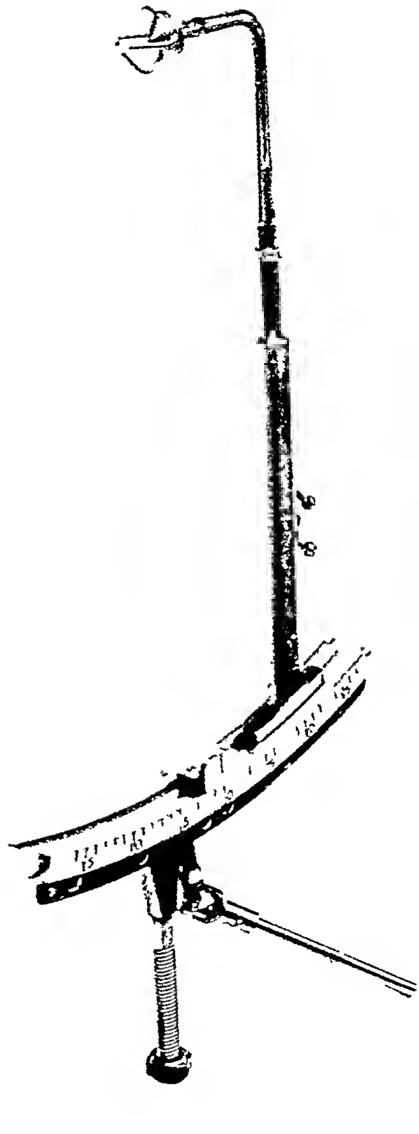
$$D = \tfrac{1}{2}\rho U_0^2 \iint [1-g-(1-g^{\frac{1}{2}})^2]\, dS_3,$$

reduces approximately to

$$D = \tfrac{1}{2}\rho U_0^2 \iint (1-g)\, dS_3 = \iint (H_0-H_2)\, dS_3.$$

Since H_1 in equation (2) can be identified with H_0, this is the result obtained by neglecting the second term in (2). In fact at a section in the wake some distance from the body the second integral of equation (2) is negligibly small, and if there is no lift, the third integral is zero and the drag becomes simply

$$D = \iint (H_1-H_2)\, dS. \tag{8}$$



Fage and Jones† have shown that for an aerofoil at a small incidence the simplified expression (8) gives an accurate measure of the drag when the section is as close to the trailing edge as one chord length.

For the drag of a three-dimensional body of revolution the approximate expression corresponding to (8) is

$$D = 2\pi \int_0^{r_0} (H_1 - H_2) r \, dr,$$

where r_0 is the radius of the edge of the wake,—i.e. the radius at which H_2 is equal to H_1.

116. The determination of wind direction.

The mean wind direction at a small region can be determined by means of a pressure direction-meter.‡ This makes use of the experimental fact that the pressure given by a total-head tube falls off as the axis of the head of the tube is given an increasing inclination to the wind (see Fig. 58). When an inclination of about 45° is reached the rate of change of pressure with angle has reached a value not far short of its maximum, so that the sensitivity of a single total-head tube used as a direction meter is a maximum at about 45° inclination to the wind direction.

The direction and velocity meter shown in Fig. 61 and Pl. 23‡ has two pairs of fine total-head tubes, one pair being in a horizontal plane and the other in the vertical plane. The axes of the two members of each pair converge as shown at an angle of 90° towards their mouths, each tube being thus nearly at the angle of maximum sensitivity to the axis of the instrument. In use, the head is rotated about the axis AB until the pressures in the mouths of the two horizontal tubes 1 and 2 are equal, this condition being indicated on a differential manometer to which the tubes are connected. A rotation is then given about the perpendicular axis CD until tubes 3 and 4 also indicate zero pressure difference. If the instrument were perfectly symmetrically constructed, the wind direction would then lie along the common axis of symmetry of the two pairs of tubes forming the direction-head. Actually, however, there will always be a small error on each pair of tubes, and the instrument is really only used to indicate changes of direction from a direction of reference

† *Proc. Roy. Soc.* A, **111** (1926), 592–603.
‡ Lavender, *A.R.C. Reports and Memoranda*, No. 844 (1923).

which is generally the axis of the wind tunnel.† The wind direction in the empty tunnel is assumed to be along the axis, and a preliminary adjustment of the instrument to zero pressure difference in the empty tunnel (i.e. before the model is in place) serves to establish the zero for direction changes in the plane of each pair of tubes. These direction changes are read off on two angle scales provided for the purpose.

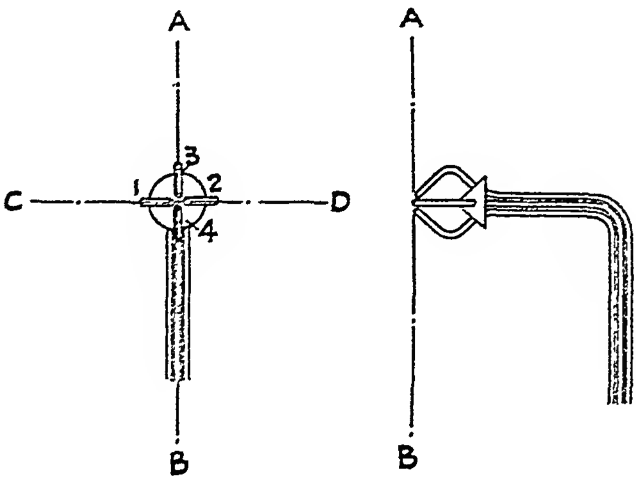


FIG. 61.

Velocity is obtained from a reading of the difference in pressure between the mouth of the fifth central tube and the mouth of any one of the other four, the reading being taken when the head has been adjusted to its null position. In order to increase the velocity reading, tube 5 opens into the space behind a small hollow cone, and is thus in a region of fairly intense suction. A preliminary calibration against a standard pitot tube has to be made to establish the relation between velocity and the differential reading obtained from tube 5 and the selected one of the other four. Tube 5 cannot be seen in Fig. 61; it is a central tube with its opening sheltered by the apex of the cone.

The accuracy of the instrument is about $\frac{1}{10}$ degree on angle and $\frac{1}{2}$ per cent. on velocity.

Hot wire direction-meters are described in § 118 below.

† The instrument would indicate absolute direction if it were reversible. To make it so, however, leads to mechanical complication and the instrument is generally used in the simpler manner described.

117. Electrical methods. The hot wire anemometer.

A type of instrument especially suitable for use as a low-speed anemometer and for recording the speed variations in turbulent flow consists of a fine electrically heated wire (0·001 to 0·005 inch diameter) stretched across the ends of two prongs. When exposed to an air-stream the wire loses heat by convection, and consequently its temperature, and therefore electrical resistance, varies with the speed and current in accordance with a law which can be established by calibration tests. In one method of using the instrument the wire is heated by a constant current and the speed is determined from a measurement of the resistance; in another, the wire is maintained at a constant temperature and the speed is determined from the measured value of the current. Either method can be used for recording low speeds, but the latter is generally used at speeds greater than about 10 feet per second on account of the increased accuracy obtainable.

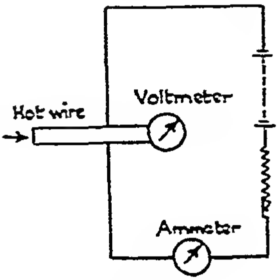


FIG. 62.

The electrical measurements are made with the wire connected to a special circuit. One form, suitable for the constant current method, is shown in Fig. 62. Here the wire is placed in series with a battery and a rheostat, the latter being adjusted to keep the current at a constant average value. A voltmeter is connected across the wire to indicate the potential drop, and so gives a reading which is related to the speed in a manner determinable by calibration. With a fine platinum wire heated to about 500° C. in still air, the method can be used to measure speeds up to 100 feet per second; the accuracy, however, although high at the low end of the range, decreases with increase of speed. If a platinum wire of 0·4 inch length and 0·005 inch diameter, heated by a current of 0·118 ampere, is used, and if measurements of current and voltage are accurate to 1 part in 500, then, from the curve of Fig. 63, it should be possible to determine a speed of 3 feet per second to within ±0·04 foot per second and a speed of 90 feet per second to within ±3·3 feet per second. These figures are not, however, realized in practice, because the calibration characteristic is subject to change, partly through the 'ageing' of the wire and partly through the accumulation of

dust, which affects the thermal conditions at the surface. For this reason hot wires must be frequently calibrated against a standard instrument.

In the second method the hot wire forms one arm of a Wheatstone bridge (Fig. 64) which has in the other three arms resistances such that the bridge is in balance when the temperature, and consequently the

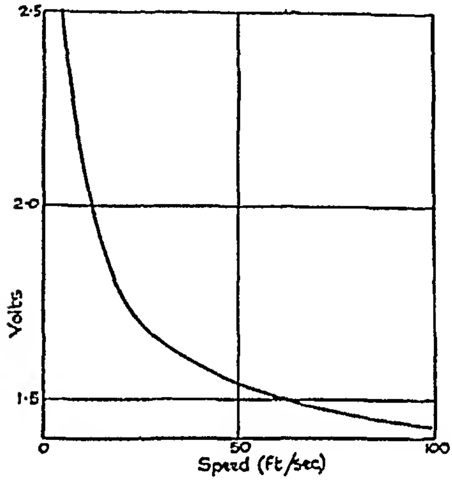


FIG. 63.

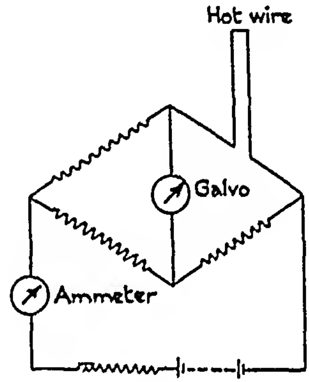


FIG. 64.

resistance, of the wire reaches some desired value. Any change in the speed necessitates an adjustment of the heating current to restore the wire to its former temperature. This is effected by means of a rheostat in series with the battery, so that when the bridge is again in balance the change in current gives a measure of the speed. For most purposes the current in the external circuit is measured, usually with a reflecting pointer type of ammeter, the readings of which can be related to speed by calibration. A typical calibration curve for the platinum wire previously referred to is shown in Fig. 65: from this it is evident that the changes of resistance resulting from the cooling are most marked at low speeds. The estimated limits of accuracy of measurements made at 3 and 90 feet per second are $\pm 0{\cdot}048$ and $\pm 0{\cdot}65$ foot per second respectively. It should be added that, for reasons already stated, these figures are probably unduly favourable.

A more nearly linear calibration curve can be obtained if, in place of the ammeter, a fine wire enclosed in a tube and connected to a

voltmeter† is used for measuring current. On the passage of the current the wire is heated to a high temperature. Any change of current affects the resistance, and so produces a proportionately greater change of potential across the wire. Thus, if the wire is appropriately chosen, it is possible to extend the range of the readings for the higher speeds, and thereby to secure a fairly uniform scale.

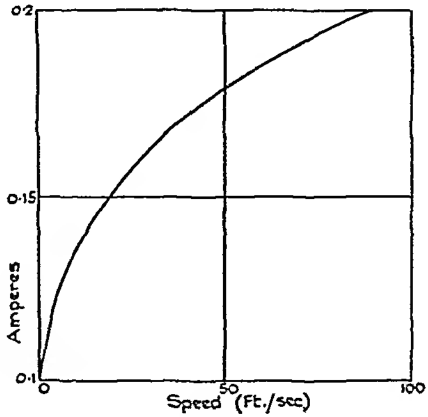


FIG. 65.

118. Electrical methods. Hot wire direction-meters.

Although a hot wire held transversely to the stream is insensitive to direction, a combination of two parallel hot wires placed close together can be used to indicate the direction of flow. The method depends on the fact that the cooling of the second wire is influenced by the wake of the first, and consequently the temperature difference between the two is a maximum when one is shielded by the other. To determine this position the wires are mounted on a support and rotated about a transverse axis until the out-of-balance current of a Wheatstone bridge, of which they constitute two adjacent arms, is a maximum. In general, the accuracy obtainable is not high, being seldom greater than $\pm 0{\cdot}25°$.

The two-wire direction-meter illustrated in Fig. 66 is a more sensitive instrument.‡ It comprises two short inclined wires fused together to the end of a vertical manganin support, the free ends

† Huguenard, Magnan, and Planiol, *Comptes Rendus*, **176** (1923), 287. Also King, *Engineering*, **117** (1924), 136, 249.

‡ Simmons and Bailey, *A.R.C. Reports and Memoranda*, No. 1019 (1926).

being similarly fused, each to a separate support. In use the wires are set roughly in a plane parallel to the flow with the common point upstream. Under these conditions equal wires, heated by the same current, are equally cooled when the line bisecting the angle between them lies along the direction of flow. To find the wind direction the wires are therefore rotated to a position where the temperature of each wire is the same, as shown by the balance of a Wheatstone

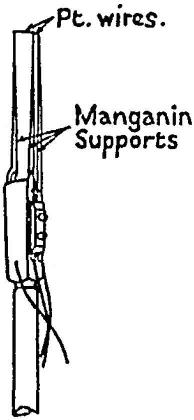


FIG. 66.

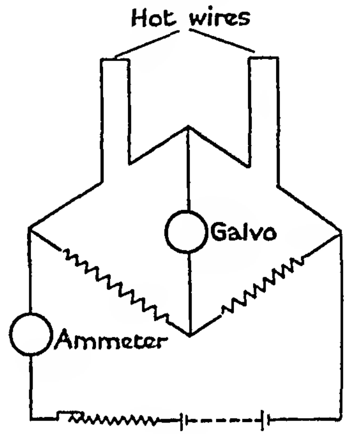


FIG. 67.

bridge, to which they are connected in the manner shown in Fig. 67. If, as usually happens, the wires differ slightly in length, in the null position the axis of symmetry will be inclined at a small angle to the stream. The error arising from this cause can be measured in the empty tunnel, wherein the direction of flow is known, and a correction applied to any subsequent measurements made in the neighbourhood of a model.

119. Electrical methods. Measurements of speed variations in turbulent flow.

A hot wire anemometer held transversely to the stream affords a convenient means of recording the variation of longitudinal speed due to turbulence.† In the method most commonly employed it is connected to a Wheatstone bridge similar to Fig. 64, which is balanced in the usual manner by adjusting the heating current until the galvanometer reading is zero. But though the average potential

† Dryden and Kuethe, *N.A.C.A. Report* No. 320 (1929). Also Mock and Dryden, *ibid.*, No. 448 (1932).

across the wire is thus neutralized, there exists a potential varying with the changes of speed. These fluctuations are too fast to be registered by the ordinary galvanometer, and too small to be measured with an A.C. instrument unless first magnified. The bridge is accordingly coupled to a valve amplifier, and the out-of-balance potential measured in terms of the variations of the output current by a thermo-junction type of ammeter or (in cases where the wave form is required) by the cathode-ray oscillograph. The speed variations are then deduced from the electrical constant of the Wheatstone bridge and the amplifier, coupled with a knowledge of the law of cooling of the wire.

Due allowance must be made for the decrease in the response of the wire at high frequencies, since even with the finest wire (of 0·0001 inch diameter) the amplitude of the potential changes are neither proportional to, nor in phase with, the speed variations at frequencies above about 100 feet per second. Electrical methods of compensating the loss of response have been developed by Dryden and Kuethe,† and, independently, by Ziegler.‡ These, it has been shown, enable a wire to record accurately small changes of speed up to a frequency of 2,000 cycles per second.

Further examples of the applications of the hot wire anemometer are contained in the works cited below.||

120. Electrical methods. Correlation measurements in turbulent flow.

With the aid of two hot wires it is possible to measure the correlation coefficient, R, between the longitudinal turbulent velocity components at two fixed points in a stream. In the method described by Prandtl and Reichardt†† a cathode-ray oscillograph is provided with two pairs of deflexion plates, and the variable potentials across the wires produced by the fluctuating velocities are applied, after magnification, to the plates of the oscillograph, so that at any instant the horizontal and vertical displacements of the beam represent the longitudinal velocity components at the two points. The beam

† *Loc. cit.*

‡ *Proc. Roy. Acad. Sci., Amsterdam*, **34** (1931), 663–672.

|| Ower, *Measurement of Air Flow* (London, 1933), Chap. X (with bibliography on p. 221); Richardson, *Les appareils à fil chaud. Leurs applications dans la mécanique expérimentale des fluides* (Inst. de Mécanique des Fluides, Paris, 1934).

†† *Deutsche Forschung*, Part **21** (1934), 110–121.

traces out an irregular path of varying size and shape, and a photographic record is taken on a plate exposed to the beam for some time. When developed this reveals a darkened area, roughly elliptical in shape; its outline is ill-defined, but a number of ellipses can be constructed whose boundaries connect points of the same optical density. The axes of the ellipses bisect the angles between axes representing horizontal and vertical displacements; and if R is the correlation coefficient between the longitudinal turbulent velocity components, as above, the ratio of the squares of the lengths of the axes of any ellipse is given by

$$\frac{b^2}{a^2} = \frac{1-R}{1+R},$$

so that

$$R = \frac{a^2-b^2}{a^2+b^2}.$$

This method may be used when the hot wires are close enough together for the correlation to be high, i.e. when $1-R$ is fairly small and the ellipses are elongated in shape. For small correlations it is not so suitable as the electrodynamometer method, described below. A description of an alternative, and very convenient, method of measuring values of R in the neighbourhood of unity follows the description of the electrodynamometer method.

In the electrodynamometer method, two hot wires are arranged in Wheatstone bridge circuits. The out-of-balance potentials (proportional to the longitudinal turbulent velocity components, u_1, u_2, at the two wires) are applied to compensated amplifiers, the output currents from which are indicated by a sensitive electrodynamometer. The quantities $\overline{u_1 u_2}$, $\overline{u_1^2}$, and $\overline{u_2^2}$ are measured separately: $\overline{u_1 u_2}$ by the deflexion, δ_1, of the dynamometer when the moving coil is energized by the output current of the first amplifier and the fixed coil by the current from the second amplifier, and $\overline{u_1^2}$, $\overline{u_2^2}$ by the deflexions, δ_2, δ_3, when the coils of the instrument are joined in series so as to measure, in turn, the mean square value of each output current. The coefficient R is then given by the ratio $\delta_1/(\delta_2\,\delta_3)^{\frac{1}{2}}$.

Fig. 68 shows the Wheatstone bridges containing the wires A and B, the amplifiers, and the output circuits arranged for measuring $\overline{u_1 u_2}$. These circuits include the fixed coil of the dynamometer, F, and the moving coil M. The compensating coils M', M'' are each made similar to M, and coils F', F'' similar to F. Accordingly, when

the steady drop of potential across $F''M''$ with a small resistance r in series is balanced against the E.M.F. of the battery E, the proportion of the alternating current output (through F in one case and M in the other) remains unchanged over the probable range of frequencies associated with the type of turbulence under examination. The deflexions δ_2 and δ_3 are then observed immediately after δ_1, and are therefore made with the same degree of amplification.

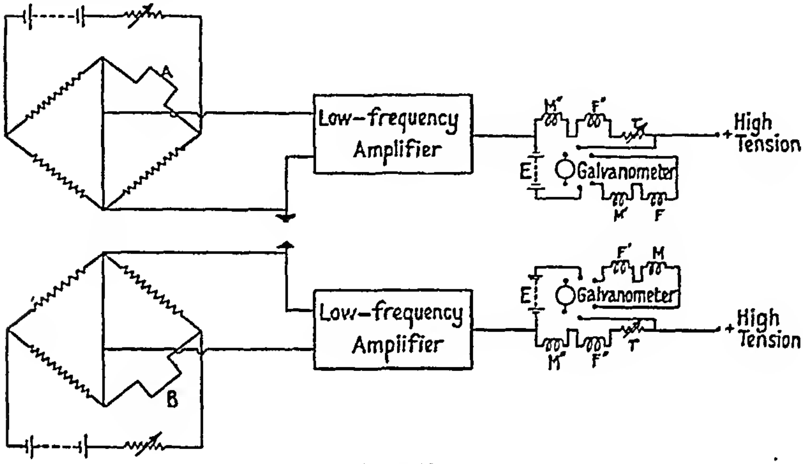


FIG. 68.

When values of R in the neighbourhood of unity are required (i.e. when the wires are near together) $1-R^2$ may be obtained directly as the ratio of two deflexions of a galvanometer by a method due to Taylor.† Two hot wires, A and B, are used in bridge circuits connected together at DF (see Fig. 69). The steady currents through the hot wires are balanced out in the ordinary way (the bridges and galvanometers used are not shown in Fig. 69), and the mean potentials at all points on the resistances CD and FE across the bridge are then identical. Let a speed variation u_1 at A produce a potential difference E_1 between C and D, and similarly a variation u_2 at B produce a potential difference E_2 between E and F. Then

$$R = \frac{\overline{E_1 E_2}}{(\overline{E_1^2})^{\frac{1}{2}}(\overline{E_2^2})^{\frac{1}{2}}}.$$

By means of sliding contacts P and Q any proportion of either potential difference E_1 or E_2 may be applied to an amplifier and

† *Proc. Roy. Soc.* A, 157 (1936), 537–546.

recorded by the output current passed through a thermo-milliameter. If $\alpha = PD/DC$, $\beta = FQ/EF$, the potential difference applied to the input of the amplifier is $\alpha E_1 - \beta E_2$, and its mean square value, to which the deflexion of the galvanometer is proportional, is

$$\alpha^2 \overline{E_1^2} + \beta^2 \overline{E_2^2} - 2\alpha\beta \overline{E_1 E_2}.$$

With P kept fixed in position, Q is adjusted until a minimum

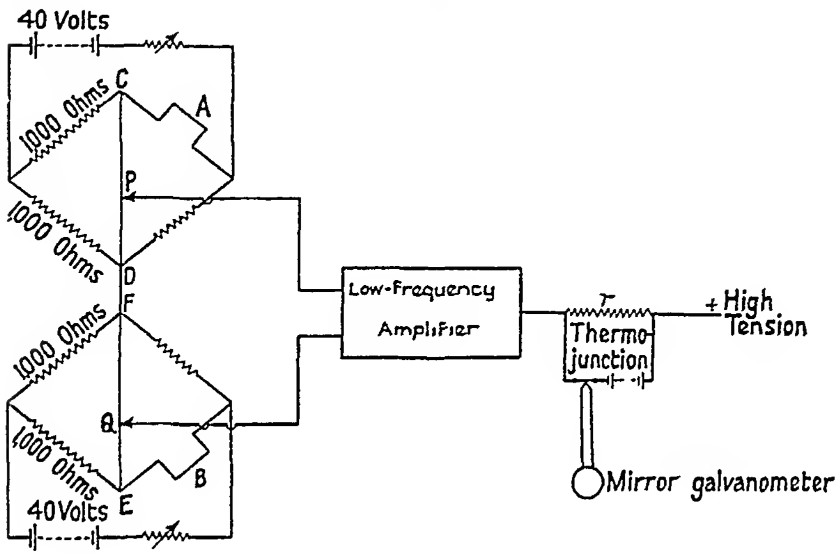


FIG. 69.

deflexion $\delta_{\min}$ is observed. The minimum value, which occurs when

$$\beta = \alpha \frac{\overline{E_1 E_2}}{\overline{E_2^2}},$$

is proportional to

$$\alpha^2 \left\{ \overline{E_1^2} + \frac{(\overline{E_1 E_2})^2}{\overline{E_2^2}} - 2\frac{(\overline{E_1 E_2})^2}{\overline{E_2^2}} \right\}$$

which is equal to $\qquad \alpha^2 \overline{E_1^2}(1-R^2)$.

Finally Q is moved to F (P remaining fixed in position), and the new deflexion δ_2 is observed with the same amplification as before. Then δ_2 is proportional to $\alpha^2 \overline{E_1^2}$, and hence

$$1 - R^2 = \frac{\delta_{\min}}{\delta_2}.$$

Since both deflexions can usually be read to an accuracy of ± 8 per cent., the error made in estimating $1-R$ when, say, $R = 0{\cdot}98$

will not exceed 16 per cent., whereas if R were found by the electro-dynamometer method, an error of 1 per cent. in δ_1, or an error of 2 per cent. in δ_2 or δ_3, would produce an error of 1 per cent. in R and an error of 50 per cent. in $1-R$.

121. Electrical methods. Determination of an energy spectrum in turbulent flow.†

Turbulent motion contains no true periodic components, but (for example) the longitudinal turbulent velocity component u at any point may be subjected to harmonic analysis by Fourier integrals. A definite fraction of the kinetic energy associated with u lies between any given limits of frequency. In this sense we can obtain a spectrum of turbulent motion. By the use of circuits called filter circuits, which cut off all frequencies either above or below a certain definite value, it is possible to find the values, $(\overline{u^2})_{n-}$ and $(\overline{u^2})_{n+}$, of $\overline{u^2}$ when all frequencies above n cycles per second in the first case, or below n cycles per second in the second case, are cut off. If these are divided by $(\overline{u^2})_t$, the value of $\overline{u^2}$ without a filter in circuit, the results are equal to $\int\limits_0^n F(n)\,dn$ or $\int\limits_n^\infty F(n)\,dn$, where $F(n)\,dn$ is the fraction of $(\overline{u^2})_t$ for frequencies between n and $n+dn$. Hence by plotting results against n and finding the slopes of the resulting curves, curves of $F(n)$ against n may be obtained.

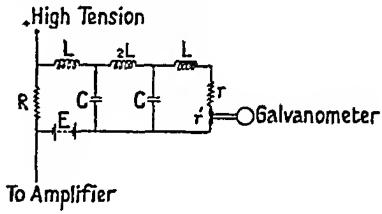


FIG. 70.

The filter circuit is inserted in the output lead of an amplifier used in conjunction with a hot wire. Of the two kinds of filters employed that shown in Fig. 70 passes currents of low frequency, but rejects currents whose frequency exceeds a value which is governed by the inductance L and the capacity C. The filter is placed as a shunt across a resistance R in the anode of the power valve of the amplifier, and the current passing through it is measured in the usual way by a thermo-milliameter. The sum of the resistances r and r' is equal to R and is also equal to the characteristic impedance, and the behaviour of the filter approaches closely to the ideal form which gives a uniform

† For references see p. 233.

response up to the critical frequency and zero response beyond. Values of $(\overline{u^2})_{n-}$ are measured in succession with a series of filters of this kind, designed to cut off at different frequencies, and $(\overline{u^2})_t$ is determined without a filter in circuit.

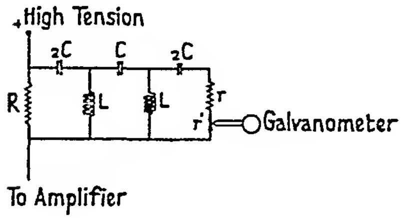


FIG. 71.

The type of filter shown in Fig. 71 produces no attenuation to currents whose frequency is higher than a critical value, and is more satisfactory than the former type for exploring the high frequency end of the turbulence spectrum.

122. Manometers. The Chattock manometer.

The Chattock manometer† is in effect a water U-tube so modified that it enables pressure differences of the order of 0·01 inch of water column to be observed to an accuracy of 1 per cent; that is, the instrument is sensitive to a differential pressure of about 0·0001 inch of water. This degree of sensitivity is achieved as a result of two distinct features. In the first place the pressure difference is not allowed to change the water levels in the two limbs of the U-tube: instead, the tube is tilted in its plane by means of a lever pivoted at one end and operated by a micrometer screw, the tilt given being just sufficient to balance the applied pressure difference. The manner in which balance is indicated is the other feature responsible for the high sensitivity; it can best be explained by reference to Fig. 72, which shows the glass-work of a Chattock manometer. The two cups A and B, which constitute in effect the two vertical limbs of the U-tube, communicate with the central vessel C. Cup A communicates directly with C through its walls, but the tube from cup B enters C from below and passes up the centre as shown, terminating in a ground chamfered end about two-thirds of the height of C from the bottom.

The lower portions of the two cups, the tubes connecting them to the central vessel, and the lower part of the latter itself are filled with distilled water. The remaining space of C is entirely filled with medicinal paraffin admitted from the small reservoir above. Medicinal

† Pannell, *Engineering*, 96 (1913), 343, 344; *A.R.C. Reports and Memoranda*, No. 242 (1915); Duncan, *ibid.* No. 1069 (1927); *Journ. Sci. Insts.* 4 (1927), 376–379.

paraffin does not mix with water, and the levels and the quantities of the two liquids admitted are adjusted so that when the free levels in the two cups are at convenient heights (i.e. about half-way up the cylindrical portions) a surface of separation between the water and the paraffin is formed on the open end of the central vertical tube in *C*. There will be another surface of separation in the annular space surrounding this tube, but that is incidental. The one formed on the end of the tube has the appearance of a bubble when viewed from

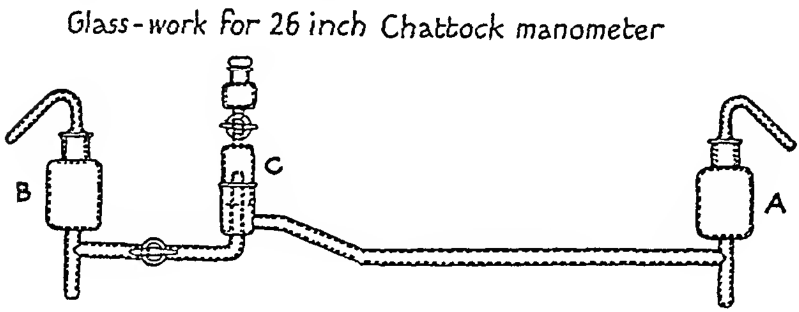


FIG. 72.

outside. If now a pressure difference is applied to the two cups, this bubble tends to become larger or smaller, and its movement, which can be observed by means of a low-power microscope and suitable illumination from behind, is arrested by giving the whole glass-work an appropriate tilt by means of the micrometer. In practice the microscope is carried by the tilting frame on which the glass-work is supported, so that the operation of the instrument merely entails keeping the image of the bubble on a fixed line in the eyepiece of the microscope and observing the applied tilt.

Two types of Chattock manometer are ordinarily employed. These differ only in the distance between the axes of the two cylindrical cups, which is approximately 26 inches in the larger size and 13 inches in the smaller. In both types the length of the lever arm which moves the tilting frame carrying the cups is 10 inches, and the pitch of the micrometer screw is 0·05 inch. About 20 turns of the screw are generally allowed in both types, which gives the larger a pressure range of about 2·6 inches of water and the smaller a range of half this amount.

Manometers having a sensitivity ten or more times that of the

Chattock have been designed for special investigations. For details the reader is referred to the papers cited below.†

123. Manometers. Large-range micromanometers.

It is not practicable to extend the range of the Chattock gauge appreciably without introducing objectionable features. A simple modification of the U-tube principle, however, enables a micromanometer to be made that has a range limited only by the length of micrometer screw that can be cut to the desired accuracy. In this type of instrument the two vertical limbs of the U-tube are connected at their lower ends by a length of flexible rubber tubing, so that one can be moved vertically relatively to the other. One limb is then held fixed, while the other is raised or lowered by means of a micrometer screw on which a special nut travels, the nut carrying the moving limb of the U-tube. A differential pressure applied to the two limbs is balanced by the appropriate vertical displacement, indicated on the micrometer head and scale, of the moving limb. Balance may be indicated in a variety of ways. The fixed limb may take the form of a cup similar to those used on the Chattock gauge, and the moving limb may communicate with it by means of a glass tube passing up the centre, as in the central vessel of the Chattock. If the upper part of the fixed cup is filled with medicinal paraffin, a 'bubble' can be formed on the mouth of the central vertical tube and used to indicate balance in the manner already described. A manometer of this type is in regular use at the N.P.L.‡ It has a range of 4 inches of water and a sensitivity of about 0·001 inch of water. This sensitivity could be increased without difficulty, but is ample for the purpose for which the instrument is used.

Alternatively the moving limb may terminate in an inclined glass tube of adjustable slope, the liquid meniscus being always brought back to a fixed mark etched on this tube. This system has been adopted in an instrument made at the University of Toronto;|| its sensitivity is stated to be 0·0002 inch of water and its range is 10 inches. Other large-range micromanometers are described in the papers cited below.††

† Fry, *Phil. Mag.* (6), 25 (1913), 494–501; Hodgson, *Journ. Sci. Insts.* 6 (1929), 153–156; Ower, *A.R.C. Reports and Memoranda*, No. 1308 (1930); *Phil. Mag.* (7), 10 (1930), 544–551; Falkner, *A.R.C. Reports and Memoranda*, No. 1589 (1934); Reichardt, *Zeitschr. f. Instrumentenkunde*, 55 (1935), 23–33.

‡ *Report of the National Physical Laboratory* (1921), p. 170.

|| Parkin, *Bull. School Engrg. Res., Toronto Univ.*, 2, No. 1 (1921), 49–51.

†† See, for example, Douglas, 'Note on a Large Range Manometer for Wind Tunnel

124. Manometers. The inclined tube manometer.

If one limb of the U-tube is made very large in cross-sectional area compared with the other, virtually all the motion of the liquid takes place in the narrower limb. If, in addition, this limb is inclined at a small angle α to the horizontal the motion is magnified in the ratio $1/\sin\alpha$. Very convenient and robust instruments may be made on this principle; although their sensitivity is not in general as good as that of a micromanometer, being of the order of 0·002 inch of water at 5° slope, it is ample for a variety of purposes. Instruments of this kind require calibration against a fundamental standard (such as a micromanometer whose readings depend only on measurable lengths and liquid density) since the motion of the liquid in the inclined tube is governed not only by its density and the slope of the tube, but also by certain other features whose effects cannot easily be determined directly, such as straightness of tube and surface tension as affected by variations of temperature and bore of tube.

125. Manometers. Multitube manometers.

In work involving measurements of the pressure distributions on the surfaces of bodies (see pp. 255, 256) a great saving of time and labour can often be effected by measuring simultaneously the pressure at a number of points on the surface. For this purpose multitube manometers have been designed. A successful type† consists of a manometer with a number of inclined tubes leading out of a common reservoir containing the manometric liquid—in this case alcohol. Each tube is connected to one of the tubes let into the surface of the model (see pp. 255, 256), while the air space above the liquid level in the reservoir is connected either to the atmosphere, or more usually to a source of static pressure at some convenient place in the wind tunnel. The various refinements and special features of construction embodied in the design confer upon this particular multitube manometer an accuracy approaching that of a Chattock gauge with cup centres 26 inches apart.

126. Surface tubes.

The instruments commonly used for the exploration of the flow in a boundary layer are the small pitot tube and the hot wire velocity-meter.

Work', *A.R.C. Reports and Memoranda*, No. 657 (1920); also 'Micrometer Water and Pressure Gauge', *The Engineer*, **151** (1931), 248.

† Warden, *A.R.C. Reports and Memoranda*, No. 1572 (1934).

Instruments of these types are not capable of measuring the velocity very close to a surface with good accuracy. The difficulty with the pitot tube of the ordinary type arises from the fact that a sufficiently close approach to a surface cannot be obtained for models of the size commonly used in wind tunnel experiments, even when the diameter of the tube is small. When an exceedingly fine hot wire is used, and its temperature is kept sufficiently low to avoid radiation loss, the heat conducted across the thin layer between the hot wire and the

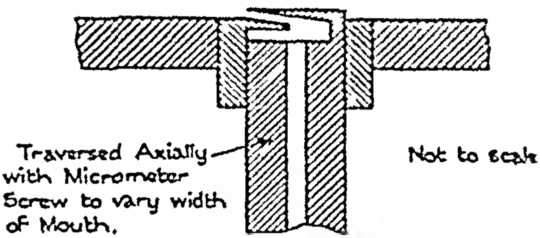


FIG. 73.

surface considerably modifies the forced convection from the wire due to the wind stream.

To allow measurements of velocity to be made very close to a surface Stanton† designed a special form of total-head tube, shown in Fig. 73, which was such that the inner wall of the tube was formed by the surface itself. The width of the opening could be varied by moving the outer wall. Owing to the extreme smallness of the opening of the tube, the speed deduced from the pressure at its mouth is not the same as that at the geometrical centre of the opening. The tube has therefore to be calibrated to determine the position of the 'effective centre' corresponding to the speed calculated from the measured pressure. This calibration is made in a long pipe of rectangular cross-section, with laminar flow at the section at which the tube is placed. The measurements made in the calibration are the pressure drop down the pipe and the difference between the pressure at the mouth of the tube and the static pressure in the pipe. From the first of these measurements, the mean rate of flow through the pipe and the velocity distribution at the surface are calculated from the known relations for stream-line flow. The second measurement gives the velocity at the mouth of the tube

† Stanton, Miss Marshall, and Mrs. Bryant, *Proc. Roy. Soc.* A, 97 (1920), 422–434.

The effective distance corresponding to the velocity at the mouth is obtained from the calculated velocity distribution at the surface.

A particular form of surface tube which has been used to measure the distribution of friction on the surface of an aerofoil† is one in

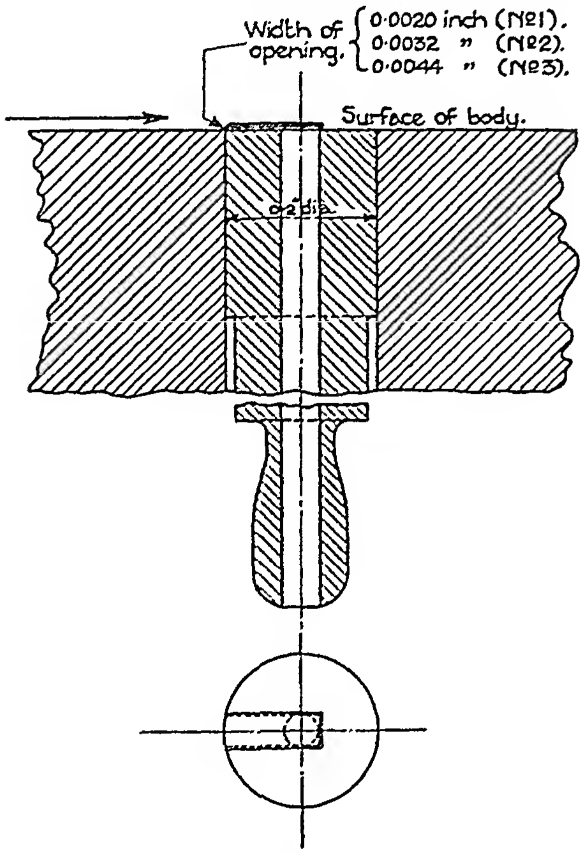


Fig. 74.

which the tube is constructed on the top of a circular rod designed to pass with a very small clearance through holes drilled in the polished surface of a metal model. The tube is mounted with the top surface of the rod flush with the model surface (see Fig. 74). The widths of the openings of three such tubes used in the research mentioned were 0·0020 inch (No. 1), 0·0032 inch (No. 2), and 0·0044

† Fage and Falkner, *Proc. Roy. Soc.* A, **129** (1930), 378–410.

inch (No. 3). A small hole drilled along the rod axis served to transmit the pressure at the mouth of the tube to a manometer.

Data obtained from the calibrations of the three tubes are given in the following table.

(W = *width of the opening of a tube.*)

Velocity calculated from the pressure at the mouth of the tube (ft./sec.)	*Effective distance (inch)*		
	No. 3 W = 0·0044 *in.*	*No.* 2 W = 0·0032 *in.*	*No.* 1 W = 0·0020 *in.*
8	0·00320	0·00320	0·00270
11	0·00298	0·00296	0·00253
14	0·00281	0·00276	0·00238
17	0·00268	0·00258	0·00224
20	0·00255	0·00241	0·00217

The effective centre of tube No. 3 is seen to be within the opening, whereas the effective centre of No. 1 is beyond the outer edge of the opening. The ratio of the effective distance to the width of the opening increases therefore as the width is decreased. It will also be observed that there is an outward movement of the effective centre of each tube as the speed at the mouth is decreased. A very interesting characteristic exhibited is that although the opening of tube No. 1 is less than one-half of that of tube No. 3, yet the effective distance is only about 15 per cent. smaller. Tube No. 1 does not, therefore, allow observations to be taken much closer to the surface than either of the tubes No. 2 or No. 3.

SECTION III

VISUALIZATION AND PHOTOGRAPHY OF FLUID MOTION

127. Stream-lines, filament lines, and particle paths.

Various methods are used to reveal to the eye details of fluid motion, and to enable such details to be photographed and analysed. The particular feature of the flow that is observed or recorded depends on the method of visualization and upon the experimental arrangement. Thus a photographic record of a type of fluid motion may show either stream-lines, filament lines, or particle paths. A filament line is the line joining the instantaneous positions of all particles that have passed through a given point in the fluid, while a particle path is the track of any particle of the fluid. In steady motion any

stream-line is at the same time a filament line and a particle path, but in unsteady motion this is not so. If we imagine a thin jet of smoke introduced at a certain point into a stream of gas or air, or a jet of an opaque liquid into a stream of colourless liquid, then an instantaneous photograph of the flow, taken under suitable illumination, will reveal a filament line.† If, on the other hand, small puffs of smoke are introduced into the air-stream, or small discrete drops or solid particles into the liquid, a photographic exposure occupying a finite interval will show, in the form of streaks, the paths of the smoke puffs or particles during that interval.

128. Miscellaneous methods of examining flow in a boundary layer. Wool tufts; coating the surface; double refraction.

Before considering the more widely used methods of examining flow by visual means, we mention briefly three methods specially designed for examining flow in a boundary layer (of which the third may also be used for other purposes). The first two methods apply to the flow of air, the third to the flow of certain liquids.

We mention first the method of wool tufts or streamers, a method which has valuable practical applications.‡ Light streamers consisting of threads of fine silk or cotton, attached at one end to a wire support or to a surface near which it is desired to explore the flow, will reveal by their behaviour whether there is present turbulence of the kind associated with separation of the boundary layer from the surface or with an eddying wake. This method is very valuable in searching for regions where the flow has separated from the surface, and has been used both in wind tunnels and on actual aeroplanes in flight. Interesting information on the stalling of aerofoils has been obtained in this way.||

In another method of examining air flow near a surface, the surface itself is coated with lead hydroxide, and sulphuretted hydrogen is mixed with the air-stream. A brownish stain develops on the surface where the gas flows along it and indicates the average path of the flow.†† Other combinations of chemical coating and vapour may also be used.

† Actually, since the jet must have a finite thickness, the picture will show a conglomeration of filament lines.

‡ Clark, *A.R.C. Reports and Memoranda*, No. 1552 (1933). Other references are given on p. 10 of that report.

|| Cambridge University Aeronautics Laboratory, *A.R.C. Reports and Memoranda*, No. 1588 (1934).

†† Clark, *loc. cit.*

The third method,† of more recent use, is applicable only to certain viscous liquids, namely those which, when in motion, exhibit the property of double refraction. When such a liquid is flowing past a solid boundary the shearing stresses set up produce an effect on the optical properties, analogous to the effect produced in photo-elasticity on a solid material such as glass or bakelite. Measurements of certain optical constants with polarized light enable the velocity distribution in two-dimensional motion to be calculated.†

129. Air flow. Smoke.

The technique involving the use of smoke for examining air flow depends on the particular problem under investigation. Certain features of slow air currents, for example, may be followed by the aid of tobacco smoke introduced into the stream (see § 127). On the other hand, a more elaborate technique is required for studying the motion of large currents in the upper atmosphere, for which purpose shells are exploded and the drift of the smoke is observed. Again, at an aerodrome the smoke obtained from oil sprayed on a hot plate, by revealing the air flow near the ground, proves useful as a direction indicator. All these are examples in which smoke formed by the incomplete combustion of organic matter is used; but when the finer details of the flow structure are under examination, as in many tunnel investigations, because of the different circumstances a more opaque medium is needed. Coloured gases such as chlorine, bromine, and iodine can be used, if safeguards are provided to protect the observer against their toxic effects. The smoke produced by mixing the gases of ammonia and hydrochloric acid, or the smokes generated by hygroscopic salts like titanium tetrachloride or stannic tetra-chloride on exposure to air, are, however, more suitable. All contain a large percentage of small water particles held in suspension, to which they mainly owe their obscuring powers. At the same time the presence of the water makes them heavier than air, causing them to sink under gravity. Therefore, unless allowance is made for the natural motion, observations taken at the slowest rates of flow are apt to prove misleading. At higher speeds, when the rate of descent is small compared with the forward speed, the indications are more reliable. In these circumstances the extreme ease with which a satisfactory source of supply can be maintained, combined

† Alcock and Sadron, *Physics* (U.S.A.), 6 (1935), 92–95.

with the high optical density, are properties which make chemical smoke especially useful in research.

The advantages mentioned apply more particularly to the use of titanium tetrachloride and stannic tetrachloride. Each is liquid at ordinary temperatures, and if brought into contact with the air combines chemically with the moisture present to form fumes containing the oxide of the salt, hydrochloric acid and water, leaving a solid deposit after evaporation. A drop of the liquid at the end of a glass rod emits a cloud of smoke lasting for several minutes in still air. When the rod is held in a reasonably steady air current the cloud is drawn into a thin trail which remains visible for some distance downstream. Persisting as it does for some time, it serves admirably as a streamer for indicating the general direction of flow, and can therefore be used for mapping the lines of flow around models in wind tunnels, and for locating the eddying regions in the wake. Again, by disclosing the changes in the flow pattern following any alteration in the shape, it can be of service in detecting the interference effects between component parts of a model. In these and in similar problems, where the flow conditions at or near the model are under examination, it is more convenient to generate the smoke from a few drops of liquid placed on the surface. This method is also frequently adopted for investigating the flow in the boundary layer, to indicate the extent of the laminar and turbulent regions and the position at which the layer separates from the surface. Care, of course, is taken to remove the solid deposit left after each application of the liquid, as its presence is likely to cause premature turbulence in the boundary layer. The best results are obtained when the models and the surrounding walls of the tunnel are painted black so that the smoke is always viewed against a dark background.

130. Air flow. The smoke tunnel.

A special form of wind tunnel constructed at the N.P.L. for smoke experiments was fitted with an optical system for projecting an enlarged image of the smoke stream on to a screen, in order to render the motion more easily visible. An improved pattern has been designed by Farren,† primarily for obtaining smoke pictures of the flow past small models at low Reynolds numbers. It comprises a

† *Journ. Roy. Aero. Soc.* 36 (1932), 454–460.

wind tunnel (see Fig. 75) fitted with a honeycomb A, a contracting inlet B and D, and guide vanes at C and F to secure a steady and uniform stream through the working section E, where the model is held. This part has a cross-section 3 inches $\times$ 3 inches and two glass sides 8 inches long through which the beam of light, used to illuminate

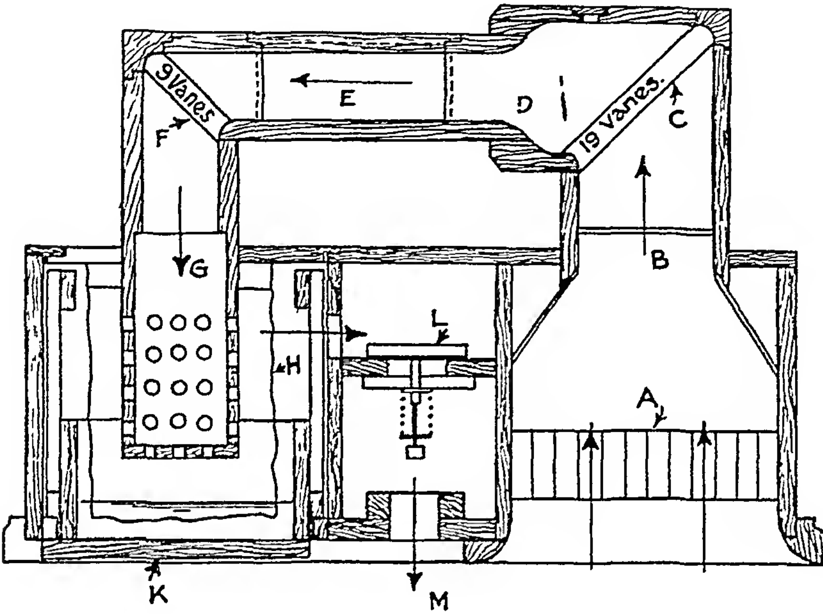


FIG. 75.

the smoke, passes. The flow through the tunnel is maintained by a fan and controlled by valves, one of which, L, can be opened or closed suddenly in order to examine conditions in the neighbourhood of the model when the air flow starts or stops. Titanium tetrachloride, applied on the surface of the model or introduced by a glass rod upstream, is used to make the flow visible. The fact that the smoke contains hydrochloric acid is a great disadvantage; nevertheless, by avoiding as far as possible the use of metal in the construction, the tunnel does not suffer damage. To ensure the steadiest conditions, it is important to exhaust the air into the room instead of into the open. Provision is made for absorbing the acid in the smoke by allowing the air to pass through a gauze curtain, H, surrounding the perforated box G and dipping into the tray K containing a weak solution of ammonia, with the result that the stream emerging from

the outlet M on its way to the exhaust fan is hardly more objectionable than tobacco smoke.

Details of the optical system are shown in Fig. 76. Light from a 250-watt metal filament lamp, A, passes through a condensing lens, B, and then to a larger condensing lens, C, supported in contact with one of the glass sides of the working section E. A good quality lens D of about 3 inches aperture is situated on the far side of the tunnel and projects an image of the model and the smoke on to the screen. No difficulty is experienced in obtaining a sharp silhouette of the

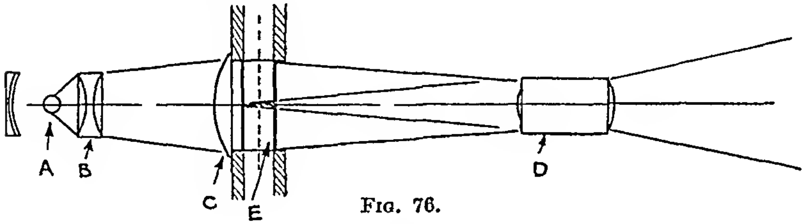


FIG. 76.

smoke streams lying in the centre plane of the tunnel, though, owing to its width, the image of the model itself is generally out of focus.

The tunnel can be used to demonstrate some of the fundamental features of fluid motion, such as the change in the character of the flow round circular cylinders between Reynolds numbers of 10 and 1,000, to quote one example. It is also useful for research purposes, constituting a valuable auxiliary to other methods of investigation. One drawback is that tests can only be made at speeds lower than 5 feet a second, since, owing to the vigorous breaking up and mixing of the smoke, it is impossible to follow anything in the nature of turbulent flow at higher speeds. This, added to the small size of the tunnel, restricts its use to Reynolds numbers very considerably less than those of general interest in practical aeronautics. Nevertheless, a critical study of the features of the flow revealed by these small-scale experiments can sometimes afford valuable help in the design of full-scale aircraft, particularly in locating sources of high drag due to breakdown of flow.

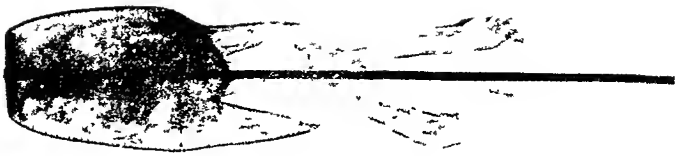
131. Air flow. Smoke photography: low and high speeds.

While much valuable information can be obtained by observing air flow in the manner already described, it is impossible, by inspection alone, to detect the finer details of eddying motion. Photography proves an invaluable aid for this purpose by providing permanent

records which can be examined at leisure. Attention is drawn to some of the more important methods in use, which, for convenience, are described separately according as they are adapted for photographing air moving at low speeds or at high speeds.

(a) *Low speeds*. The smoke tunnel needs little adaptation to make it suitable for photographic purposes. Instead of the screen and projecting lens, all that is required is a camera fitted with a wide aperture lens together with a subsidiary lens which concentrates a beam of light after it passes through the tunnel, so that when the camera is focused on the central plane it receives the maximum quantity of light. As is to be expected, the best results are obtained in cases where the model extends from wall to wall, the flow at the mid-section being approximately two-dimensional. Successful photographs of the eddying motion in the wakes of cylinders, aerofoils, etc., with time exposures of one-hundredth of a second, have been produced in this way, as well as kinematograph films showing the successive stages of development of the eddies. The records appear identical with those taken at the same Reynolds number in water. Flow pictures at low speeds but at higher Reynolds numbers are obtainable by the same method in ordinary wind tunnels, larger models being used for the purpose. As before, the highest speed at which photographs can be taken is determined by the rate at which the smoke can be supplied. An abundant supply can be secured by blowing air through titanium tetrachloride contained in a flask, but the presence of the tube used for conveying the smoke into the tunnel upstream of the model introduces disturbances which make the method unsuitable for many investigations. Some success has attended efforts to maintain a continuous supply of liquid on the model by means of a tube having its open end flush with the surface. In such cases it is found advisable to add an equal volume of carbon tetrachloride, as the mixture is then less liable to block the mouth of the tube by leaving a solid residue projecting above the surface. The smoke produced by this process, though less effective than that of the undiluted liquid, provides sufficient contrast for photographs taken with relatively long exposures.

Instantaneous photographs are generally more useful for studying vortex motion, because of the improved definition. The technique required is somewhat different from that previously described, since a mechanical shutter cannot give the extremely short exposure



a. Disk normal to wind: $D = 0{\cdot}6$ inch, $V = 1{\cdot}25$ feet per second

b. Smoke jet at a wind speed of 40 feet per second

necessary. Photographs are therefore taken in a dark room with an open camera exposed to a brilliant light lasting for a brief period of time. The light is produced by the spark discharge of a condenser, placed on the far side of the wind tunnel immediately opposite the camera. A copy of a photograph taken at the N.P.L.† with a spark lasting less than one-millionth of a second for the purpose of tracing the regions of vorticity generated by a disk, is reproduced in Pl. 24*a*. Such photographs are easily taken by charging an oil insulated condenser of 1 microfarad capacity until the voltage is high enough to cause a spark to jump the small gap between two strips of magnesium ribbon connected across it. A convenient method of charging the condenser is by means of a Ruhmkorff coil, the secondary of which is joined in series with the condenser and to the plate of a 500-watt power valve, the latter serving as a diode to rectify the alternating current in the secondary coil. The time taken before the spark occurs varies with the condition of the points; and though magnesium gives a light rich in actinic value, it oxidizes fairly rapidly, and in consequence the width of the gap changes. Usually, however, it is possible to arrange for the spark to take place from one to one and a half minutes after the coil is started. At low wind speeds this interval is sufficiently long to enable the liquid to be dropped on to the model before the plate is exposed to the flash; but at speeds above about 10 feet per second the smoke disperses too quickly to leave an adequate margin between the application of the liquid and the occurrence of the spark.

(*b*) *High speeds*. An adequate supply of smoke for the photography of air flow at high speeds cannot easily be maintained without disturbances being introduced into the flow in the form of eddies generated by the tube carrying the smoke into the stream, or without the speed of the smoke issuing from the tube exceeding that of the neighbouring air flow. In some problems the disturbances have little effect. A good example concerns the correct shaping of the roof of a building, in order to reduce the extent of the eddying region over the top. Here it is possible to examine the conditions of flow from photographs recording the path of a smoke stream as it issues from a tube some distance upstream and passes over the model.

Ammonium chloride smoke used in one set of experiments‡ was

† Simmons and Dewey, *A.R.C. Reports and Memoranda*, No. 1334 (1931).
‡ Bryant and Williams, *ibid.* No. 962 (1925).

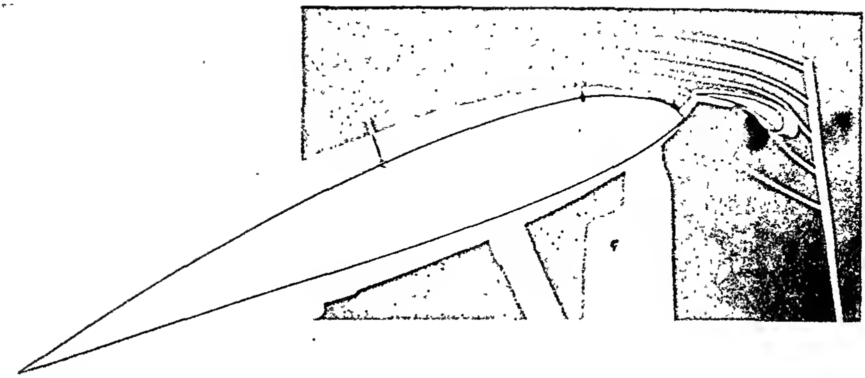
prepared in a flask by the mixing of two air streams, one saturated with hydrochloric acid vapour, the other with ammonia. A pipe connected the flask to a small open-ended tube facing the model; and by means of compressed air a stream of smoke was injected into the wind tunnel, the rate of the supply being adjusted until the jet could be clearly seen after it was deflected by the model. Any change in the rate could easily be detected by the appearance of the stream; for the smoke rapidly became invisible when the speed was too low, and had a feathery appearance when it was too high. The illumination was provided by two arc lamps, and photographs were taken by reflected light, up to wind speeds of 60 feet per second, with a camera mounted so that the optical axis coincided with a line passing through the upper edge of the model. Most of the exposures given were between 2 and 5 seconds, according to the density of the smoke. Thus the photographs recorded the average shape of the discontinuous boundary of the eddying region, but gave no indication of the changes that occur from time to time within that region.

Instantaneous photographs of the jet cannot be taken by the direct illumination of a spark placed behind it and on a level with the camera: attempts to do so invariably lead to negatives which show no trace of the jet. Satisfactory results can, however, be obtained if the spark is placed in a position where the smoke reflects light into the camera and so produces a bright image on the plate. The underlying principle is the same as that whereby smoke, introduced into a beam of light in a dark room, is seen best when the line of vision is inclined at about 45° to the beam. From this it follows that the spark should always be placed on the far side of the smoke, on one or other of two lines inclined at 45° to the axis of the camera. When circumstances permit, it should be held within a few inches of the smoke, with a screen supported near it to intercept the rays which would otherwise enter the camera without illuminating the smoke. Pl. 24*b*† is reproduced from a spark photograph taken with a wide aperture lens ($f = 2{\cdot}8$), and illustrates the billowy appearance of a smoke jet projected into an air current moving at a speed of 40 feet per second.

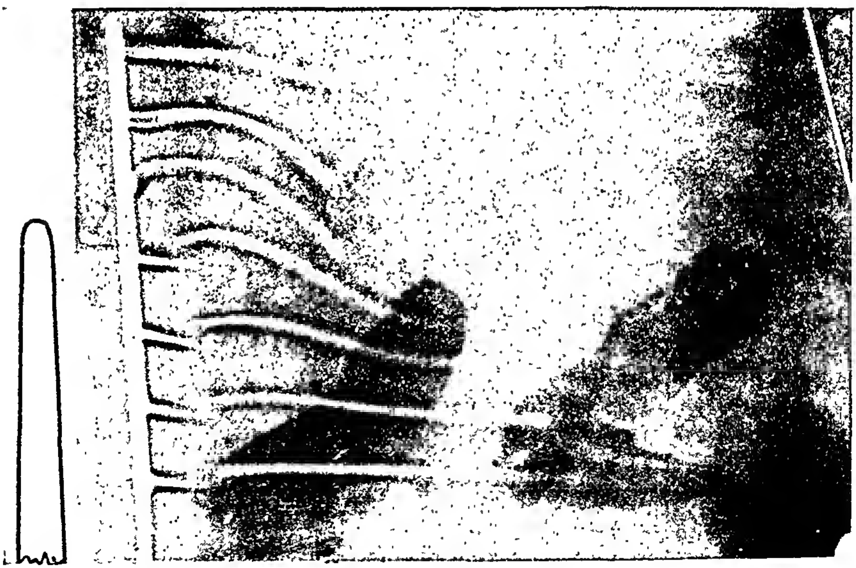
132. Air flow. Change of refractive index.

Any local change in the refractive index of the air, though not directly visible, may be made so by suitable illumination. A well-

† *Report of the National Physical Laboratory* (1931).



a. Flow round R.A.F. 31 (Slotted) with Hot Wires. $\alpha = 18°$, $V = 40$ feet per second (infinite aspect ratio)



b

known example is that of the photography of sound waves in which the wave front is revealed by its altered density.

133. Air flow. Hot wire shadows.

The motion of an air-stream may be made visible by mounting in it a fine wire which is heated by passing an electric current through it.† The filament of heated air in the wake of the wire, though it cannot be seen directly, may be detected by either the simple shadow method of Dvořák or by the 'Schlieren' (striae) method.‡

In the shadow method the field of flow is illuminated from the side by a small arc lamp, without any lenses, which casts a shadow of the heated filament of air on a screen. The temperature of the wire is unimportant: a platinum wire about ½ inch long and 0·002 inch diameter heated to a dull red is suitable. and produces a filament line several inches in length.

The length of the shadow depends mainly on the degree of turbulence of the stream, and may vary from an inch or two when very turbulent to 15 inches or more when steady. However, a shadowgraph made on a process plate with an exposure of about 0·001 second may give a longer record for turbulent motion than appears to the eye.

There is practically no upper limit to the air speed at which shadows may be observed, but at very low velocities there is a convection effect, though this is usually negligible above about 2 feet per second. The method is particularly useful for studying transient motions, e.g. the early stages of the flow round an aerofoil, etc.

Pl. 25*a*† is a simple shadowgraph, obtained directly on gaslight printing paper with an exposure of about 7 seconds, of the flow around a slotted aerofoil at a wind speed of 40 feet per second. The wires are ½ inch apart.

Pl. 25*b*‖ shows the flow behind an airscrew 19 inches in diameter developing a fairly high thrust at a forward speed of 15 feet per second. This shadowgraph was obtained by light passing through a slit in a rotating disk driven at airscrew speed, and is equivalent to a snapshot.

Pl. 26*a* shows the flow past a rotating cylinder 1 inch in diameter.

† Townend, *A.R.C. Reports and Memoranda*, No. 1349 (1931).
‡ Töpler, *Ann. d. Phys. u. Chem.* 131 (1867), 33–35. See also Wood, *Phil. Mag.* (5), 48 (1899), 218–227; Taylor and Waldram, *Journ. Sci. Insts.* 10 (1933), 378–389; Townend, *ibid.* 11 (1934), 184–187; Schardin, *Ver. deutsch. Ing., Forschungsheft* 367 (1934).
‖ *A.R.C. Reports and Memoranda*, No. 1434 (1932).

In steady motions, such as Pl. 25*a*, the filament lines obtained by a hot wire shadowgraph are identical with the paths of particles and with the stream-lines, but this is not so in periodic motions such as that in Pl. 25*b*, where successive particles of air passing the hot wire do not follow the same paths. Thus in Pl. 25*b*, on account of the thrust of the airscrew, particles passing on opposite sides of a blade receive radial velocity increments of opposite sign, and this causes breaks in the filament lines that widen as the motion proceeds.

134. Air flow. Spark shadows. The 'Schlieren' method. Kinematography. The determination of velocity distributions and measurements of turbulence: accuracy.

In cases of the foregoing kind a hot wire will not yield the path of a particle directly, but this can be obtained if the heat is produced by a periodic electric spark instead of a wire.† The possibility of obtaining records of the motions of small masses of air heated in this way enables measurements to be made of the instantaneous velocity at a point in the air-stream.‡ Pl. 26*b*|| shows the flow behind an airscrew. In this photograph shadowgraphs of a hot wire and of a series of sparks may be compared.

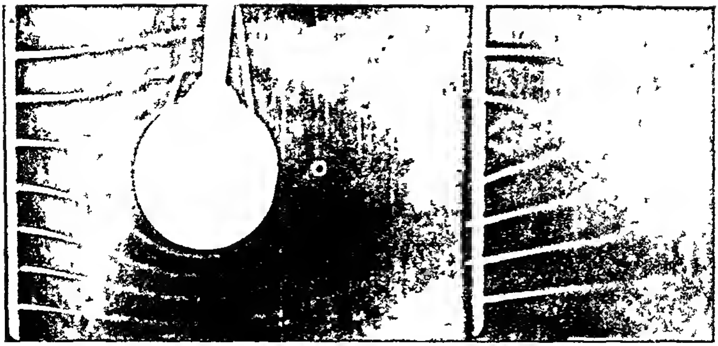
Although instantaneous shadowgraphs of the hot spots may be made as described above, much better records are obtained by the use of the 'Schlieren' method. The extra optical sensitivity of this method permits smaller sparks to be used, and this is important when the displacements to be measured are small. The principle of the method may be understood from Fig. 77, which shows the arrangement used for photographing air flow.

An arc lamp is focused on to the straight edge of a stainless steel mirror D. An inverted image of this portion of the mirror is formed, by reflection in the concave mirror M, on the edge of a diaphragm d. The working edges of D and d are close together, and are placed near the centre of curvature of the mirror M. By means of an adjusting screw nearly all the light is intercepted by d, only that from the

† Townend, *Phil. Mag.* (7), **14** (1932), 700–712; *Journ. Aero. Sciences*, **3** (1936), 343–352.

‡ Alternatively, when it is desired to determine the velocity of air from kinematograph records, suspended particles may be photographed. See C. Chartier and J. Labat, *Comptes Rendus*, **202** (1936), 729, 730 (aluminium powder); U. Schmieschek, *Zeitschr. f. tech. Physik*, **17** (1936), 98–100 (commercial variety of polymerized acetaldehyde). Soap bubbles have been used by H. Redon and F. Vinsonneau, *Aérotechnique*, **15** (1936), 60–66.

|| *A.R.C. Reports and Memoranda*, No. 1434 (1932).



a

b

extreme edge of D being allowed to pass over into the camera C. This light forms in the camera a uniformly illuminated image of the mirror M. Any optical disturbance in front of the concave mirror, such as that produced by the heating of a small mass of air at a, will deflect some rays of light so that they are intercepted by d, and others which are normally intercepted will be thrown clear of d and pass into the camera. Thus there will be formed on the plate an 'image' of a which is bright on one side and dark on the other. In the arrangement shown the rays pass through a twice, and this increases the optical sensitivity of the method.

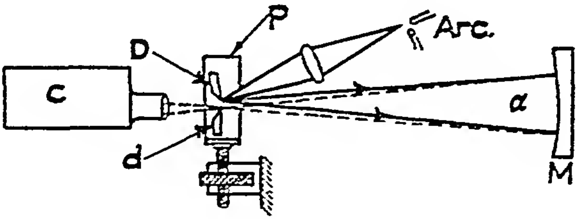


FIG. 77.

The camera C may be a kinematograph camera, and in that case the light entering it should be intercepted by a rotating disk situated as close behind d as possible, and having a narrow slit near its edge. The speed of the disk should be synchronized with that of the spark generator. It is unnecessary to synchronize the camera, since the exposure is so short (about $\frac{1}{4,000}$ sec.) that the mechanism for moving the film intermittently through the camera may be removed and the film driven at constant speed by a small motor.

A spark may be used as light source instead of an arc lamp, and this spark may be supplied by the same magneto or coil that provides the spark used as heat source in the air-stream. The rotating shutter is then unnecessary.†

The velocity distribution at different points in a flow may be determined by using several spark gaps. Pl. 27 shows the distribution in a 3-inch square pipe using seven spark gaps. The sparks were discharged in series. The following conditions of flow are depicted. (a) and (b) Laminar motion near the entry of the pipe, velocity distribution uniform across the pipe; (c) laminar motion 60 diameters downstream, velocity distribution parabolic; (d) and

† See *Report of the National Physical Laboratory* (1933), pp. 196–198.

(*e*) turbulent motion 60 diameters downstream. In records (*b*) and (*e*) not only are the hot spots due to the sparks visible, but the filament lines springing from the ends of the electrodes themselves, which are kept hot by the stream of sparks, are also apparent. This is because the edge of diaphragm *d*, Fig. 77, was parallel to the direction of flow. In the other records the edge was at right angles to the direction of flow, and then the filament lines are practically suppressed, because the deflexion of the light rays is mainly transverse to the filament lines themselves and so does not alter the amount of light passing over the edge of *d*.

Records of the kind shown in Pl. 27 (*d*) have been used to make a statistical analysis of the turbulent velocities in the flow through a pipe.† By measuring the displacements of a spot relative to the spark gap for several hundred pictures, the mean velocity at the point can be found, and also the root-mean-square values of the lateral and axial components of the turbulence. Quantitative estimates of the turbulence are thus obtainable, and the distribution of turbulence across the field may be measured.‡

In non-turbulent motion the position of the centre of a hot spot may be estimated to about 0·02 inch at a distance of 5 or 6 inches downstream, since the spot does not change its shape. When measuring turbulence much less accuracy is obtainable owing to dispersion of the spot. If, however, the turbulence is small and incidental to the measurements required, as in exploring a steady flow in a normal wind tunnel, the uncertainty introduced is small.

Convection becomes appreciable below 2 feet per second, but can be corrected for, when necessary, by measuring it in a uniform stream.||

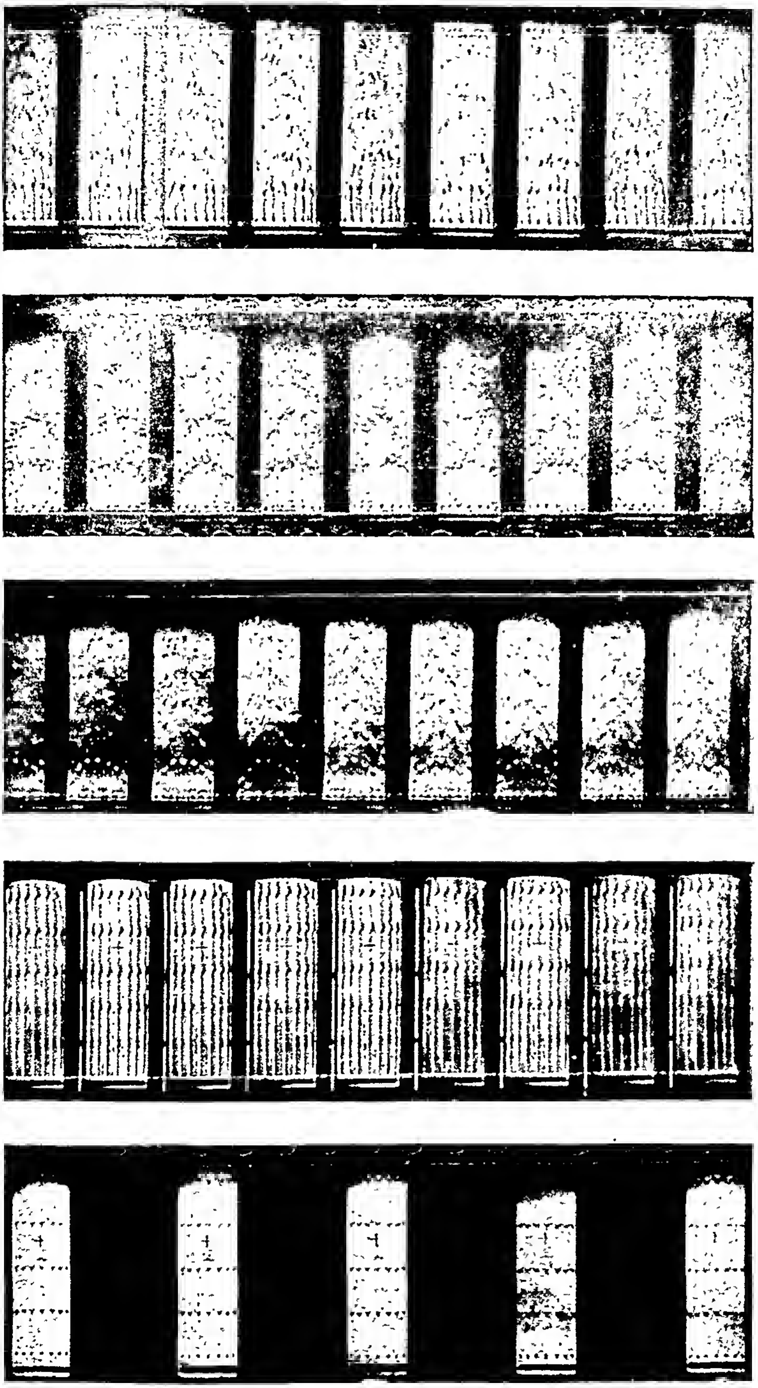
135. Water flow. Water channels and tanks.

The water channel provides a simple method of obtaining pictures of the flow around a body. The body is either held fixed in a moving stream or is moved through stationary water. A common method of making the flow visible is to inject colouring matter, such as red

† Townend, *Proc. Roy. Soc.* A, **145** (1934), 180–211.

‡ Information concerning turbulent fluctuations may also be obtained by utilizing the effect of the air-stream on a glow discharge between two electrodes. See F. C. Lindvall, *Electrical Engineering*, **53** (1934), 1068–1073.

|| For later developments of the methods described in this section see Townend, *A.R.C. Reports and Memoranda*, No. 1803 (1937.)



e

d

c

b

a

ink, in the form of a fine jet, the velocity of efflux being the same as the local water velocity. Another method is to add to the water small drops of oil, and to illuminate them by a beam of light from an arc lamp. When the drops are viewed at the appropriate angle they appear as bright points of light, which can easily be photographed. If the time of exposure is sufficient, the image of each illuminated particle traces a line on the plate, and the velocity of the particle can be determined from the length of this line and the time of exposure. To eliminate the effects of gravity the drops used are obtained from a mixture of two oils in such proportions that the specific gravity is unity. The best results are obtained when the refractive index of the oil drops is such that the angle between the incident and emergent rays is 90°, for then all the illuminated drops can be brought into focus on the photographic plate at the same time, if the plane of illumination is sufficiently thin. A mixture of olive oil and nitrobenzene has been found suitable for this purpose.†

Minute particles of aluminium or lycopodium powder scattered on the surface of the water have been used to reveal the flow past a two-dimensional obstacle when it projects beyond the surface.‡ Very fine flakes of mica‖ on the surface may also be used to reveal the flow. In such experiments the surface has to be kept very clean to minimize capillary effects. A test for cleanliness is to sprinkle aluminium powder on the surface and to spread the powder by blowing gently on it. If the particles do not collect together, the surface is clean. Movement of the aluminium particles away from the obstacle, under capillary action at the junction, can be prevented by coating the obstacle with a thin layer of paraffin. Motion of a regular pattern is clearly revealed by this method because a great number of the particles have the same orientation.

An apparatus has been designed at Cambridge†† which allows observation of the flow when a model is given an impulsive start from rest. This apparatus consists of an enclosed tank filled with water and having parallel sides 6 inches apart, between which the

† Relf, *Adv. Comm. for Aeronautics, Reports and Memoranda*, No. 76 (1913).

‡ Ahlborn, *Abhandl. Gebiete Naturwiss.* **17** (1902), 8–37; Rubach, *Forschungsarbeiten des Ver. deutsch. Ing.*, No. 185 (1916), 1–35.

‖ Prandtl and Tietjens, *Die Naturwissenschaften*, **113** (1925), 1050–1053; Prandtl, *Verhandlungen des dritten internationalen Mathematiker-Kongresses, Heidelberg*, 1904 (Leipzig, 1905), pp. 490, 491.

†† Jones, Farren, and Lockyer, *A.R.C. Reports and Memoranda*, No. 1065 (1927). See also Walker, *ibid.*, No. 1402 (1932).

model can be moved through the water, which is at rest apart from the disturbance created by the model. Oil drops suspended in the water reflect light into a camera focused on the plane of flow to be examined, and the movements of the drops due to the disturbance set up by the model are photographically recorded. When the exposure of a plate is of short but finite duration, the photograph records motions relative to the undisturbed water, and a short trace made by a drop can be taken as a vector giving the fluid velocity at its middle point. The energy required to give an impulsive start to the carriage carrying the model is obtained from a flywheel.

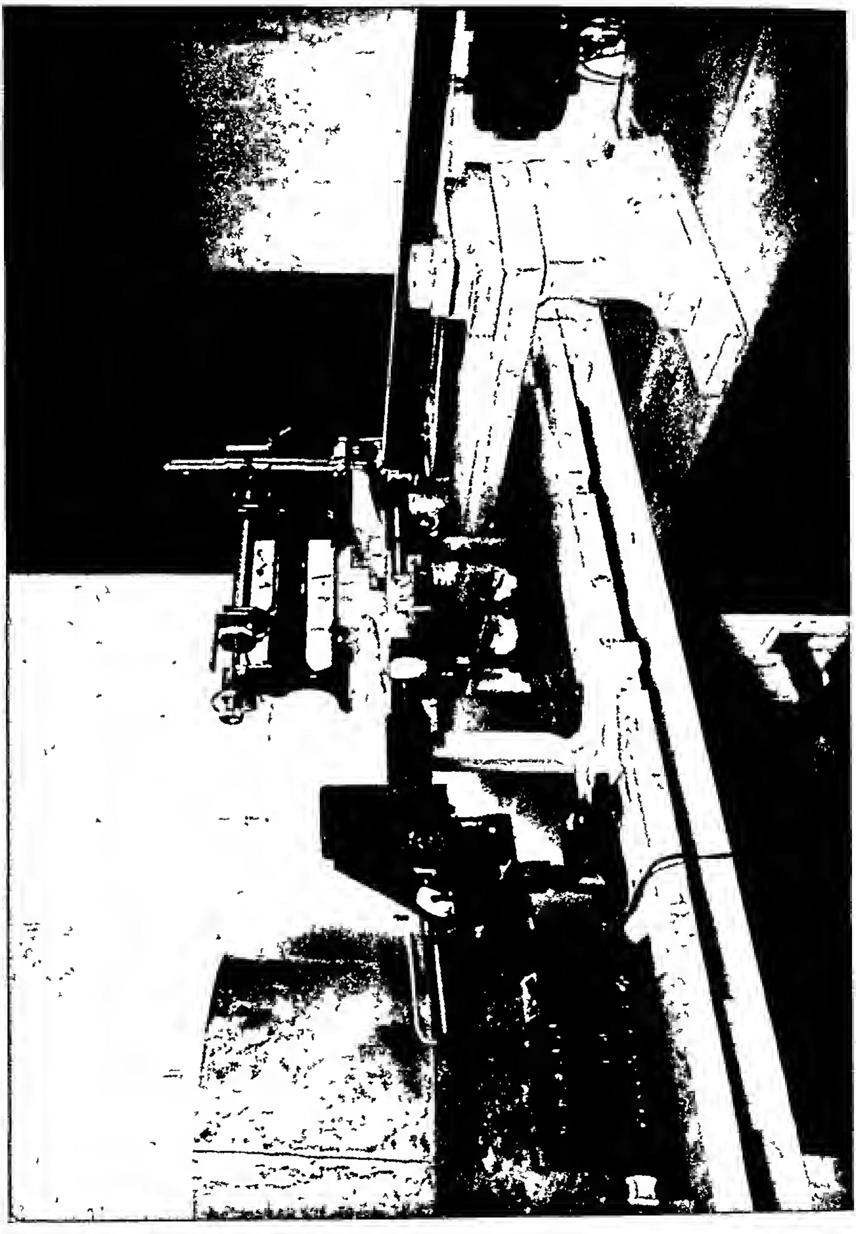
136. Water flow. The ultramicroscope. Ultramicroscope photography.

The above methods involve the introduction of particles into the water, and reliable views of the flow are obtained when the motions of relatively large molar masses of water are concerned. Some doubt arises, however, if particles are added for the examination of micro-turbulence, especially near the boundary of the fluid where the scale of the turbulence is small, since such particles may be comparable in size with the molar masses, and then the internal motions of these masses would not be faithfully represented. The ultramicroscope† affords a means of obtaining reliable information on minute details of fluid flow without any possible interference with the motion, since neither particles nor measuring instruments are introduced into the fluid.

The principle of the ultramicroscope depends on the fact that very small particles usually present in most fluids, but invisible in ordinary light even under the most powerful microscope, become visible when intensely illuminated provided they are seen against a dark background. Even particles smaller than the wave length of light become visible as bright points of light, if the intensity of the light beam is sufficiently great. A photograph of the ultramicroscope arranged to examine turbulent flow in a square pipe is given in Pl. 28. The water flowing through the pipe is illuminated through a glass window let into one side of the pipe, and observation is made through a window let into an adjacent side. The light from an arc lamp taking 5 amperes is brought to a focus by a single condensing lens, and

† Fage and Townend, *Proc. Roy. Soc.* A, 135 (1932), 656–677. See also Nisi and Porter, *Phil. Mag.* (6), 46 (1923), 754–768.

PLATE 28



then passed through a compound lens and through the glass window into the water. A small cylindrical lens is interposed between the image of the arc and the compound lens in order to convert the conical incident beam into a wedge-shaped beam: the width of the beam of illumination is thus increased up to the diameter of the field of the microscope without an increase in depth. The illumination of particles well outside the focal plane of the microscope, which would impair the darkness of the background, is thus prevented, and the light available is also conserved. A fine slit placed in the focal plane of the first condensing lens can be used to obtain a very thin illuminating beam. The height of the incident beam can be adjusted by mounting the lens system on an optical bench pivoted at one end and provided with a screw adjustment at the other: this adjustment, used in conjunction with lateral and vertical movements of the microscope, allows observation to be made at any selected point in the fluid. Observation through the microscope can conveniently be made under a magnification of about 50 (except very near a surface, when a higher magnification to increase sensitivity is necessary in order to show the finer details of the motion).

The particles passing through any fixed point in a completely eddying stream move in different directions. When the radii of curvature of the sinuous paths of these particles are large compared with the diameter of the field of the microscope, only short lengths of the paths are seen when the particles are illuminated. These short lengths appear as bright rectilinear streaks inclined at various angles to the mean direction of flow, and at high speeds they appear to intersect each other, owing to persistence of vision. The direction of a path of a particle can be measured by mounting in the focal plane of the eyepiece a fine platinum wire which can be rotated about the axis of the microscope by means of a pointer moving over an angular scale. Observation is facilitated if this wire is rendered luminous by heating it electrically to a dull red glow.

To measure the speed of a particle, use is made of the principle that the particle appears as a bright stationary point when viewed at the speed at which it is moving. Instead of moving the microscope as a whole, the same effect is obtained, over a limited region, if the eyepiece and the microscope tube are held fixed and the objective is moved in the same direction as the particle. The objective is therefore carried on a wheel which is rotated about an offset axis, so that

once in every revolution the axis of the objective coincides with the axis of the microscope tube. The position of the axis of rotation is chosen so that at the instant of this coincidence the direction of motion of the objective is parallel to the mean direction of flow. The factor for obtaining the speed of a particle from the speed of rotation is easily obtained by direct calibration. In turbulent flow the successive views seen when the objective rotates differ on account of the fluctuations in the velocity of the stream at any point in the fluid: the velocity component in the direction of mean flow fluctuates continually between minimum and maximum values, so that particles can only appear as points provided that the corresponding speed of the objective lies between these limits. Hence the minimum speed at a point is obtained by slowly increasing the speed of rotation until particles first appear, and the maximum speed until they just cease to appear. The mean velocity is taken as the mean of the maximum and minimum velocities measured at the point, and the maximum velocity fluctuations as one-half the difference between these values.

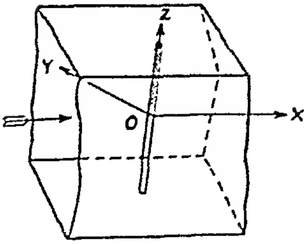
The maximum lateral components of the turbulent velocity can be deduced from observations of the maximum angular deviations of the particle paths from the mean direction of flow.†

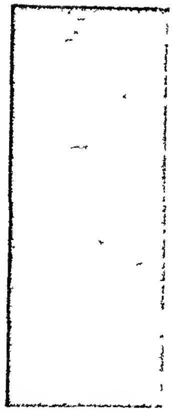
It has been found possible to take photographs under a reduced magnification of some of the views seen with an ultramicroscope.‡ The kind of photograph obtained is illustrated in Pl. 29,|| where views are given of ordinary tap water (without any particles added) flowing past a long circular cylinder at very low Reynolds numbers ($U_0\,d/\nu$). These photographs were taken with a small camera having an $f/3$ lens ($f = 2$ inches). Photograph (*a*) shows the steady nature of the flow in the empty tunnel. Photographs (*b*)–(*h*) were taken on the median plane at right angles to the axis of the cylinder at values of $U_0\,d/\nu$ lying within the range 17·7 to 170. Photographs (*i*), (*j*), (*k*) were taken on the plane passing through the axis of the cylinder: in photograph (*j*), the flow in the standing vortices at $U_0\,d/\nu = 38{\cdot}0$ is of a spiral character.

† Fage, *Phil. Mag.* (7), **21** (1936), 80–105.
‡ Fage, *Proc. Roy. Soc.* A, **144** (1934), 381–386.
|| *Report of the National Physical Laboratory* (1933).

CIRCULAR CYLINDER

Diameter = 0·01 inch

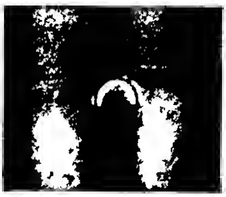




a. Free stream

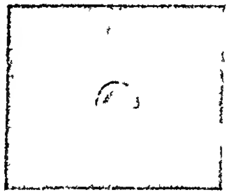
Views at O on Plane XOY

b. $\frac{U_0 D}{\nu} = 17{\cdot}7$

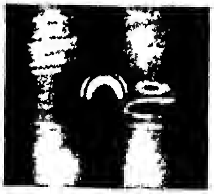


c. = 21

d.



e. = 32

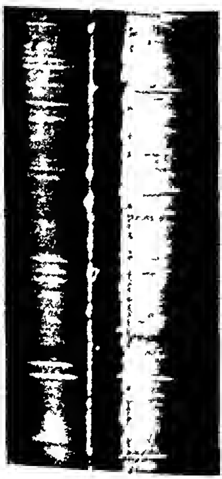


f. = 47

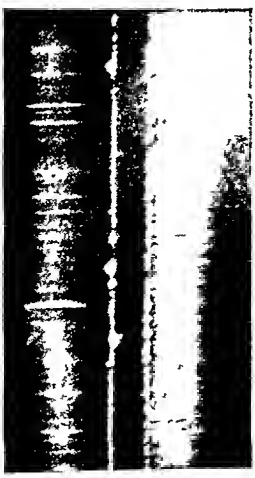
g. = 104

h. = 170

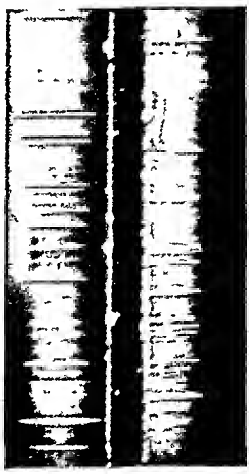
Views at O on Plane XOZ



i. = 24·3



j. = 38



k. = 137

VII

FLOW IN PIPES AND CHANNELS AND ALONG FLAT PLATES

137. Introduction.

The subject-matter of this and the following chapter may be conveniently divided into three sections:—(i) laminar flow; (ii) the transition from laminar to turbulent flow: (iii) fully developed turbulent flow: (i) and (ii) will be discussed in the present chapter and (iii) in Chapter VIII. In each section those problems which have received theoretical consideration will be discussed first, since they are, in general, fundamental. When theoretical solutions have not been obtained, experimental results will be reduced by dimensional considerations to forms which facilitate application to other problems of the same type.

The Reynolds number is defined for flow in a pipe or channel of any section as $4mu_m/\nu$, where m is the hydraulic mean depth (defined as the area of the cross-section divided by its perimeter), u_m is the average velocity over a section, and ν is the kinematic viscosity. This definition will always be used in the absence of a specific statement to the contrary. (For a circular pipe m is a quarter of the diameter d, and the Reynolds number is $u_m d/\nu$: for two-dimensional flow between parallel walls at a distance $2h$ apart m is h, and the Reynolds number is $4u_m h/\nu$.)

In steady flow along straight pipes and channels, far away from the inlet and the exit (i.e., where the flow at any section is similar to that at any other), the skin-friction τ_0 and the pressure drop† Δp per unit length are related by the equation

$$A\Delta p = L\tau_0, \tag{1}$$

where A is the area of the section and L its perimeter. (For problems of turbulent flow only mean values are considered.) In terms of the hydraulic mean depth m equation (1) may be written

$$m\Delta p = \tau_0. \tag{2}$$

For any length l of a straight pipe or channel a non-dimensional resistance coefficient γ is defined by the equation

$$\gamma = \frac{p_1 - p_2}{\frac{1}{2}\rho u_m^2}\frac{m}{l}, \tag{3}$$

† The pressure is taken throughout as the difference of the actual pressure and the hydrostatic pressure.

where p_1 and p_2 are the pressures at the end sections of the length considered.

For a fully developed flow (at a sufficient distance from the ends) the pressure gradient is constant, so that $(p_1-p_2)/l$ is Δp and the definition of γ in (3) is equivalent to

$$\gamma = \tau_0/(\tfrac{1}{2}\rho u_m^2). \tag{4}$$

SECTION I

LAMINAR FLOW

138. Flow through a straight pipe of circular cross-section.

The theoretical solution of this problem has been given in Chap. I, § 6 (p. 20). The following results were obtained on the assumption of no slip at the walls:—

(i) The velocity distribution across a section is parabolic and such that the mean velocity, u_m, is half the maximum.

(ii)
$$u_m = -\frac{a^2}{8\mu}\frac{\partial p}{\partial x}, \tag{5}$$

where a is the radius of the pipe, μ is the viscosity, and $-\partial p/\partial x$ is the pressure gradient; the total flux, $\pi a^2 u_m$, therefore varies directly as the fourth power of the radius and the first power of the pressure gradient.

(iii)
$$\gamma = 16/R. \tag{6}$$

It was also pointed out that the velocity distribution is not parabolic all the way from the entry; a certain length—the inlet length—is required before the parabolic distribution is attained. The flow in the inlet length will be considered in § 139.

The theoretical results have been compared with experiment by taking measurements of the flux and obtaining its variation with change of radius and pressure gradient. The easiest way in practice to overcome the difficulty of end effects in the measurement of the pressure gradient is to observe the pressures at holes in the pipe wall in the fully developed region; otherwise the pipe used must either be sufficiently long for the inlet length to be negligible, or some end correction must be applied.

Hagen† and Poiseuille,‡ using water in capillary tubes, found by

† *Poggendorff's Annalen d. Physik u. Chemie* (2), **46** (1839), 423–442.

‡ *Comptes Rendus*, **11** (1840), 961–967; 1041–1048; **12** (1841), 112–115; *Mémoires des Savants Étrangers*, **9** (1846), 433–543.

experiment the proportionality of the flux to the pressure gradient and to the fourth power of the radius; later workers, by applying end corrections to the experimental results of Poiseuille, and also by many further experiments with various fluids in tubes of various materials and widely different radii, have obtained excellent agreement between theory and experiment. Stanton and Pannell,† using water and oil in smooth brass and steel pipes, have made a few velocity measurements and obtained good agreement with the predicted values. Thus the collected results of various experimenters verify the theoretical predictions, and so are in accord with the assumption of no slip at the walls.

139. Flow in the inlet length of a circular pipe.

With a well-designed short trumpet-shaped intake, if care is taken to avoid disturbances‡ the velocity at the entry to a circular pipe will be practically constant over the cross-section. The velocity at the wall is, however, zero, so that an infinitely thin boundary layer is formed round the walls of the pipe; the thickness of this layer increases as we pass downstream until it becomes equal to the radius of the pipe. Until this happens there is a core of fluid practically uninfluenced by viscosity, and in it the total head may be considered constant. Since the flux across any section is constant, and since the boundary layer thickness is increasing, this core is accelerated, and there is a corresponding fall in pressure. The fully developed parabolic distribution is theoretically attained only asymptotically; but we are now interested in the distance which it is necessary to travel downstream before the difference from the parabolic distribution becomes less than the least experimental error. It should be remembered that this state may not be reached simultaneously with the boundary layer thickness becoming equal to the pipe radius: calculations on pp. 304–308 indicate that the whole of the fluid across a section becomes influenced by viscosity some distance before the parabolic distribution is approached.

We require expressions for (i) the pressure difference between any two sections; (ii) the velocity distribution at any section; and (iii) the value of x for which the fully developed parabolic flow may be said to be attained.

† *Phil. Trans.* A, **214** (1914), 199–224.
‡ The influence of friction in the intake is neglected.

If the fluid enters the pipe from a cistern in which the pressure P is maintained and in which the velocity is negligible, and if p_L is the pressure at a distance L from the entry to the pipe, at which distance the permanent régime may be considered attained, then a usual approximate result for the pressure difference $P-p_L$ is given by equation (8) below, whilst a more accurate result (but still obtained by neglect of friction in the intake before the entry to the pipe) is given by equation (44) with $x = L$.

We obtain a first approximation to the difference between the pressure at the entry ($x = 0$) and the pressure when the final velocity distribution may be said to be attained ($x = L$) by means of the kinetic energy end-correction.† In this approximation the dissipation of energy in the inlet length is supposed to be equal to the dissipation in the same length when the velocity distribution is parabolic. If p_0, p_L are the pressures at $x = 0$ and at $x = L$, then the rate at which the pressures are doing work is

$$\pi a^2 p_0 u_m - p_L\Big[2\pi \int_0^a ur\,dr\Big]_{x=L} = \pi a^2 u_m(p_0-p_L),$$

since the velocity at $x = 0$ is constant and equal to the average velocity, u_m, over a section. The rate of inflow of kinetic energy at $x = 0$ is $\frac{1}{2}\pi a^2 \rho u_m^3$, and the rate of outflow at $x = L$ is

$$2\pi\rho \int_0^a \tfrac{1}{2}u^3 r\,dr = \pi\rho a^2 u_m^3.$$

Thus the difference in the rate of inflow and the rate of outflow of kinetic energy gives rise to an additional pressure drop of $\frac{1}{2}\rho u_m^2$ between $x = 0$ and $x = L$. On these grounds it is assumed that

$$\frac{p_0-p_L}{\frac{1}{2}\rho u_m^2} = \frac{16\mu u_m L}{\rho u_m^2 a^2}+1 = \frac{32L}{aR}+1. \tag{7}$$

If the fluid enters the pipe from a large cistern in which the pressure is P and the velocity is negligible, there is a pressure drop of $\frac{1}{2}\rho u_m^2$ between the cistern and entry (the influence of friction in the intake being neglected), so that

$$\frac{P-p_L}{\frac{1}{2}\rho u_m^2} = \frac{16\mu u_m L}{\rho u_m^2 a^2}+2 = \frac{32L}{aR}+2. \tag{8}$$

† Hagenbach, *Poggendorff's Annalen d. Physik u. Chemie* (4), 109 (1860), 385–426; Couette, *Ann. de Chimie et de Physique* (6), 21 (1890), 433–510; Prandtl and Tietjens, *Hydro- und Aeromechanik*, 2 (Berlin, 1931), 25, 26.

An approximation can also be obtained by treating the retarded layer in a manner analogous to that used by Pohlhausen in his discussion of flow in a boundary layer (Chap. IV, § 60 (p. 157)). We suppose that the retarded layer has a thickness δ at any cross-section; we write $y = a-r$, and then, following Schiller,† we put

$$\frac{u}{u_1} = 2\frac{y}{\delta}-\frac{y^2}{\delta^2} \tag{9}$$

in the retarded layer, where u_1 is the velocity in the core. It will be noted that the assumption that the velocity distribution becomes parabolic when $\delta = a$ is inherent in this method.

The equation of continuity and the momentum equation give two relations between u_1, x and δ. Elimination of δ leads to a relation between u_1 and x which, with

$$\chi = u_1/u_m-1, \tag{10}$$

integrates to

$$x/(aR) = f(\chi), \tag{11}$$

where

$$R = 2au_m/\nu$$

and

$$f(\chi) = \frac{1}{8}\left\{\frac{58}{15}\chi-\frac{22}{5}\log(1+\chi)-\frac{17}{15}\sqrt{(4+2\chi-2\chi^2)}\right.$$
$$\left.-\frac{16}{5}\left(\frac{4-2\chi}{1+\chi}\right)^{\frac{1}{2}}-\frac{37\sqrt{2}}{10}\sin^{-1}\frac{2\chi-1}{3}+\frac{26}{3}-\frac{37\sqrt{2}}{10}\sin^{-1}\frac{1}{3}\right\}.\ddagger \tag{12}$$

A graph of $f(\chi)$ is shown in Fig. 78. In the parabolic flow $u_1 = u_{\max}$, and since $u_{\max} = 2u_m$, $\chi = 1$; this occurs first when

$$x = 0{\cdot}0575aR;$$

so according to this approximation the permanent régime is established after a finite distance $0{\cdot}0575aR$. The approximate nature of the calculation is illustrated by the fact that the maximum velocity in the inlet length does not join on smoothly to its final value $2u_m$, since $d\chi/dx$ does not vanish at $\chi = 1$. (Cf. Fig. 80, p. 304.)

If p_1 and p_2 are the pressures at the sections x_1 and x_2, it follows from the constancy of the total head in the core, since

$$u_1 = u_m(\chi+1),$$

that

$$(p_1-p_2)/(\tfrac{1}{2}\rho u_m^2) = [2\chi+\chi^2]_{x_2}-[2\chi+\chi^2]_{x_1} = \Delta(2\chi+\chi^2) \tag{13}$$

† *Zeitschr. f. angew. Math. u. Mech.* 2 (1922), 96–106; *Handbuch der Experimentalphysik*, 4, part 4 (Leipzig, 1932), 48–57.

‡ This is a simplified form of Schiller's result.

(say). The resistance coefficient γ defined in equation (3) is therefore equal to

$$\frac{a\Delta(2\chi+\chi^2)}{2(x_2-x_1)}. \tag{14}$$

The theory can be checked against experiment by finding γ for

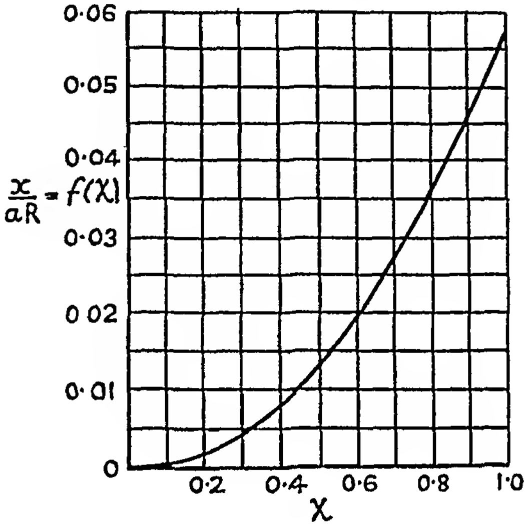


FIG. 78.

fixed x_1 and x_2 and various values of R. Typical results found by Schiller are shown in Fig. 79, where the straight line gives the uncorrected theoretical result.

For $x \leqslant 0{\cdot}0575aR$,

$$p_0-p = \tfrac{1}{2}\rho u_m^2(2\chi+\chi^2), \tag{15}$$

where p_0 is the pressure at the entry, and p and χ are evaluated at the section at the distance x from the entry. When $x = 0{\cdot}0575aR$, $\chi = 1$ and

$$p_0-p = 1{\cdot}5\rho u_m^2. \tag{16}$$

For $x > 0{\cdot}0575aR$,

$$-\partial p/\partial x = 8\mu u_m/a^2 = 16\rho u_m^2/(aR),$$

and hence $$p_0-p = 1{\cdot}5\rho u_m^2+\frac{16\rho u_m^2}{aR}(x-0{\cdot}0575aR),$$

so that $$(p_0-p)/(\tfrac{1}{2}\rho u_m^2) = 1{\cdot}16+32x/(aR). \tag{17}$$

For flow out of a cistern where the pressure is P,

$$(P-p)/(\tfrac{1}{2}\rho u_m^2) = 2{\cdot}16+32x/(aR), \tag{18}$$

friction in the intake being neglected. The term 2·16 corresponds to the term 2 of the kinetic energy end-correction.

Schiller† also tested his results by measuring the value of C in the relation

$$(P-p)/(\tfrac{1}{2}\rho u_m^2) = C+32x/(aR);\ddagger \tag{19}$$

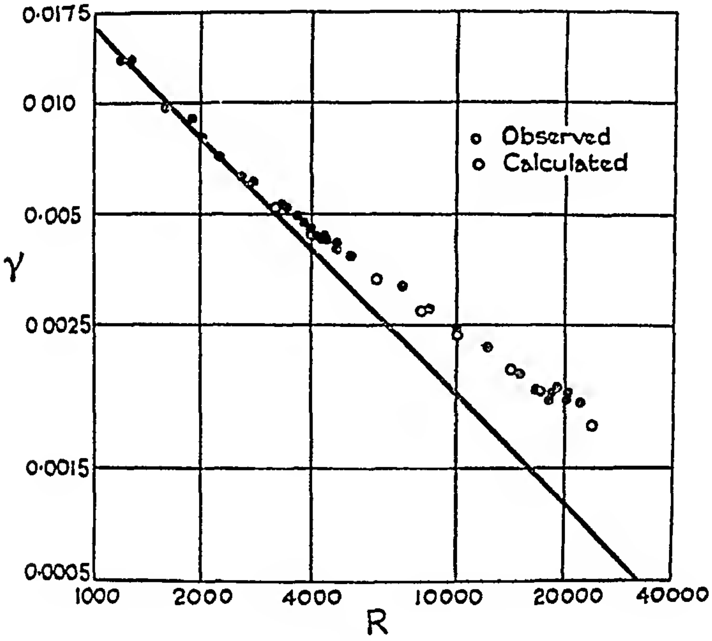


FIG. 79.

he found $C = 2{\cdot}115$, $2{\cdot}35$, $2{\cdot}36$, $2{\cdot}45$ in four series of experiments. The mean of these results is 2·32, but Schiller attaches more weight to the first one than to the other three. It seems, in fact, that no sufficiently accurate experiments‖ to determine C have yet been performed; if they are it will be necessary to take into account friction in the intake. This may be done by assuming an increase in the length of the pipe and determining this increase together with C experimentally.

Velocity measurements in the inlet length have been made by

† *Forschungsarbeiten des Ver. deutsch. Ing.*, No. 248 (1922), pp. 29–33; *Handbuch der Experimentalphysik*, 4, part 4 (Leipzig, 1932), 56, 57.

‡ The value of C is of importance in the measurement of viscosity of liquids. For further details see Hatschek, *The Viscosity of Liquids* (London, 1928), chap. II.

‖ Results of other experiments are given by Hatschek, *loc. cit.*

Nikuradse, and the results for $r/a = 0$, 0·2, 0·4, 0·6, 0·7, 0·8, and 0·9 are reproduced in Fig. 80,† which also contains results calculated by Schiller's method.

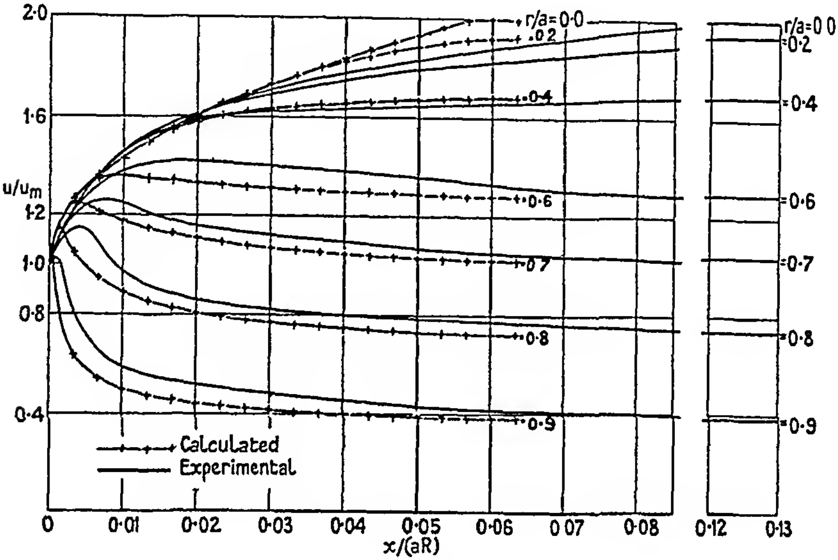


Fig. 80.

A more accurate solution near the pipe entry may be obtained by starting from the equations of motion:‡

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial x^2}\right), \tag{20}$$

$$u\frac{\partial v}{\partial r}+v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}+\frac{\partial^2 v}{\partial x^2}\right), \tag{21}$$

in which use is made of the symmetry of the motion about the axis, and v is the radial component of velocity (which must be present since u changes with x).

The equation of continuity is

$$\frac{\partial}{\partial x}(ru)+\frac{\partial}{\partial r}(rv) = 0, \tag{22}$$

so a stream-function ψ exists such that

$$u = \frac{1}{r}\frac{\partial \psi}{\partial r}, \qquad v = -\frac{1}{r}\frac{\partial \psi}{\partial x}.$$

† The experimental results in Fig. 80 are reproduced from a small-scale graph in Prandtl and Tietjens, *Hydro- und Aeromechanik*, **2** (Berlin, 1931), 28. Further details do not appear to have been published.

‡ Atkinson and Goldstein (unpublished). The method is a variation of a method due to Schlichting for the corresponding problem in two dimensions. See p. 309.

The boundary conditions are

$$\begin{aligned} u = u_m, \quad v = 0 \quad \text{at} \quad x = 0, \\ u = 0, \quad v = 0 \quad \text{at} \quad r = a. \end{aligned} \tag{23}$$

We are concerned only with large Reynolds numbers, and the approximations usual in boundary layer theory may be made. The term $\nu\partial^2 u/\partial x^2$ in the first equation of motion may be neglected, and $\rho^{-1}\partial p/\partial x$ may be taken to be a function of x only. So long as there is a core uninfluenced by viscosity we put

$$\xi = \left(\frac{2x}{aR}\right)^{\frac{1}{2}}, \quad y_1 = \frac{1}{2}\left(1-\frac{r^2}{a^2}\right), \quad \eta = \frac{y_1}{2\xi}, \quad \text{and} \quad -\frac{1}{\rho}\frac{\partial p}{\partial x} = u_1\frac{du_1}{dx}, \tag{24}$$

where u_1 is the velocity in the core. We assume that

$$u_1 = u_m\{1+K_1\xi+K_2\xi^2+...\}, \tag{25}$$

where the K's are constants to be determined: we shall see that with the assumption (25) the K's can be determined so that the flux across a section is constant.

A solution can then be obtained by generalizing Blasius's solution of the two-dimensional boundary layer equation. We put

$$\psi = -a^2u_m[\xi f_1(\eta)+\xi^2 f_2(\eta)+...], \tag{26}$$

so that

$$u = \tfrac{1}{2}u_m[f_1'(\eta)+\xi f_2'(\eta)+\xi^2 f_3'(\eta)+...]. \tag{27}$$

We substitute in the equation of motion and equate coefficients of powers of ξ. This yields the following equations for the f's:

$$f_1'''+f_1 f_1'' = 0, \tag{28}$$

$$f_2'''+f_1 f_2''-f_1' f_2'+2f_1'' f_2 = -4K_1+4\eta f_1'''+4f_1'', \tag{29}$$

$$f_3'''+f_1 f_3''-2f_1' f_3'+3f_1'' f_3 = -8K_2-4K_1^2+f_2'^2-2f_2 f_2''+4\eta f_2'''+4f_2'', \tag{30}$$

. . . .

. . . .

where (so long as a core exists)

$$f_n(0) = f_n'(0) = 0, \qquad f_1'(\infty) = 2, \qquad f_n'(\infty) = 2K_{n-1} \quad (n > 1). \tag{31}$$

The necessary K's can be determined, as below, before the respective f's are calculated.

For large values of η, $f_1 \sim 2\eta+A_1$, $f_2 \sim 2K_1\eta+A_2$, $f_3 \sim 2K_2\eta+A_3,...$, where the A's are determined by the numerical integration of the equations (28), (29), etc.

For constancy of flux the difference in the values of $2\pi\psi$ at $r = a$ (i.e. $\eta = 0$) and at $r = 0$ (i.e. $\eta = 1/(4\xi)$) must be constant and equal to the flux $\pi a^2 u_m$. The stream-function ψ vanishes at $\eta = 0$, and so long as there is a core uninfluenced by viscosity, $1/(4\xi)$ must be sufficiently large for the above approximations to hold in finding the value of ψ at $\eta = 1/(4\xi)$. The condition of constancy of flux therefore becomes

$$\xi\left(\frac{1}{2\xi}+A_1\right)+\xi^2\left(\frac{K_1}{2\xi}+A_2\right)+\xi^3\left(\frac{K_2}{2\xi}+A_3\right)+... = \tfrac{1}{2}. \tag{32}$$

Hence $K_1 = -2A_1, \quad K_2 = -2A_2, \quad K_3 = -2A_3,$ etc. (33)

Thus a solution is obtained for small values of $x/(aR)$. It is found that the K's increase rapidly ($K_1 = 3{\cdot}4415$, $K_2 = -9{\cdot}0938$, $K_3 = 141{\cdot}982$, $K_4 = -2788$). Beyond about $\xi = 0{\cdot}05$ the series (25) up to the term $K_4\xi^4$ does not give sufficiently accurate results, whilst some allowance must be made for the following terms even for $\xi = 0{\cdot}05$. It appears that for $\xi \leqslant 0{\cdot}05$ the values of u/u_m calculated by Schiller's method are accurate to within 1 per cent. for $r/a \leqslant 0{\cdot}8$ and to within 5 per cent. for $r/a = 0{\cdot}9$. It follows that the experimental results in Fig. 80 do not agree with the accurate solution for very small values of $x/(aR)$. The singularity in the solution at $x = 0$ will cause some discrepancy very near the entry; whether this is sufficient to explain the actual discrepancy between the observed and calculated values we cannot tell in the absence of further experimental details.

The solution for small values of $x/(aR)$ may be continued by a method due to Boussinesq.† If we write

$$X = 2x/(aR), \quad Y = r^2/a^2, \quad w = Rrv/(2a) \quad \text{and} \quad R = 2u_m a/\nu, \tag{34}$$

the equation of continuity becomes

$$\frac{\partial u}{\partial X}+2\frac{\partial w}{\partial Y} = 0,$$

so that

$$2w = \int\limits_Y^1 \frac{\partial u}{\partial X}\,dY \tag{35}$$

(since $w = 0$ at $Y = 1$). Hence equation (20) (with $\nu\partial^2u/\partial x^2$ neglected) becomes

$$u\frac{\partial u}{\partial X}+\frac{\partial u}{\partial Y}\int\limits_Y^1 \frac{\partial u}{\partial X}\,dY = -\frac{1}{\rho}\frac{\partial p}{\partial X}+4u_m\frac{\partial}{\partial Y}\left(Y\frac{\partial u}{\partial Y}\right). \tag{36}$$

Differentiation with regard to Y gives the equation for u; also, from (36),

$$-\frac{1}{\rho}\frac{\partial p}{\partial X} = \left[u\frac{\partial u}{\partial X}-4u_m\frac{\partial}{\partial Y}\left(Y\frac{\partial u}{\partial Y}\right)\right]_{Y=0},$$

since $\partial p/\partial X$ is independent of Y and

$$\int\limits_0^1 \frac{\partial u}{\partial X}\,dY = 0$$

$\left(\text{because} \int\limits_0^1 u\,dY \text{ is constant}\right)$.

In the permanent régime $\quad u = 2u_m(1-Y),$

so we now put $$u = u_m\{2(1-Y)+\varpi\}. \tag{37}$$

For a first approximation, ϖ_1, we suppose ϖ small and neglect squares; we then find that

$$\frac{1-Y}{Y}\frac{\partial}{\partial X}\left(Y\frac{\partial\varpi_1}{\partial Y}\right) = 2\frac{\partial^2}{\partial Y^2}\left(Y\frac{\partial\varpi_1}{\partial Y}\right), \tag{38}$$

which is an equation for $Y\,\partial\varpi_1/\partial Y$.

† *Comptes Rendus*, **113** (1891), 9–15; 49–51. Actually Boussinesq applied his method right from the entry, but it is more accurate when used to continue a solution such as that described above.

Since $\int_0^1 u\,dY$ is constant, $\int_0^1 \varpi_1\,dY = 0$; and since $\varpi_1 = 0$ at $Y = 1$,

$$\int_0^1 Y\frac{\partial\varpi_1}{\partial Y}\,dY = [Y\varpi_1]_0^1 - \int_0^1 \varpi_1\,dY = 0.$$

If we put
$$Y\,\partial\varpi_1/\partial Y = ce^{-2\lambda X}\phi(Y) \tag{39}$$
in (38) we find that
$$\phi''+\lambda\Big(\frac{1-Y}{Y}\Big)\phi = 0. \tag{40}$$

The boundary conditions are $\phi(0) = 0$ and $\int_0^1 \phi\,dY = 0$. Equation (40) with these boundary conditions shows that λ has one of a series of characteristic values. If $\lambda_1,\ldots,\lambda_n,\ldots$ are the characteristic values, then the complete solution is
$$Y\,\partial\varpi_1/\partial Y = c_1 e^{-2\lambda_1 X}\phi_1(Y)+c_2 e^{-2\lambda_2 X}\phi_2(Y)+\ldots. \tag{41}$$
Hence
$$-\varpi_1 = c_1 e^{-2\lambda_1 X}\Phi_1(Y)+c_2 e^{-2\lambda_2 X}\Phi_2(Y)+\ldots, \tag{42}$$
where
$$\Phi_r(Y) = \int_Y^1 \phi_r(Y)\frac{dY}{Y}. \tag{43}$$

The condition to determine the c's is that

$$\int_0^1 (c_1\Phi_1+c_2\Phi_2+\ldots+c_n\Phi_n+\varpi_0)^2\,dY$$

is a minimum, where ϖ_0 is the value of ϖ_1 when $X = 0$, which we take to be the section from which the solution is started.

Atkinson and Goldstein took into account the terms in c_1 and c_2 in (42), and also found a second approximation by substituting the result so obtained into the neglected terms in (36) and solving again, taking into account in this second approximation only the term in c_1^2. This solution was then joined to the series solution (27) at $\xi = 0{\cdot}05$ by making the mean-square difference a minimum. The results for $x/(aR) \geqslant 0{\cdot}015$ are shown in Fig. 81. For smaller values of $x/(aR)$ they are certainly inaccurate.

The calculated values of $(p_0-p)/(\frac{1}{2}\rho u_m^2)$ are shown in Table 12 below. The

TABLE 12

$\frac{x}{aR}$	$\frac{p_0-p}{\frac{1}{2}\rho u_m^2}$	$\frac{x}{aR}$	$\frac{p_0-p}{\frac{1}{2}\rho u_m^2}$
0	0	0·015	1·36
0·001	0·32	0·020	1·63
0·002	0·46	0·025	1·88
0·003	0·56	0·030	2·10
0·004	0·65	0·040	2·51
0·005	0·73	0·05	2·88
0·007	0·87	0·06	3·24
0·009	1·00	0·07	3·59
0·011	1·11	0·08	3·93
0·013	1·22	0·09	4·26
0·015	1·33	0·10	4·59
		0·11	4·92

values in the first two columns are found by Schiller's method; those in the third and fourth by Atkinson and Goldstein's extension of Boussinesq's method. At $x/(aR) = 0{\cdot}015$ the results differ by about 2 per cent. For more accurate results the values of u/u_m found from the series (27) would have to be continued for larger values of $x/(aR)$ by step-by-step calculations, and the

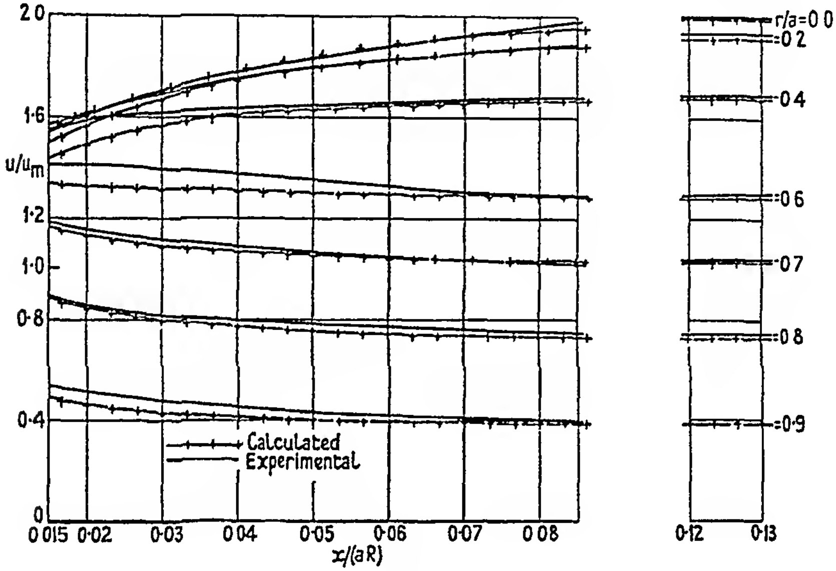


FIG. 81.

extension of Boussinesq's solution joined to the values so found at a larger value of $x/(aR)$ than is at present possible.

For larger values of $x/(aR)$ than those shown in the table

$$\frac{p_0-p}{\frac{1}{2}\rho u_m^2} = \frac{32x}{aR}+1{\cdot}41,$$

and for flow out of a cistern where the pressure is P

$$\frac{P-p}{\frac{1}{2}\rho u_m^2} = \frac{32x}{aR}+2{\cdot}41. \tag{44}$$

140. Two-dimensional flow through a straight channel.

At sections far from the ends the velocity distribution for steady flow in a channel of breadth $2h$ may be obtained by integrating the equation of momentum for a symmetrically situated rectangular slab of fluid of breadth $2y$, with the boundary condition $u = 0$ at $y = h$: the result is

$$u = -\frac{1}{2\mu}\frac{\partial p}{\partial x}(h^2-y^2),$$

where y is the distance from the middle of the channel. The mean velocity over a section is

$$u_m = -\frac{h^2}{3\mu}\frac{\partial p}{\partial x}. \tag{45}$$

The mean velocity is two-thirds the maximum, and the velocity distribution is parabolic.

The hydraulic mean depth, m, is equal to the half-width, h. As Reynolds number we therefore take

$$R = 4hu_m/\nu,$$

and for the resistance coefficient γ we obtain the equation

$$\gamma = \tau_0/\tfrac{1}{2}\rho u_m^2 = 24/R, \tag{46}$$

where τ_0 is the shearing stress at a wall.

141. Two-dimensional flow in the inlet length of a straight channel.

By calculations similar to those in § 139 the velocity distribution, the pressure drop and the skin-friction in the inlet length may be determined for two-dimensional pressure flow between parallel walls. The equation for the pressure drop from the entry to any section where the parabolic flow is fully established, when calculated by Schiller's method, is

$$(p_0-p)/(\tfrac{1}{2}\rho u_m^2) = 24x/(hR)+0{\cdot}626.\dagger \tag{47}$$

The corresponding result from the kinetic energy end-correction is

$$(p_0-p)/(\tfrac{1}{2}\rho u_m^2) = 24x/(hR)+0{\cdot}543. \tag{48}$$

A solution‡ by a method similar to that described on pp. 304–308 for a circular pipe leads to the result

$$(p_0-p)/(\tfrac{1}{2}\rho u_m^2) = 24x/(hR)+0{\cdot}601: \tag{49}$$

the velocity distributions calculated by this method are shown in Figs. 82 and 83. The broken line in Fig. 82 is the Poiseuille parabola: the broken lines in Fig. 83 give the asymptotic values for the parabolic flow. In Table 13, $(p_0-p)/(\frac{1}{2}\rho u_m^2)$ is tabulated against $\nu x/(h^2 u_m)$, where p_0 is the pressure at the entry and p is the pressure at a section distant x downstream.

† Schiller gave 0·614 in place of 0·626. See Schlichting, *Zeitschr. f. angew. Math. u. Mech.* 14 (1934), 372.

‡ Schlichting, *op. cit*, pp. 368–373.

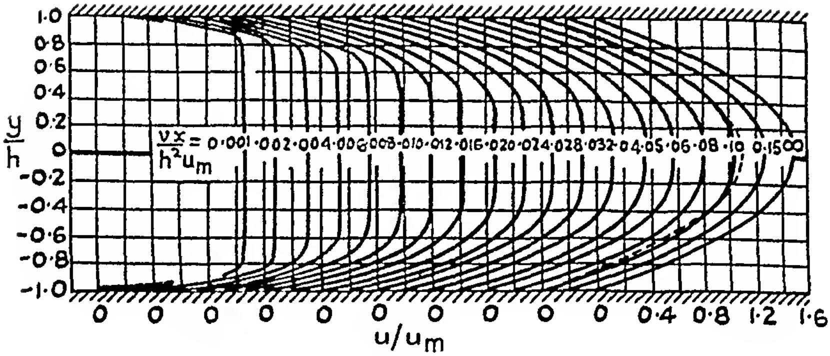


Fig. 82.

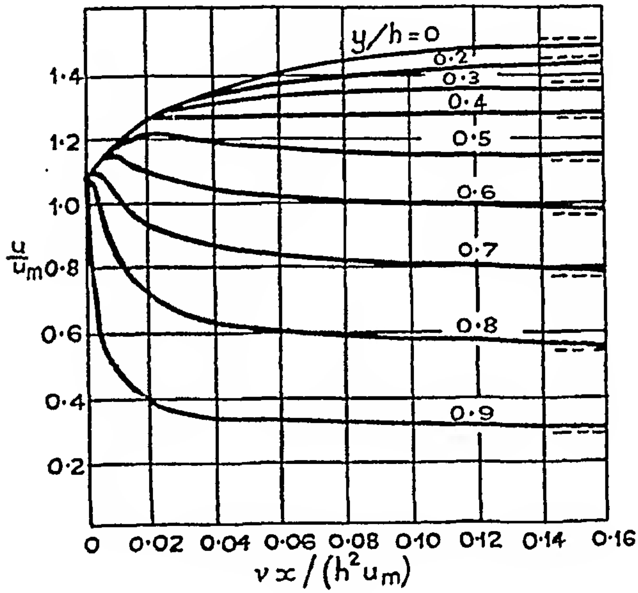


Fig. 83.

TABLE 13

$\dfrac{\nu x}{h^2 u_m}$	$\dfrac{p_0-p}{\frac{1}{2}\rho u_m^2}$	$\dfrac{\nu x}{h^2 u_m}$	$\dfrac{p_0-p}{\frac{1}{2}\rho u_m^2}$
0	0	0·040	0·688
0·001	0·1100	0·050	0·772
0·002	0·1600	0·060	0·850
0·004	0·2322	0·080	1·002
0·006	0·291	0·100	1·145
0·008	0·338	0·125	1·315
0·010	0·378	0·150	1·478
0·012	0·411	0·250	2·101
0·016	0·466	0·500	3·601
0·020	0·510	1·000	6·601
0·025	0·558	∞	∞
0·032	0·620		

142. The effect of roughness on laminar flow in pipes and channels.

We can estimate the order of magnitude of the maximum size of protuberances which will not alter the character of the flow in a circular pipe.† If ϵ is the maximum permissible height of a protuberance, then (ϵ being small) the velocity at its tip is

$$u_\epsilon = 2u_m\{1-(1-\epsilon/a)^2\} \doteqdot 4u_m\,\epsilon/a$$

to the first order. (It is assumed that the presence of the roughness has not altered the character of the flow.) Therefore

$$R_\epsilon\,(= \epsilon u_\epsilon/\nu) = 2(\epsilon/a)^2 R, \quad \text{where} \quad R = 2au_m/\nu.$$

For uniform flow past an obstacle a critical Reynolds number R_c exists such that for $R > R_c$ a vortex wake forms behind the obstacle, whereas for $R < R_c$ the flow closes up behind. If the flow past the protuberances which form the roughness in the pipe is such as to cause the production of vortex wakes, then the form of the flow in the pipe will be considerably different from the flow in a smooth pipe. In order to obtain an order of magnitude for the size of the protuberances which do not produce great variation in the flow we may therefore express the condition that R_ϵ should be less than the value of R_c for flow past an obstacle of the shape of the protuberances. For a circular cylinder, for example, we may take $R_c = 50$,‡ and this value leads to the condition

$$\epsilon/a < 5/R^{\frac{1}{2}}.$$

If for a flat plate normal to a disturbed stream we assume that $R_c = 30$, we obtain $\epsilon/a < 4/R^{\frac{1}{2}}$ as the condition for sharp-edged roughnesses.

If we write U_τ for $\surd(\tau_0/\rho)$, where τ_0 is the shearing stress at the wall, and if we assume that τ_0 is unaltered, then the condition can be expressed in the form that $\epsilon U_\tau/\nu$ must be less than $\surd R_c$, or, with $R_c = 30$, less than about 5·5. This order of magnitude appears to be in satisfactory agreement with experiment.‖

For a channel $R_\epsilon = \frac{3}{4}R(\epsilon/h)^2$, and ϵ/h must be less than about $6/R^{\frac{1}{2}}$ for sharp-edged roughnesses to be without effect. $\epsilon U_\tau/\nu$ must again be less than $\surd R_c$.

† Schiller, *Handbuch der Experimentalphysik*, 4, part 4 (Leipzig, 1932), 191, 192.

‡ Cf. Chap. IX, § 184.

‖ Schiller, *op. cit.*, pp. 189–192; see also p. 377 and Nikuradse, *Ver. deutsch. Ing., Forschungsheft* 361 (1933). It should be observed that, with a disturbed entry, the above estimate will not apply till the disturbance has died down.

143. Flow through curved pipes.

We confine our attention to flow through pipes of circular cross-section whose axes are bent into circles. We take coordinates as shown in Fig. 84 (O being the centre of curvature of the axis of the pipe), and denote by L the radius of curvature of the axis, by a the radius of the pipe, and by U, V, and W the velocities in the directions of r, ϕ, and θ increasing. In the region where the flow is fully developed U, V and W are independent of θ. The equations of motion referred to this system of coordinates are to be found in a paper by Dean;† they are complicated. If a/L is small and if terms of order $1/L$ are neglected compared with terms of order $1/r$, the equations simplify. By reducing these simplified equations to non-dimensional form it may be shown that dynamical similarity depends on the parameter

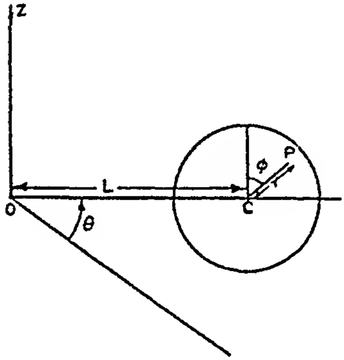


Fig. 84.

$$K' = \left(\frac{a}{L}\right)^{\frac{1}{2}}\left(\frac{2W_0 a}{\nu}\right) \tag{50}$$

only, where W_0 is the mean velocity in flow through a straight pipe under the same pressure gradient as that along the pipe axis in the curved pipe. Thus the ratio of the flux F_c through the curved pipe to the flux F_s through a straight pipe under the same pressure gradient is a function of K' only for small values of a/L; further, if γ_c and γ_s are the resistance coefficients in the curved pipe‡ and in a straight pipe when the flux is F_s in both cases, it may be shown that $F_c/F_s = \gamma_s/\gamma_c$. Expanding in powers of K', Dean finds that

$$\frac{\gamma_s}{\gamma_c} = \frac{F_c}{F_s} = 1-0{\cdot}03058\left(\frac{K'^2}{1152}\right)^2+0{\cdot}01195\left(\frac{K'^2}{1152}\right)^4+\dots. \tag{51}$$

The error when only the three terms shown are taken into account increases with K'. It is probably not serious at any rate for $K' < 30$.

To a first approximation the parameter K' is equal to K, defined as $(a/L)^{\frac{1}{2}}(2W_m a/\nu)$, where W_m is the mean velocity in the flow through

† *Phil. Mag.* (7), 4 (1927), 208–223; 5 (1928), 673–695.

‡ γ_c has a definition equivalent to the definition (3) for γ_s, viz.

$$\gamma_c = -\frac{1}{L}\frac{\partial p}{\partial \theta}\frac{a}{\rho W_m^2}.$$

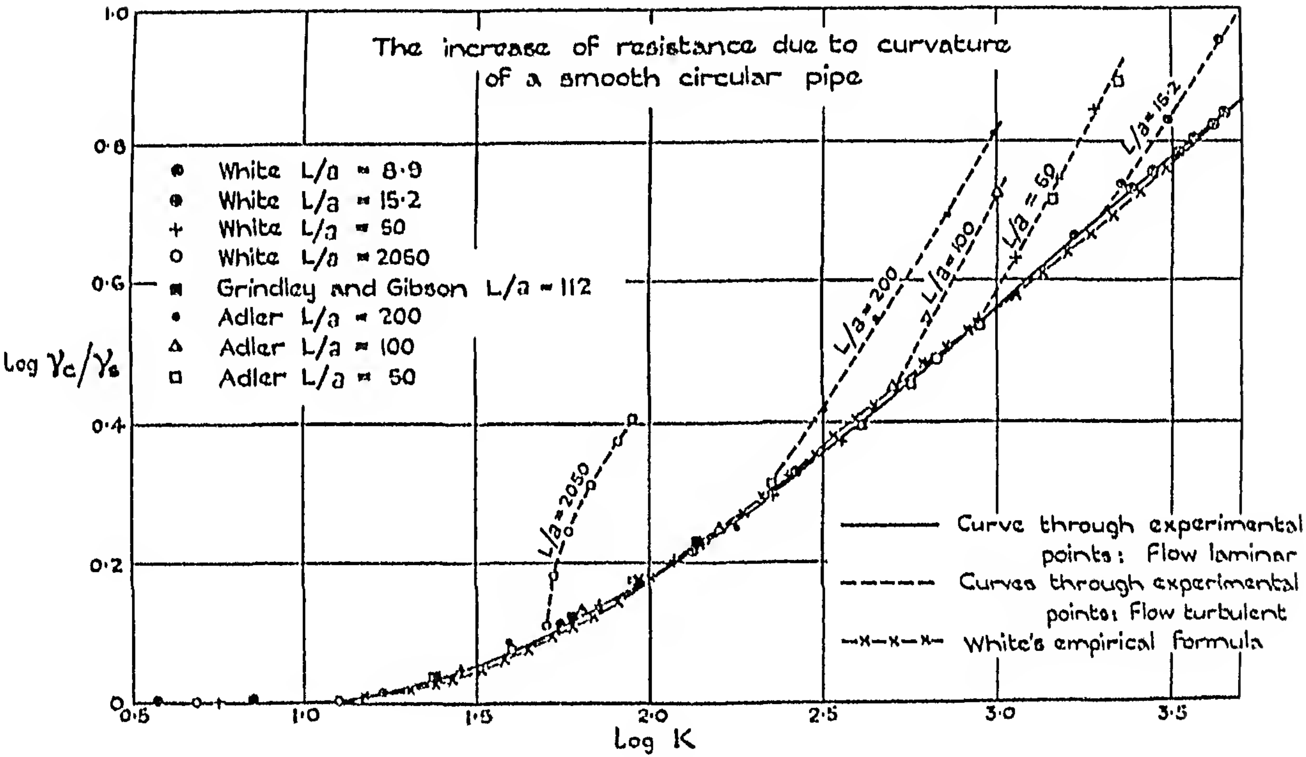

The increase of resistance due to curvature of a smooth circular pipe
White L/a = 8·9
White L/a = 15·2
White L/a = 50
White L/a = 2050
Grindley and Gibson L/a = 112
Adler L/a = 200
Adler L/a = 100
Adler L/a = 50
L/a = 2050
L/a = 200
L/a = 100
L/a = 60
L/a = 15·2
Curve through experimental points: Flow laminar
Curves through experimental points: Flow turbulent
White's empirical formula
Log γ_c/γ_s
Log K
0
0·2
0·4
0·6
0·8
1·0
0·5
1·0
1·5
2·0
2·5
3·0
3·5

FIG. 88.

the curved pipe: it follows immediately from the definition of K that it is in any case a function of K' only (since W_m/W_0 is a function of K' only), so dynamical similarity depends on the value of K only. Experimenters have usually preferred to use K instead of K'.

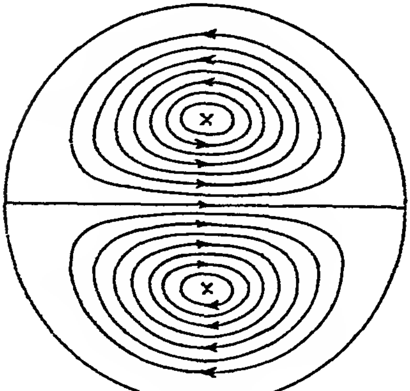


FIG. 86.

Experiments have been performed by White† and Adler,‡ whose results are shown in Fig. 85 by curves with $\log_{10}(\gamma_c/\gamma_s)$ as ordinates and $\log_{10} K$ as abscissae. White remarks that once turbulence sets in the parameter K is not sufficient to define the ratio γ_c/γ_s; this is supported by Adler's measurements. It appears, therefore, that the value of $\log K$ at which turbulence sets in for any particular pipe is the abscissa of the point where the corresponding observed curve in Fig. 85 leaves the universal curve.

The empirical equation

$$\frac{\gamma_c}{\gamma_s} = \left[1 - \left\{1 - \left(\frac{11 \cdot 6}{K}\right)^{0 \cdot 45}\right\}^{2 \cdot 22}\right]^{-1} \quad (11 \cdot 6 < K < 3000), \qquad (52)$$

due to White, gives a good representation of the observed non-turbulent results. The corresponding curve is included in Fig. 85.

Adler (*loc. cit.*) has made a large number of measurements of the velocity distribution, for which reference may be made to the original paper.

The calculated projections of the particle paths on a cross-section of the pipe are shown in Fig. 86 for a very small value of K; they were calculated by neglecting terms of order K^4, but serve to show the secondary motion which is set up.|| White and Adler both found this secondary motion experimentally for laminar flow:†† the former observed that it was by no means as evident once the flow became turbulent. This he ascribed to the fact that in turbulent flow the

† *Proc. Roy. Soc.* A, 123 (1929), 645–663. The results when $L/a = 8 \cdot 9$ were obtained after the publication of the paper, and kindly communicated to us by Prof. White.

‡ *Zeitschr. f. angew. Math. u. Mech.* 14 (1934), 257–275.

|| Dean, *loc. cit.* For a qualitative explanation, see Chap. II, § 28 (p. 84).

†† This secondary flow has also been confirmed in a striking manner by Taylor using a colour thread (*Proc. Roy. Soc.* A, 124 (1929), 243–249).

velocity distribution is much more nearly uniform than in laminar flow, and correspondingly the pressure gradients producing the secondary motion are less.

144. Two-dimensional flow through curved channels.

We confine our attention to a channel formed by two coaxial cylindrical walls, and consider two-dimensional flow perpendicular to the common axis due to a pressure gradient parallel to the walls and perpendicular to the common axis.

If we denote by W the velocity parallel to the walls and perpendicular to the common axis, and by W_m the mean value of W, we find from the equations of motion that

$$W = -\frac{1}{2\mu}\frac{\partial p}{\partial\theta}\left[\frac{(b^2\log b - a^2\log a)}{(b^2-a^2)}r - \frac{a^2b^2}{(b^2-a^2)}\left(\log\frac{b}{a}\right)\frac{1}{r} - r\log r\right], \quad (53)$$

and

$$(b-a)W_m = -\frac{1}{2\mu}\frac{\partial p}{\partial\theta}\left[\frac{(b^2-a^2)}{4} - \frac{a^2b^2}{(b^2-a^2)}\left(\log\frac{b}{a}\right)^2\right], \quad (54)$$

where r is distance measured from the common axis, θ is angular distance measured round the common axis, the walls are at $r = a$ and $r = b$ $(b > a)$, and p is the pressure. ($\partial p/\partial\theta$ may be shown to be a constant independent of r and θ.) The other velocity components vanish: there is a pressure gradient across the channel which just balances the centrifugal force.

We may obtain alternative forms for (53) and (54) by writing $b = a+2h$, $r = a+h+y$, $h/a = \alpha$. These forms are

$$W = -\frac{1}{2\mu}\frac{\partial p}{\partial\theta}\frac{h}{\alpha}\left\{\left[\frac{(1+2\alpha)^2\log(1+2\alpha)}{4\alpha+4\alpha^2}\right]\left[1+\alpha+\frac{\alpha y}{h} - \frac{1}{1+\alpha+\alpha y/h}\right] - (1+\alpha+\alpha y/h)\log(1+\alpha+\alpha y/h)\right\}, \quad (55)$$

$$W_m = -\frac{1}{4\mu}\frac{\partial p}{\partial\theta}\frac{h}{\alpha^2}\left\{\alpha+\alpha^2 - \frac{(1+2\alpha)^2}{4(\alpha+\alpha^2)}\left[\log(1+2\alpha)\right]^2\right\}. \quad (56)$$

Curves in Fig. 87 show W/W_m plotted against y/h for $\alpha = 2$, 0·5, 0·1, and 0 (straight channel). The effect of the curvature is very small when $\alpha = 0{\cdot}1$.

The fact that no secondary flow occurs has been pointed out previously.† The pressure gradient across the channel due to

† Chap. II, § 28 (p. 87).

centrifugal force produces no secondary flow in the absence of walls parallel to it, no retarded layers being produced on which the pressure gradient could act.

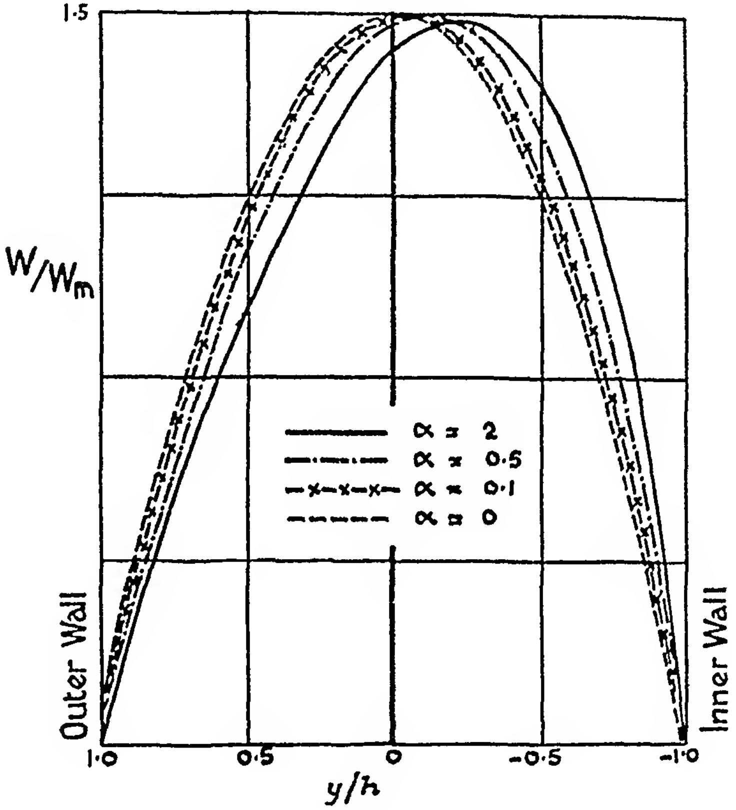


FIG. 87.

145. Flow along a flat plate. Roughness.

The laminar flow along a flat plate placed edgeways to a steady stream has been calculated on the assumptions of the boundary layer theory in Chap. IV, § 53 (pp. 135, 136). The velocity distribution has been measured experimentally by Hansen† in an open jet wind tunnel, and his results are compared with theory in Fig. 88: there is quite good agreement. (In Fig. 88, u_1 is the velocity outside the boundary layer, x is distance from the leading edge, and u the forward velocity at a distance y from the plate.) The experimental

† *Zeitschr. f. angew. Math. u. Mech.* 8 (1928), 185–199.

values of u/u_1 obtained by Dryden† in a closed tunnel, when plotted against $y\sqrt{(u_1/\nu x)}$, are also in very satisfactory agreement with the theoretical curve; whereas the results of measurements in a closed tunnel by Burgers and Van der Hegge Zijnen‡ are consistently greater than those of theory. Burgers attributes the discrepancy to the fact that the main stream may not have been quite steady,

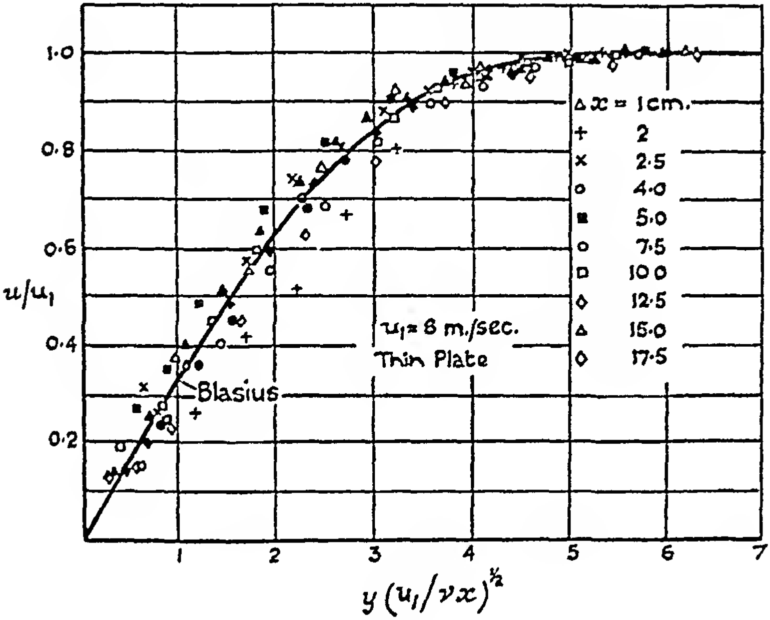


FIG. 88.

and suggests that the interchange which subsequently arises in the boundary layer would lead to these increased values: on the other hand, it is possible that the acceleration of the main stream due to the growth of the boundary layer along both the plate and the channel walls would account for the discrepancy.||

In Chapter IV it was pointed out that for flow along a cylinder open at both ends (with generators parallel to the main flow) the same results should hold as for a flat plate provided that the cylinder is

† *N.A.C.A. Report* No. 562 (1936). See especially Fig. 17.

‡ *Proc. 1st Internat. Congress for Applied Mechanics, Delft*, 1924 (Delft, 1925), pp. 113–128.

|| Due account was taken of the pressure gradient in the direction of flow in reducing the observations, but no account was taken of the pressure gradient in the theoretical calculations.

short enough. Experiments have been carried out on small rings of this type by Miss Marshall,† who obtained the skin-friction from measurements in the wake; the results were larger than those of theory.

The resistance of a flat plate has been obtained by Fage‡ by measurements in the wake in an open jet tunnel; he too finds values which are larger than those calculated.

It may be remarked that considerable fluctuations in the velocity in the 'laminar' portion of the boundary layer along a flat plate have been observed.|| These fluctuations are due to fluctuations in the main flow. The fluctuating components of the velocities are uncorrelated, and so do not give rise to any transfer of momentum or 'apparent stresses'. The fluctuations are of amplitude several times the amplitude in the main stream, but are much slower.

A limit can be set to the permissible height of roughnesses in order that they may not materially affect the flow. From equations (44) and (46) of Chapter IV

$$u = \tfrac{1}{2}u_1\left(\alpha\eta - \frac{\alpha^2\eta^4}{4!} + \dots\right),$$

where $\alpha = 1{\cdot}32824, \qquad \eta = \tfrac{1}{2}\left(\dfrac{u_1}{\nu x}\right)^{\frac{1}{2}} y,$

so that at the tip of a projection of height ϵ,

$$u_\epsilon = \frac{\alpha}{4}\frac{u_1^{\frac{3}{2}}}{\nu^{\frac{1}{2}}}\frac{\epsilon}{x^{\frac{1}{2}}}$$

to the first order. Thus

$$R_\epsilon \left(= \frac{\epsilon u_\epsilon}{\nu}\right) = \frac{\alpha}{4}\frac{u_1^{\frac{3}{2}}\epsilon^2}{\nu^{\frac{3}{2}}x^{\frac{1}{2}}} = R_x^{\frac{3}{2}}\left(\frac{\epsilon}{x}\right)^2\frac{\alpha}{4},$$

where $R_x = u_1 x/\nu$.

Using the same estimates of the critical value (R_c) of R as in § 142, namely $R_c = 50$ for rounded roughnesses and 30 for sharp ones, we see that we require

$$R_x^{\frac{3}{2}}(\epsilon/x)^2 \leqslant 200/\alpha \doteqdot 150$$

or

$$\epsilon/x \leqslant 12{\cdot}2(R_x)^{-\frac{3}{4}}$$

for the former, and

$$R_x^{\frac{3}{2}}(\epsilon/x)^2 \leqslant 120/\alpha \doteqdot 90$$

or

$$\epsilon/x \leqslant 9{\cdot}5(R_x)^{-\frac{3}{4}}$$

† *A.R.C. Reports and Memoranda*, No. 1004 (1926).

‡ *A.R.C. Reports and Memoranda*, No. 1580 (1934), pp. 1–7.

|| Dryden, *Proc. Fourth Internat. Congress for Applied Mechanics, Cambridge*, 1934 (Cambridge, 1935), p. 175; *Journal of the Washington Academy of Sciences*, 25 (1935), 106; *N.A.C.A. Report* No. 562 (1936).

for the latter, if the effect of the roughness is to be inappreciable. It is evident that a given size of roughness is more likely to have a disturbing effect near the leading edge than elsewhere.

In terms of U_τ the condition is $\epsilon U_\tau/\nu < \sqrt{R_c}$, exactly as for pipes and channels (§ 142).

SECTION II

THE TRANSITION FROM LAMINAR TO TURBULENT FLOW

146. The transition to turbulence in smooth pipes and channels.

The phenomena in a pipe or channel are closely associated with the entry conditions. With disturbed conditions at entry the disturbances die out after a certain length if R is small enough. As R is increased a critical value R_{crit} is reached such that for $R > R_{\text{crit}}$ the disturbances are no longer damped out and the flow in the pipe is turbulent. There is a minimum value of R_{crit} such that for R less than this minimum value all disturbances, however great, are damped out sufficiently far downstream. This minimum value of R_{crit} is determined experimentally by imposing very disturbed conditions at entry: the values obtained by various experimenters are shown in the following tables for straight and for curved pipes.

Straight Pipes

R_{crit}	*Shape of cross-section*	*Experimenter*	*Method*
2000	Smooth circular	Reynolds†	pressure drop and flux (water)
2100	,, ,,	Ruckes‡	,, ,, ,, ,, (air)
2100	,, ,,	Stanton and Pannell‖	,, ,, ,, ,, (air and water)
2000	,, ,,		$u_m/u_{\max}$ and $au_{\max}/\nu$ (air and water)
1900	,, ,,	Barnes and Coker††	axial temperature (with pipe surrounded by jacket) and flux (water)
2800	Rectangular pipe: ratio of sides 104:1–165:1	White and Davies‡‡	resistance and flux (water)
2100	Square	Schiller‖‖	,, ,, ,, ,,
1600	Rectangular pipe: ratio of sides 2·83:1	Schiller‖‖	,, ,, ,, ,,
2400	Annular pipe: ratio of radii 0·818:1	Fage†††	pressure and flux (water)

† *Phil. Trans.* **174** (1883), 935–982; *Scientific Papers*, **2**, 51–105.
‡ *Ann. d. Phys.* (4), **25** (1908), 983–1021.
‖ *Phil. Trans.* A, **214** (1914), 199–224.
†† *Proc. Roy. Soc.* A, **74** (1905), 341–356.
‡‡ *Ibid.* **119** (1928), 92–107.
‖‖ *Zeitschr. f. angew. Math. u. Mech.* **3** (1923), 2–13.
††† *Proc. Roy. Soc.* A, **165** (1938), 520–525.

The two quantities mentioned in the last column were in each instance plotted against each other, and a sudden change in the relationship was taken to indicate the onset of turbulence. The resistance was obtained, in those methods where it was used, from pressure measurements; there is therefore effectively no difference between the two methods. The experimental fluid is mentioned in brackets.

Curved Pipes of Circular Cross-Section

R_{crit}	a/L	*Experimenter*	*Method*
7590†	0·066	White‡	Curves of γ_c/γ_s plotted against K (see pp. 313, 314). Point where particular curve leaves universal one taken to indicate onset of turbulence.
6020†	0·020	White‡	
5620	0·020	Adler‖	
4730	0·010	Adler‖	
3980	0·005	Adler‖	
2270†	0·00049	White‡	

White (*loc. cit.*) remarks that turbulence arises, in all his experiments, when $\gamma_c = 0{\cdot}0090$. It is rather difficult to take values accurately from Adler's graph; it appears that, in the order given in the table, his results give $\gamma_c = 0{\cdot}0095$, $0{\cdot}0090$, $0{\cdot}0084$. The difference between 0·0090 and the first and last of these results is probably greater than the error in estimating values from the graphs. Even so, $\gamma_c = 0{\cdot}0090$ gives a rough idea of the value of R_{crit}.

147. The effect of roughness on the critical Reynolds number.

If the disturbances produced by the roughnesses are less than those introduced in the entry, roughness may be expected to have no effect on R_{crit}: if they are bigger, roughness will certainly affect R_{crit}. No systematic effects of roughness on the minimum value of R_{crit} are apparent from experimental data available, and no great differences between the minimum values of R_{crit} for smooth and rough pipes have been obtained.

148. The entry length. Experimental results for smooth entry conditions in straight pipes.

In determining the values of R_{crit} experimentally a sufficient entry length must be allowed in order to determine whether an initial disturbance will increase or be damped out. This length depends

† These figures are given by Taylor (*Proc. Roy. Soc.* A, **124** (1929), 243–249), who took them from White's results. He also confirmed these values of R_{crit} by means of colour threads.

‡ *Loc. cit. ante* (p. 314).

‖ *Loc. cit. ante* (p. 314).

largely on the nature of the disturbances. Schiller† found that a length of 130 diameters was sufficient, whereas one of 65 diameters was not. For flow in a channel (a rectangular pipe of large breadth/depth ratio) Davies and White‡ found that 54 times the depth was a sufficient entry length.

As the disturbances at entry decrease, R_{crit} increases. (There does not seem to be any evidence concerning a possible systematic dependence of R_{crit} on the *form* of the disturbances.) Reynolds believed

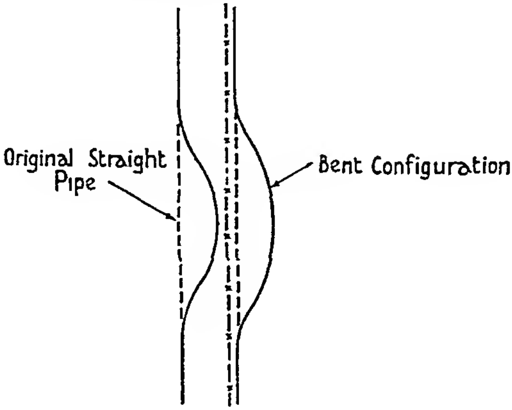


FIG. 89.

that an upper critical number exists for flow in pipes such that for greater numbers the flow is necessarily turbulent: experimentally, using a colour thread, he found a value 6,000 for such a number. The available evidence seems to show, however, that R_{crit} could be increased indefinitely if the disturbances present could be decreased indefinitely. Barnes and Coker (*loc. cit.*) obtained laminar flow at a Reynolds number greater than 2×10^4, and Schiller‖ reached about the same value: Ekman,†† using Reynolds's original apparatus, obtained laminar flow in one case at a Reynolds number as high as 5×10^4. Taylor, in some unpublished experiments with a very long inlet, found that laminar flow was possible up to a Reynolds number of $3{\cdot}2 \times 10^4$, and also showed that when the pipe was bent by an amount nearly equal to the diameter, the flow still remained laminar. Moreover, a filament line (shown —×—×— in Fig. 89) appeared nearly straight.

† *Zeitschr. f. angew. Math. u. Mech.* 1 (1921), 436–444.

‡ *Proc. Roy. Soc.* A, 119 (1928), 99.

‖ *Op. cit.*, p. 442.

†† *Ark. f. mat., astr. och. fys.* 6 (1910), No. 12.

149. The instability of laminar flow in curved channels.

The theoretical and experimental results obtained by Taylor for the stability of the flow between two rotating cylinders were described in Chap. V, § 75 (pp. 196, 197).† Taylor's experimental work was confined to water and glass surfaces; Lewis‡ has covered a larger range by using xylene, nitrobenzene, and a mixture of the two as experimental fluids and by using a lead or silver inner surface together with a glass outer tube.

We discussed steady laminar flow in a curved channel on pp. 315, 316: Dean|| has made calculations concerning its stability, assuming infinitesimal disturbances symmetrical about the common axis and periodic along it. If $2h$ and a denote respectively the distance between the walls and the radius of the inner wall, as before, then Dean found for small values of h/a a critical Reynolds number given by $2W_m h/\nu = 25{\cdot}45(a/h)^{\frac{1}{2}}$. For greater Reynolds numbers the system is unstable, the instability first manifesting itself for a disturbance with a particular wave-length in the direction parallel to the axis.

The type of disturbance for which instability is produced in this problem does not produce instability in flow through a straight channel. It is evident, therefore, that curvature has a marked effect on the stability of steady flow.†† This result is borne out by experiments on flow in straight and curved pipes. When turbulence appears in straight pipes there is a sudden increase in the loss of head, which is not observed in curved pipes. A possible explanation is that curved flow becomes unstable for infinitesimal disturbances, whereas straight flow becomes unstable for finite disturbances only.

The stability of the flow due to a pressure gradient parallel to their common axis between two rotating cylinders has been investigated theoretically‡‡ and experimentally.|||| If the radii of the cylinders are a, $a+2h$ and their angular velocities Ω_1, Ω_2, where $\Omega_2/\Omega_1 = \alpha$, then for small values of h/a stability depends, for a given value of the Reynolds number R of the flow parallel to the axis, on the value of

$$(1+\alpha)\frac{\Omega_1^2 h^3}{\nu^2}\frac{a^2-\alpha(a+2h)^2}{a+h}.$$

† For further information concerning critical speeds for flow between rotating cylinders see also pp. 388–390.

‡ *Proc. Roy. Soc.* A, **117** (1928), 388–407.

|| *Ibid.* **121** (1928), 402–420.

†† Cf. also pp. 388–390.

‡‡ Goldstein, *Proc. Camb. Phil. Soc.* **33** (1937), 41–61.

|||| Cornish, *Proc. Roy. Soc.* A, **140** (1933), 227–240; Fage, *ibid.* **165** (1938), 513–517.

If α is not nearly equal to unity, this reduces approximately for small values of h/a to

$$\frac{(1-\alpha^2)\Omega_1^2 h^3 a}{\nu^2}.$$

The theoretical numerical results have been evaluated for the outer cylinder at rest and for small values of R: for $R = 0$ they agree with Taylor's result. The experiments also were performed with the outer cylinder at rest. Apart from a rapid fall in the calculated value of the critical rotational speed Ω_c near the highest value of R at which calculations were made (which was probably due to too drastic mathematical approximations), the calculated results are consistent with the results of Fage's experiments. These experiments show a continual increase in Ω_c with increasing R. The value of Ω_c was found by measurements of the pressure drop down a test length, and the results of these measurements, at each fixed value of R, show that as Ω_1 is increased beyond Ω_c the difference of the pressure drop from its theoretical value for laminar flow is at first very small and increases very slowly, and that this stage is followed by a much more rapid increase at higher speeds.

150. Phenomena associated with a disturbed entry. Schiller's theory of the transition to turbulence in straight pipes and channels.

The form of the disturbances associated with the transition from laminar to turbulent flow in a straight pipe of circular cross-section with various types of entry has been examined experimentally† by introducing a coloured indicator. Three distinct types of flow were observed.

(i) For the smallest Reynolds numbers, even though the flow before entering the pipe is disturbed, the flow in the pipe is smooth and the colour thread straight right from the entry. For a straight circular pipe with a sharp entry Naumann gives $R = 280$ as the Reynolds number at which this régime breaks down when the water from the reservoir is very disturbed.

(ii) At higher Reynolds numbers the colour filament assumes a wave-like form in what is roughly the inlet length as calculated by Schiller (see p. 301), but becomes rectilinear farther downstream.

† Schiller, *Proc. 3rd Internat. Congress for Applied Mechanics, Stockholm*, 1930, 1, 226–233; *Zeitschr. f. angew. Math. u. Mech.* 14 (1934), 36–42; Naumann, *Forsch. Ingwes.* 2 (1931), 85–98; Kurzweg, *Ann. d. Phys.* 18 (1933), 193–216.

Apparently a vortex-sheet, arising from the edge of the entry to the pipe, encloses with the wall a dead water region. It seems probable that this vortex-sheet is unstable for sufficiently high speeds and that the wave-like form is due to this instability. When the colour filament is moved towards the centre of the pipe the point at which waves begin moves continuously farther from the entry.

With increasing velocity the amplitudes of the waves increase. Finally, with still further increase of velocity, the vortex-sheet rolls up periodically and discrete eddies are formed. The disturbances still die away downstream and the flow is laminar sufficiently far from the entry. This type of flow continues for a sharp entry until R is between 1,600 and 1,700. The greater the value of R in the range 280–1,600 the greater the distance the disturbances travel before being damped out: this distance is always roughly the same as the inlet length found by Schiller.

(iii) When R is between 1,600 and 1,700 a second critical stage is reached. The vortex-sheet rolls up into a single large stationary cylindrical eddy which extends from the pipe entry to a distance L downstream and is of thickness d ($L \gg d$). At a distance of about

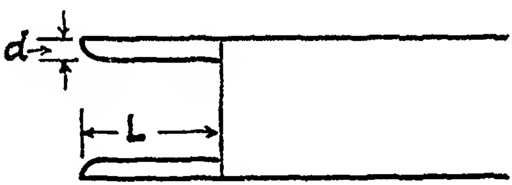


FIG. 90.

$\frac{2}{3}L$ from the entry this elongated eddy exhibits a gradually increasing constriction which leads finally to the separation of a disturbance eddy passing off downstream. This casting off occurs periodically. The distance between consecutive eddies is much greater than in stage (ii).

With the formation of this elongated eddy damping-out of the initial disturbances no longer occurs and the flow is wholly *turbulent*.

Similar results were found for a circular pipe with an annular cover—i.e. a plane annulus placed over the entry, effectively reducing the diameter there—and also with a semicircular cover.

In the neighbourhood of the critical Reynolds number, Schiller and Naumann observed (as did Reynolds) that the laminar flow was occasionally interrupted for a short distance by a vigorous eddying

motion. Reynolds called such regions of turbulence 'flashes', but no explanation of them has yet been given.

Naumann† has discussed two-dimensional flow in a straight channel, and finds results of the same kind as for a pipe. White and Davies, independently of Schiller, discovered by means of pressure measurements the existence of the three separate régimes (i), (ii), and (iii) for a rectangular channel of great breadth/depth ratio.‡ They also found 280 as the Reynolds number at which the first régime breaks down.

In a circular pipe with a sharp entry the vorticity ζ in the disturbance is about circles round the axis of the pipe. The circulation Γ per unit length in the disturbance is defined as $\int \zeta\, dS$, the integral being taken over the area enclosed by the rectangular circuit which is formed by a line of unit length along the axis, two parallel radii at its ends, and the intercept they make on the pipe wall. Schiller and Kurzweg suggest that turbulence sets in when

$$a\Gamma = 1{,}170\nu,$$

where a is the radius of the pipe.|| Naumann† obtained analogous results for flow in a channel.

151. The transition to turbulence in flow along a flat plate.

For sufficiently high velocities or sufficiently long plates laminar flow in the boundary layer near the leading edge of a plate at zero incidence is followed by a transition to turbulence farther downstream. The flow does not become a fully developed turbulent flow immediately it ceases to follow the laws of laminar flow: there is a finite transition region. The shearing stress at the plate decreases with increasing distance downstream both in the laminar and in the fully developed turbulent regions; in the transition region, however, the shearing stress at the upstream end is considerably less than that at the downstream end. General statements concerning the length of the transition region, or any of the average conditions within it, cannot yet be made with confidence. Burgers and van der Hegge Zijnen†† have investigated the conditions experimentally, and Fig. 91

† *Forsch. Ingwes.* 6 (1935), 139–145.

‡ *Proc. Roy. Soc.* A, 119 (1928), 92–107.

|| Hahnemann (*Forsch. Ingwes.* 8 (1937), 226–237) has verified this criterion for sufficiently small disturbances at entry, such that R_{crit} is greater than 3,200. When the entry disturbances are increased, so that R_{crit} falls from 3,200 to the lower critical value 2,320, $(a\Gamma/\nu)_{\mathrm{crit}}$ rises considerably.

†† *Proc. 1st Internat. Congress for Applied Mechanics, Delft*, 1924, pp. 113–128.

gives their measured values of the velocity gradient α at the surface of a glass plate in the transition region, plotted against the distance x from the leading edge for various values of U_1, the stream velocity outside the boundary layer.

Dimensional considerations suggest that in any particular experi-

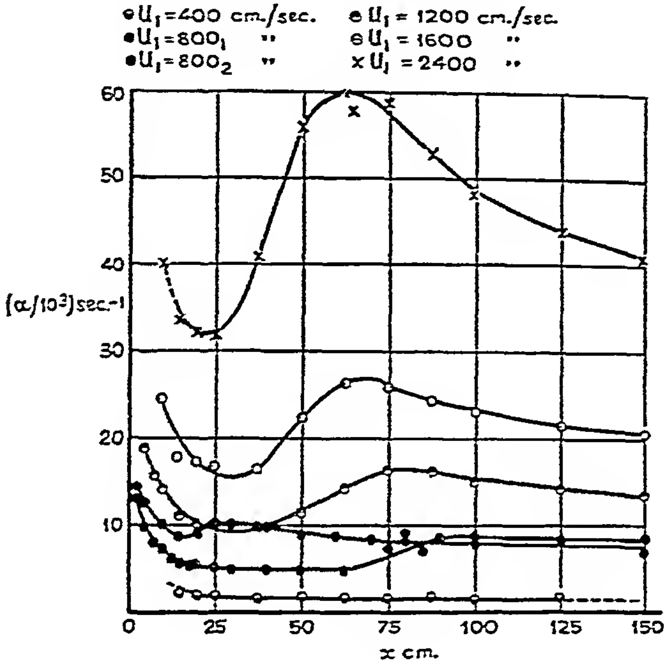


FIG. 91.

ment the transition region starts when R_x $(= U_1 x/\nu)$ reaches a certain value dependent on the turbulence in the main stream. Since R_x and R_δ $(= U_1\delta/\nu$, where δ is the thickness of the boundary layer) are related, any particular value of R_x corresponds to a particular value of R_δ. The values found experimentally for R_{xT}, the value of R_x at the commencement of the transition region, vary between 9×10^4 and $1{\cdot}1\times 10^6$;† the corresponding range for $R_{\delta T}$ is about 1,650

† Dryden (*Journ. Aero. Sciences*, 1 (1934), 71, 72; *Proc. 4th Internat. Congress for Applied Mechanics, Cambridge*, 1934, p. 175: *N.A.C.A. Report* No. 562 (1936), p. 21) has reported that as a result of experiments in the Bureau of Standards tunnel he found that $R_{xT} = 1{\cdot}1\times 10^6$ for $u/U_1 = 0{\cdot}005$ (free tunnel) and $R_{xT} = 10^5$ for $u/U_1 = 0{\cdot}03$ (stream behind wire screen of $\frac{1}{4}$-in. mesh). u denotes the root-mean-square of the turbulent velocity component in the direction of the main stream. The corresponding values of $U_1\delta_1/\nu$ were 1,700 and 560, where δ_1 denotes the displacement thickness (Chap. IV, eqn. (15), p. 123). See p. 330 for recent results.

to 5,750. There is no evidence that these numbers are upper and lower limits to R_{xT} or $R_{\delta T}$, or that such limits exist.

Small pressure gradients in the air-stream greatly affect the value of $U_1 x/\nu$ at transition, an accelerating gradient delaying the transition.† The value of $U_1 \delta_1/\nu$ is, however, less sensitive. (δ_1 denotes the displacement thickness: see Chap. IV, equation (15), p. 123.)

Prandtl‡ observes that the point where transition starts oscillates with time. Cathode ray oscillograph records|| of the fluctuations of the velocity component parallel to the stream show that transition is actually a sudden phenomenon, with an intermittent change from laminar to eddying flow in the transition region, eddying flow occurring at infrequent intervals and for only a short fraction of the time near the beginning of the region and at more and more frequent intervals and for an increasing fraction of the time as the downstream end of the region is approached. Near the downstream end there are only short, infrequent occurrences of flow of the type in the laminar portion of the boundary layer (with slow, uncorrelated fluctuations: see p. 318).

If transition is controlled by variations in the pressure gradient due to turbulence in the main stream, then fluctuations in the pressure distribution will cause fluctuations in transition; but turbulence will occur more frequently the thicker the boundary layer and so the farther downstream the section we are considering, until, sufficiently far downstream, it will be practically permanent. A satisfactory definition of a point of transition can, then, be given only on a statistical basis. The ideas described below furnish a beginning in this direction.

We have remarked that the value of $U_1 x/\nu$ or $U_1 \delta/\nu$ at the commencement of the transition region depends on the turbulence in the main stream. It appears, however, that the value of $U_1 \delta/\nu$ at transition is not a function of u/U_1 only (where u is the root-mean-square of the turbulent velocity component in the direction of the main stream) but depends also on the scale of the turbulence-producing mechanism. G. I. Taylor†† seeks to find the correct functional relation by associating the two suggestions that the disturbance necessary to produce turbulence is a function of $U_1 \delta/\nu$ and that

† Dryden, *loc. cit.*

‡ *Aerodynamic Theory* (edited by Durand), **3** (Berlin, 1935), 152.

|| Dryden, *Journal of the Washington Academy of Sciences*, **25** (1935), 105–107; *N.A.C.A. Report* No. 562 (1936).

†† *Proc. Roy. Soc.* A, **156** (1936), 307–310.

transition is controlled by variations in the pressure gradient due to turbulence in the main stream. If $\partial u/\partial x$ is the root-mean-square of the x-derivative of the turbulent velocity component along the plate, $\partial p/\partial x$ the root-mean-square of the x-derivative of the turbulent variation in static pressure, Taylor argues that, with isotropic turbulence, $\partial p/\partial x$ is of the same order of magnitude as $\rho u\, \partial u/\partial x$, and puts these two expressions proportional to one another.† From Chap. V, § 91, equations (76), (79) and (80), we see that $\partial u/\partial x$ is u/λ; and if M is a characteristic length of the turbulence-producing mechanism, then (Chap. V, § 92, equation (84)) λ is proportional to $\nu^{\frac{1}{2}}M^{\frac{1}{2}}u^{-\frac{1}{2}}$ for sufficiently large values of Mu/ν. Hence

$$\frac{\partial p}{\partial x} = \text{constant}\, \rho u^{\frac{5}{2}} M^{-\frac{1}{2}} \nu^{-\frac{1}{2}}. \tag{57}$$

Now in the Kármán-Pohlhausen approximate method of considering flow in a laminar boundary layer (Chap. IV, § 60), the velocity distribution at any section depends only on the parameter

$$\Lambda = -\frac{\delta^2}{\rho\nu U_1}\frac{\partial p}{\partial x}, \tag{58}$$

where U_1 is the velocity in the main stream just outside the boundary layer, and $\partial p/\partial x$ is the pressure gradient. This result (which rests on approximations of a rather drastic nature) makes the velocity distribution depend only on the pressure gradient at the section considered and not on the state of affairs upstream (except in so far as this affects δ): it seems reasonable to suppose that for fluctuating pressure gradients the result may still be applied with the same degree of approximation as before, and that the velocity distribution depends on $\Lambda+\Lambda'$, where

$$\Lambda' = -\frac{\delta^2}{\rho\nu U_1}\frac{\partial p}{\partial x}. \tag{59}$$

For a flat plate $\Lambda = 0$, and δ^2 is proportional to $\nu x/U_1$. Hence

$$\Lambda' = \text{constant}\left(\frac{u}{U_1}\right)^{\frac{5}{2}}\left(\frac{xU_1}{\nu}\right)^{\frac{1}{2}}\left(\frac{x}{M}\right)^{\frac{1}{2}}. \tag{60}$$

If it is supposed that the critical value of Λ' necessary to produce turbulence is a function of $\delta U_1/\nu$ or xU_1/ν, it follows that xU_1/ν at the point of transition to turbulence is a function of $(u/U_1)(x/M)^{\frac{1}{5}}$.

Owing to its complexity, we cannot formulate a mathematical

† See *Proc. Roy. Soc.* A, 151 (1935), 476, 477; *Proc. Camb. Phil. Soc.* 32 (1936), 382–384.

theory of the flow in the transition region. In problems where it is necessary to consider a boundary layer containing both laminar and turbulent portions it is customary to neglect the length of the transition region. The point where, for mathematical purposes, the instantaneous transition is imagined to take place (say $x = X$) will presumably lie within the actual transition region. Suppose now that the actual flow is fully turbulent for $x \geqslant L$: the problem arises as to how X and the conditions at $x = X$ may be determined in order that calculated and observed values of the velocity and of the wall friction may agree for $x \geqslant L$. We assume that X is determined by a condition $U_1 x/\nu = C$, where C depends on the turbulence in the main flow. The equation of momentum may be written†

$$\frac{d\vartheta}{dx} = \frac{\tau_0}{\rho U_1^2}, \tag{61}$$

where τ_0 is the wall friction and ϑ the 'momentum thickness' (Chap. IV, eqn. (37)). For an instantaneous transition ϑ is therefore continuous at $x = X$. If we take the flow as laminar on one side of a sudden transition, and as the completely developed turbulent flow on the other side, then if we calculate ϑ in terms of the boundary layer thickness δ for the laminar flow by the Kármán-Pohlhausen approximate theory (Chap. IV, § 60) and for the turbulent flow by means of formulae given in Chap. VIII, § 163, we find that making ϑ continuous implies a fairly small discontinuity in δ. On the other hand, it implies a very large increase in τ_0—a much larger increase than is usually observed between the beginning and the end of the transition region. It may be possible, then, to obtain better agreement with observed values, especially for the wall friction, by making a different *ad hoc* assumption at a point where transition is assumed to take place. Prandtl's assumption,‡ that for $x \geqslant X$ the turbulent layer behaves as though it had been turbulent right from the leading edge, implies a considerably greater discontinuity in δ and consequently a much smaller discontinuity in τ_0 than making ϑ continuous; and if X is taken as the abscissa of the point where the flow ceases to be laminar this assumption seems to give fairly good agreement with experiment in the fully developed turbulent region. When some assumption is required we shall then, for flow along a flat plate, use Prandtl's condition.

† Chap. IV, equation (38), p. 133.

‡ *Ergebnisse der Aerodynamischen Versuchsanstalt zu Göttingen*, 3 (1927), 1–5.

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